An Introduction to Counting
Lesson Plan

Cube Fellow: DJ Wells **Teacher Mentor**: Beth McNabb

Goal: Students will solve counting problems.

KY Standards: MA-08-4.1.1

Objectives: Students will understand the basic counting principles (Addition and Multiplication principles). They will apply these principles to count things. They will understand the mathematical notions of permutation and combination, and appropriately apply the related counting formulas to counting problems.

Resources and Materials needed: Student Worksheet: "Intro to Counting.pdf"; Teacher Guide: "Intro top Counting with Solutions.pdf"

Description of Plan: The worksheet serves as a complete problem set and lesson notes. Working through the packet in order, the instructor should solve each example, and students should independently solve each Problem. There are several unnumbered items with a space for students to record a statement of a counting principle or formula. These appear after the principle or formula has been demonstrated, so the instructor should try to elicit a formulation of the principle or formula from the students.

The additional problems on the final page is intended as an independent learning check to be completed independently, either at home or during a subsequent class. Note that the last two assume that the students already have a basic understanding of probability. If this is not the case, the same counting problem can be asked without referencing probability.

Lesson Source: Adapted from supplemental notes first prepared for a 100 level college math course

Instructional Mode: Teacher-guided with student participation

Date Given: 26 February 2009 **Estimated Time:** 85 min

Counting

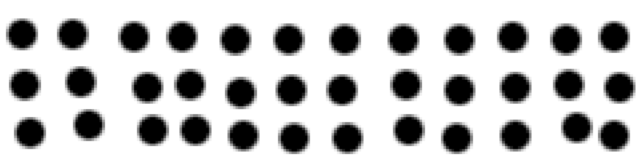
Problem 1. How many dots are there?

Addition Principle

Example 1.1. Suppose you are going to pick a card from a standard playing card deck. How many ways can you pick a a black jack or a heart?

Problem 2. Experiment: Roll a die OR flip a coin.
How many outcomes are there?

Problem 3. How many dots are there?



Multiplication Principle

Example 3.1. John has just bought 5 pairs of pants and 6 shirts (and thrown away all of his old clothes). He has no fashion sense, but does have a strange compulsion to never wear the same outfit twice. If he wears one outfit (consisting of a pair of pants a shirt) each day, how long until he will need to buy more clothes?

Problem 4. Jane (John's wife) is just as crazy as John. She has just bought 4 skirts, 5 blouses, and 3 pairs of shoes (and thrown away all of her old stuff). She has great fashion sense and carefully picked her clothes so that every combination looks great. She also shares John's strange compulsion to never wear the same outfit twice. If she wears one outfit (consisting of skirt, blouse, and shoes) each day, how long until she will need to buy more clothes?

Problem 5. How many twelve digit whole numbers are there?

Problem 6. Experiment: Flip 1000 coins.
Event: Get tails every time.
What is the probability of this event?

Example 6.1. Three tiles marked with letters "a", "b", and "c" are in a bag.
Experiment: Pull out the letters one at a time (without replacing them) to spell a "word". How many outcomes are there?

Permutations

Definition 6.2. *A* permutation *of a set of things is*

Definition 6.3. *To* permute *a set of things is to*

Definition 6.4. *To shuffle a set (for example, a deck of cards) is to pick a random permutation or permute randomly.*

Problem 7. Three students's report cards get sealed in envelopes before they are addressed. The secretary is lazy and randomly mails the three report cards to the three students' homes. What is the probability that all three students get thier own card?

Problem 8. If you are one of the three students above, what is the probability that *you* get the right report card?

Example 8.1. How many permutations are there of the set $\{a, b, c, d, e\}$?

Permutations

Lets compute a few factorials:

Problem 9. What is the probability that you shuffle a deck of cards and get the cards all in order (A, 2, 3, $\ldots$ J, Q, K of Spades, A, 2, 3, $\ldots$ J, Q, K of Hearts, A, 2, 3, ... J, Q, K of Clubs, A, 2, 3, ... J, Q, K of Diamonds)?

Problem 10. How many anagrams of "rose" are there?

Combinations

Example 10.1. Imagine that I have a tangled mess of ropes. I want to know how many ropes there are, but I can't get them untangled or even tell if two spots are on the same rope or not. What would I do?

Problem 11. Several men were traveling to St. Ives. Each man had seven wives. Each wife had seven sacks. If there were 70 sacks, how many wives were there?

Now for a harder problem where we actually have to count something.

Example 11.1. How many anagrams (rearangements of the letters, not carring if the result is a real word) of the word "flosses" are there?

Now its your turn:

Problem 12. Seven kids compete in the dance catagory of a talent show. In how many different ways can the judges give out first, second, and third place prizes to them?

Example 12.1. How many ways can you choose 5 cards from a deck of 52?

Problem 13. What is $\frac{52!}{5!\cdot 47!}$?

Combinations*:*

Problem 14. How many different pairs of students could you make from the students in the room right now?

Problem 15. How many 10 digit binary numbers have exactly four 1s?

Problem 16. Suppose McNabb were to randomly pick three students in class right now to give "get out of homework free" cards to. What would the probability be of you getting a card?

Wrapping it Up

So in summary:

If there are m ways for one thing to happen and n ways for another to happen, then there are $m+n$ ways for one **or** the other to happen (assuming that they cannot both happen at once!) and $m \cdot n$ ways for one to happen **and then** the other to happen.

There are $n!$ ways to **permute** or order n objects.

There are $C_{n,k} = \binom{n}{k}$ ("n choose k") ways to choose k things from a pool of n things.

A Few More Problems

Experiment: Roll a ten-sided die and then flip three coins. How many outcomes are there?

How many eight-digit whole numbers are there?

How many ten digit numbers have all their digits different? (Hint: not 10!.)

Roll a die 4 times. How many outcomes are there?

To play the Kentucky Lottery's "Pick 3" game, you pick three digits (numbers 0 to 9). What is the probability of winning?

Some lotteries have you pick 5 different numbers from 1 to 50. What is the probability of winning that kind of lottery?

Counting - Teachers Guide

Ask students to quietly answer the first question on their own. Allow a minute or two and then ask for a volunteer to share their solution.

Problem 1. How many dots are there?

Solution: There are 6 dots in the first group, 5 in the second, 7 in the third, and 4 in the last. All together there are $6 + 5 + 7 + 4 = 22$ dots.

This is a very basic example of what I will call the "Addition Principle" We want to know how to count things so that we can calculate probability. This means the counting problems we see will be phrased like: "In how many ways can event E happen?" or possibly as "In how many ways can X be done?" Let me state this principle in this language.

Addition Principle*If there are m ways event A can happen, and n ways B can happen (and they cannot both happen at once!!!), then there are $m + n$ ways for event A or B to happen.*

Now lets look at this principle in action:

Example 1.1. Suppose you are going to pick a card from a standard playing card deck. How many ways can you pick a a black jack or a heart?

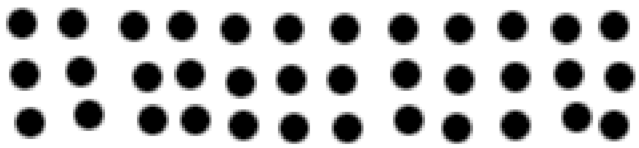
Solution: There are 2 ways to pick a black jack, and 13 ways to pick a hearts. Since there are no black jack of hearts, there are $2 + 13 = 15$ was to pick a black jack or a heart.

Notice that if I had asked how many ways can you pick a jack or a Heart, you could not have done the same computation, since some jacks are hearts. You could either use your own way to divide the options up (# non-heart jacks + # hearts) or use the full formula for the cardinality of a union (# jacks + #hearts - # jack of hearts).

Problem 2. Experiment: Roll a die OR flip a coin.
How many outcomes are there?

Now consider this example:

Problem 3. How many dots are there?



Solution: You could use brute force and count all the dots or you could use the Addition Principle and count each of the rows and add those numbers together. But most likely you noticed that there are the same number of dots in each row, and found the total number of dots by multiplying the number of rows by the number of dots in each row. There are $3 \cdot 12 = 36$ dots.

Solution: Use Addition Principle. There are $12 + 12 + 12 = 36$ dots.

So how can we make use of this 'new' principle? Well, when we have a rectangular array of dots, we can identify a specific dot by picking one row and one column. So the total number of dots is the same as the number of ways to pick one of the rows *and then also* one of the columns. This leads us to the Multiplication Principle (called the Basic Counting Law in the book).

Multiplication Principle*If there are m ways for A to happen and n ways for B to happen, then there are $m \cdot n$ ways For A and B to happen.*

Example 3.1. John has just bought 5 pairs of pants and 6 shirts (and thrown away all of his old clothes). He has no fashion sense, but does have a strange compulsion to never wear the same outfit twice. If he wears one outfit (consisting of a pair of pants a shirt) each day, how long until he will need to buy more clothes?

Solution: We want to know in how many ways John can choose a pair of pants and then choose a shirt. Since he can choose the pants in 5 ways and the shirt in 6 ways, he can choose a pair of pants and then a shirt in $5 \cdot 6 = 30$ ways.

Problem 4. Jane (John's wife) is just as crazy as John. She has just bought 4 skirts, 5 blouses, and 3 pairs of shoes (and thrown away all of her old stuff). She has great fashion sense and carefully picked her clothes so that every combination looks great. She also shares John's strange compulsion to never wear the same outfit twice. If she wears one outfit (consisting of skirt, blouse, and shoes) each day, how long until she will need to buy more clothes?

Problem 5. How many twelve digit whole numbers are there?

Solution: Think about how you can make up a twelve digit number. I would pick a first digit, and then a second, and then a third, ... How many ways can we pick a first digit? 9 (1,2,3,4,5,6,7,8,9). How about the second digit? 10 (0,1,2,3,4,5,6,7,8,9). And the third? 10, the fourth? 10, ... So there are $9*10*10*10*10*10*10*10*10*10*10*10 = 9*10^{11} = 900000000000$ twelve digit numbers.

Problem 6. Experiment: Flip 1000 coins.
Event: Get tails every time.
What is the probability of this event?

Example 6.1. Three tiles marked with letters "a", "b", and "c" are in a bag.
Experiment: Pull out the letters one at a time (without replacing them) to spell a "word". How many outcomes are there?

Solution: Well, we could just list the words. abc // acb // bac // bca // cab // cba There are 6 outcomes.

Solution: Use Multiplication Principle: Think of the experiment one letter at a time. There are 3 ways to pick the first letter. Then there are only two letters left for the second pick. Finally there is just one letter left for the final pull. So there are 3*2*1=6 ways to pick a first letter and then a second, and finally a third.

Permutations

What we did in the last example is a very common counting task. We had to find the number of ways a set can be arranged or ordered. Each of these arrangements is called a **permutation**.

Definition 6.2. *A* permutation *of a set of things is one of the various ways to put them in a line.*

Definition 6.3. *To* permute *a set of things is to put its elements into some (usually diggerent) order.*

Definition 6.4. *To shuffle a set (for example, a deck of cards) is to pick a random permutation or permute randomly.*

Problem 7. Three students's report cards get sealed in envelopes before they are addressed. The secretary is lazy and randomly mails the three report cards to the three students' homes. What is the probability that all three students get thier own card?

Problem 8. If you are one of the three students above, what is the probability that *you* get the right report card?

That was relatively easy for sets with just three elements. Did the number of permutations depend on what the objects being permuted were?

Now lets try a larger set.

Example 8.1. How many permutations are there of the set $\{a, b, c, d, e\}$?

Solution: Let's break down the process of picking a particular permutation. We can make a list of decisions we must make, and keep track of how many options we have at each decision stage. We then know, by the Multiplication Principle, that the number of ways to make all of the decisions is the product of the number of options we have at each stage.

Decision	# of ways
Choose the first letter	5
Choose the second letter	4
Choose the third letter	3
Choose the fourth letter	2
Choose the fifth letter	1
total permutations	120

When we choose the first letter, we can choose any of the five letters. But at the second stage, we have already used one letter leaving only four options. Likewise, at each step we have one fewer option.

Will this strategy work for any number of objects? (Yes.)

The number of permutations of a set with n elements is $n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (2) \cdot (1)$. Since this is such a common counting task, we have a special notation for this computation. We define n **factorial** denoted $n!$ to be the product of all the natural numbers from 1 to n. So we have this fact:

Permutations *The number of permutations of n distinct elements is $n!$.*

Lets compute a few factorials:

n		n!
0	(special definition for convience)	1
1	$1 =$	1
2	$2 \cdot 1 =$	2
3	$3 \cdot 2 =$	6
4	$4 \cdot 6 =$	24
5	$5 \cdot 24 =$	120
6	$6 \cdot 120 =$	720
7	$7 \cdot 720 =$	5040
8	$8 \cdot 5040 =$	40320
9	$9 \cdot 40320 =$	362880
10	$10 \cdot 362880 =$	3628800
11	$11 \cdot 3628800 =$	39916800
12	$12 \cdot 39916800 =$	479001600
$\vdots$		$\vdots$
30		265252859812191058636308480000000

As you can see, factorials get very large very quickly.

Problem 9. What is the probability that you shuffle a deck of cards and get the cards all in order (A, 2, 3, ... J, Q, K of Spades, A, 2, 3, ... J, Q, K of Hearts, A, 2, 3, ... J, Q, K of Clubs, A, 2, 3, ... J, Q, K of Diamonds)?

Problem 10. How many anagrams of "rose" are there?

Combinations

The next common counting task I would like to tackle is the question of how many ways you can choose k objects from a total pool of n objects. For example, how many five card hands (the order *does not* matter in a hand of cards) are there? How many ways can you choose five toppings on a pizza?

To answer this we will need a new counting technique first.

Example 10.1. Imagine that I have a tangled mess of ropes. I want to know how many ropes there are, but I can't get them untangled or even tell if two spots are on the same rope or not. What would I do? (Cutting the ropes apart wouldn't work, since that changes the number of ropes!)

Solution: Well, I can count the ends with no problem. Suppse I found that there were 12 ends. How many ropes are there? Since each rope has two ends, there must be $12/2 = 6$ ropes.

This idea of over-counting is exactly what we need to tackle the problem of Combinations. The general technique is this. Instead of directly counting the things we want, we count something else that is easier to count and has the special property that for each of the things we want to count, there are exactly the same number of things that we did count. Then we can divide to get our answer.

Problem 11. Several men were traveling to St. Ives. Each man had seven wives. Each wife had seven sacks. If there were 70 sacks, how many wives were there?

Now for a harder problem where we actually have to count something.

Example 11.1. How many anagrams (rearangements of the letters, not carring if the result is a real word) of the word "flosses" are there?

Solution: Well, counting the number of anagrams is 'hard' since there are three of the same letter. Instead lets count the number of anagrams of the word "flo$s_1 s_2$es_3". Since we have now labeled the three s's so that we can tell them apart, we know that there are a total of 7! permutations of these 7 symbols. But each anagram of $flosses$ has 3! different ways that we could label the three s's and we want to count all 3! as the same. For example: "lo$s_1 s_2 s_3$ef", "lo$s_1 s_3 s_2$ef", "lo$s_2 s_1 s_3$ef", "lo$s_2 s_3 s_1$ef", "lo$s_3 s_1 s_2$ef", and "lo$s_3 s_2 s_1$ef" are all losssef after we strip off the subscripts.

So using the same overcounting principle as before, we know that there are $7!/3! = 840$ anagrams of "flosses".

Now its your turn:

Problem 12. Seven kids compete in the dance catagory of a talent show. In how many different ways can the judges give out first, second, and third place prizes to them?

Ok, now lets tackle our example of combinations.

Example 12.1. How many ways can you choose 5 cards from a deck of 52?

Solution: This is almost the same as the talent show! In the talent show we didn't care about the order of the losers. Now we don't care about the order of "losers" (the cards we didn't pick) or the order of the winners (the ones in our hand).

Think about how one usually does this in real life. One shuffles the deck, then takes the first 5 cards, then arranges the 5 cards in numerical order (thus ignoring the order the 5 cards were dealt in). The player neither knows nor cares what order the remaining 8 cards are in. So if we start with an order of the whole 52 card deck that gives a specific 5 card hand. But if we rearrange in any of the 5! possible ways the first 5 cards in the deck, we still get the same hand. Likewise, we can rearrange in any of the 47! ways the last 47 cards without changing which 5 cards we will get. So for each 5 card hand, there are $5! \cdot 8!$ different shuffles that us give that same hand.

We know that the total number of shuffles (permutations) of the deck of 52 cards is 52!, So there are $\frac{52!}{5! \cdot 47!}$ different ways to choose 5 cards from a deck of 52.

Problem 13. What is $\frac{52!}{5! \cdot 47!}$?

Solution: Your calculators are not as smart as you are. They can't handles this porblem and you can. You need to write out what all the !s mean and then cancel things that show up in both the top and the bottom.

We can generalize this solution to get a formula for the number of combinations of k objects chosen from a pool of n objects. Look at our answer for the previous example:

$$\frac{13!}{5! \cdot 8!}$$

The 13 was the total number of objects we had to coose from, 5 was the number of objects we were choosing, and $8 = 13 - 5$ was the number of objects left over. So we can generalize this to say the number of combinations of n objects taken k at a time or the number ways to choose k objects from a pool of n is

$$C_{n,k} = \binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}.$$

We read $C_{n,k}$ and $\binom{n}{k}$ as "n choose k".
Combinations*:* There are $\binom{n}{k}$ (read: "n choose k") ways to choose k objects from a pool of k.

Problem 14. How many different pairs of students could you make from the students in the room right now?

Problem 15. How many 10 digit binary numbers have exactly four 1s?

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