Whitney Stratifications and Combinatorics

Margaret Readdy, U Kentucky & IAS.

Joint with

Richard Ehrenborg
U Kentucky & IAS

and

Mark Goresky
IAS.

Thanks to the Simons Foundation.

P n-dim'l polytope

The <u>f-vector</u> $(f_0, \ldots, f_{n-1})$

$f_i = \#\ i$-dim'l faces.

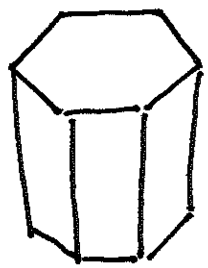
[Steinitz 1906] Characterized f-vectors of 3-dim'l polytopes

Open Q: Characterize f-vectors of n-dim'l polytopes, $n \geq 4$

[Stanley 1978; Billera-Lee 1980]. Done for simplicial polytopes.

P, n-dim'l polytope

The <u>flag f-vector</u> f_S



$$h_S = \sum_{T \subseteq S} (-1)^{|S-T|} f_T,$$

the <u>flag h-vector</u>

S	f_S	h_S	u_S
$\emptyset$	1	1	aaa
0	12	11	baa
1	18	17	aba
2	8	7	aab
01	36	7	bba
02	36	17	bab
12	36	11	abb
012	72	1	bbb

[Stanley] $h_S = h_{\bar{S}}$

The ab-index

$$\Psi(P) = \sum_{S} h_S \cdot w_S.$$

ex $\Psi($  $) =$

$$1\,aaa + 11\,baa + 17\,aba + 7\,aab + 7\,bba + 17\,bab + 11\,abb + 1\,bbb$$

$$= (a+b)^3 + 10(baa + aba + bab + abb) + 6\,(aba + aab + bba + bab)$$

$$= (a+b)^3 + 10\,(ab+ba)(a+b) + 6\,(a+b)(ab+ba)$$

$$= c^3 + 10\,dc + 6cd,$$

where $c = a+b$, $d = ab+ba$. The cd-index

Theorem: [Bayer–Klapper 1991; Stanley 1994].

P polytope then $\Psi(P) \in \mathbb{Z}\langle c,d\rangle$

P Eulerian poset then $\Psi(P) \in \mathbb{Z}\langle c,d\rangle$.

Eulerian: $\mu(x,y) = (-1)^{\rho(x,y)}$ for every interval $[x,y]$ in a graded poset P.

Equivalently, in each non-trivial interval $[x,y]$:

elts of even rank = # elts of odd rank.

A brief cd-history.

[Bayer - Billera 1985] Generalized Dehn - Sommerville relations.

[Bayer - Klapper 1991] Ψ removes all linear relations among flag vector entries.

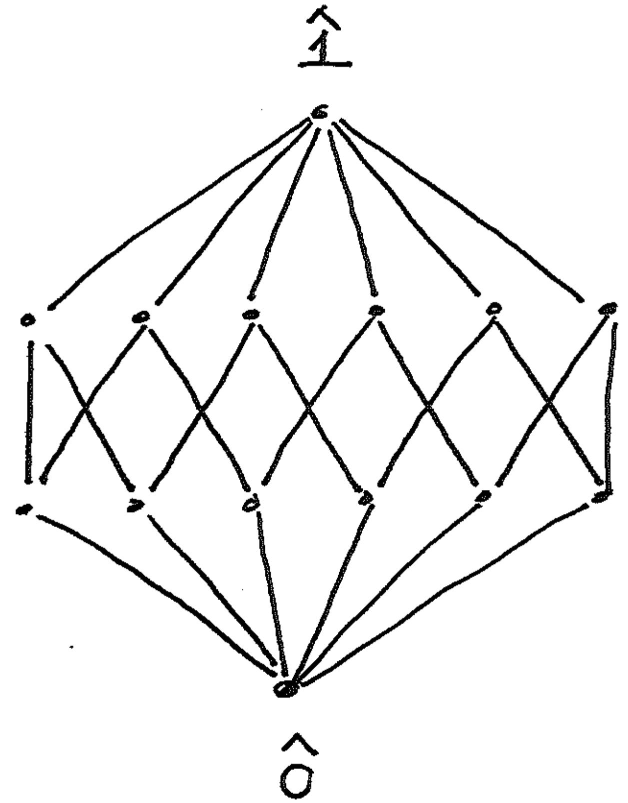
[Stanley 1994]. $\Psi \geq 0$ for $\mathcal{L}$(polytope), more generally, S-shellable face poset of regular CW-complex.

[Purtill 1993]. n-simplex $\Longleftrightarrow$ André & n-cube $\Longleftrightarrow$ signed André perms

[Ehrenborg - R 1998] coalgebraic techniques.

ex. The n-gon $(n \geq 2)$

S	f_S	h_S	w_S
$\emptyset$	1	1	aa
0	n	$n-1$	ba
1	n	$n-1$	ab
01	$2n$	1	$bb.$



$\Phi($  $) = c^2 + (n-2)d.$

ex 1-gon

S	f_S	h_S
$\emptyset$	1	1
0	1	0
1	1	0
01	1	0

Not Eulerian.

Try again ...

$\text{link}_e(v) = \cdot\ \cdot$

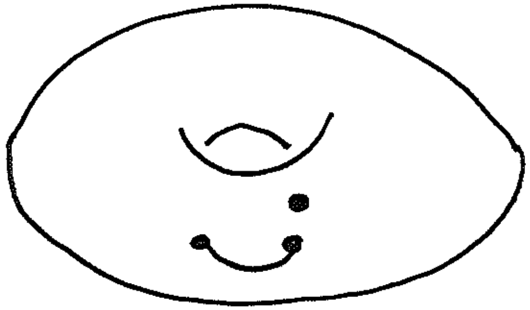
$\chi(\cdot\ \cdot) = 2$, the Euler characteristic.

S	$\overline{f}_S$	$\overline{h}_S = \sum_{T \subseteq S} (-1)^{\lvert S-T \rvert} \overline{f}_T$
$\emptyset$	1	1
0	1	0
1	1	0
01	2	1

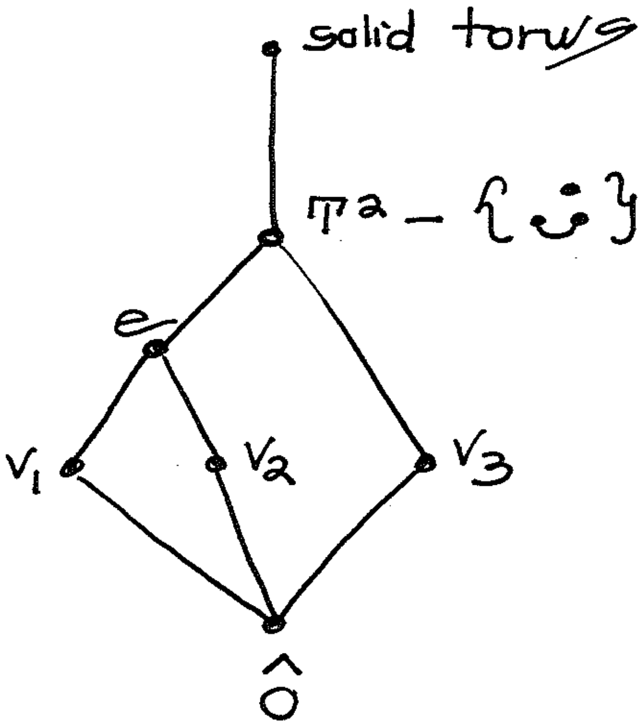
$\Psi($  $) = aa + bb$

$= c^2 - d.$

ex.



Face poset



Chain $c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$

in the face poset weighted by.

$$\bar{\eta}(c) = \chi(x_1) \cdot \chi(\mathrm{link}_{x_2}(x_1)) \cdots \chi(\mathrm{link}_{x_k}(x_{k-1}))$$

ex. (cont'd).

S	$\bar{f}_S$	$\bar{h}_S$	$3dc$	$-2cd$
$\emptyset$	0	0	0	0
0	3	3	3	0
1	1	1	3	-2
2	-2	-2	0	-2
01	2	-2	0	-2
02	2	1	3	-2
12	2	3	3	0
012	4	0	0	0

$\Psi($  $) = 3dc - 2cd.$

These are examples of

Whitney stratifications

Subdivide space into strata:

$$W = \dot{\bigcup_{X \in P}} X$$

Condition of the frontier:

$$X \cap \overline{Y} \neq \emptyset \iff X \subseteq \overline{Y} \iff X \leq_P Y$$ in face poset P.

Whitney conditions A + B:

No fractal behavior

No infinite wiggling ex. $\sqrt{x} \cdot \sin\left(\frac{1}{\sqrt{x}}\right)$

$\Rightarrow$ The links are well-defined.

The Fine Print

Definition Let W be a closed subset of a smooth manifold M, and suppose W can be written as a locally finite disjoint union

$$W = \bigcup_{X \in \mathcal{P}} X$$

where $\mathcal{P}$ is a poset. Furthermore, suppose each $X \in \mathcal{P}$ is a locally closed subset of W satisfying the *condition of the frontier*:

$$X \cap \overline{Y} \neq \emptyset \iff X \subseteq \overline{Y} \iff X \leq_{\mathcal{P}} Y.$$

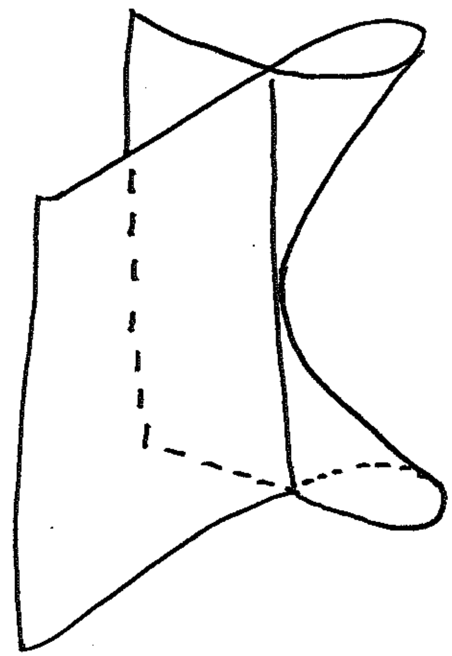
This implies the closure of each stratum is a union of strata. We say W is a *Whitney stratification* if

1. Each $X \in \mathcal{P}$ is a locally closed smooth submanifold of M (not necessarily connected).
2. If $X <_{\mathcal{P}} Y$ then Whitney's conditions (A) and (B) hold: Suppose $y_i \in Y$ is a sequence of points converging to some $x \in X$ and that $x_i \in X$ converges to x. Also assume that (with respect to some local coordinate system on the manifold M) the secant lines $\ell_i = \overline{x_i y_i}$ converge to some limiting line ℓ and the tangent planes $T_{y_i}Y$ converge to some limiting plane τ. Then the inclusions

$$\text{(A) } T_x X \subseteq \tau \qquad \text{and} \qquad \text{(B) } \ell \subseteq \tau$$

hold.

ex. The Whitney cusp.



Whitney stratifications (their face posets) are examples of ...

A <u>quasi-graded poset</u> $(P, \rho, \bar{\zeta})$ consists of

i. P finite poset with $\hat{0} + \hat{1}$ (not necessarily graded)

ii. $\rho : P \to \mathbb{N}$ order-preserving ($x < y \Rightarrow \rho(x) < \rho(y)$)

iii. $\bar{\zeta} \in I(P)$, the weighted zeta function satisfying $\bar{\zeta}(x,x) = 1 \quad \forall x \in P$.

def. $(P, \rho, \bar{\xi})$ <u>Eulerian</u> if

$$\sum_{x \le y \le z} (-1)^{\rho(x,y)} \cdot \bar{\xi}(x,y) \cdot \bar{\xi}(y,z) = \delta_{x,z}.$$

Remark: $\bar{\xi} = \xi$ gives the <u>classical Eulerian condition</u>

$$\sum_{x \le y \le z} (-1)^{\rho(x,y)} = \delta_{x,z}.$$

Define

$$\Phi(P, \rho, \bar{\xi}) = \sum_{S} \bar{h}_S \cdot w_S$$

with.

$$\bar{\xi}(c) = \bar{\xi}(x_0, x_1)\,\bar{\xi}(x_1, x_2) \cdots \bar{\xi}(x_{k-1}, x_k).$$

for a chain $c: \underset{\hat{0}}{\underset{\shortparallel}{x_0}} < x_1 < \cdots < x_k = \hat{1}$.

Theorem: $(P, \rho, \bar{\xi})$ an Eulerian quasi-graded poset.

Then

$$\Psi(P, \rho, \bar{\xi}) \in \mathbb{Z}\langle c, d\rangle.$$

Theorem: M manifold with a Whitney stratified boundary,

Then the face poset is quasi-graded + Eulerian,

where

$$\rho(x) = \dim(x) + 1.$$

$$\bar{\xi}(x, y) = \chi(\operatorname{link}_y(x)).$$

Thank you!