Pick's Theorem
Lesson Plan

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Goal: Students will draw lattice polygons with a given number of interior lattice points, compute the area of those polygons, and count the number of boundary lattice points. The students will record this data in multiple forms, and draw conclusions regarding the relationship between the area and the number of interior and boundary lattice points.

KY Standards: MA-08-2.1.1; MA-08-3.3.1; MA-08-4.1.1; MA-08-5.1.1; MA-08-5.1.2

Objectives: The students will be able to collect data and represent that data in tables and graphs. The students will draw conclusions from examples. The students will be able to quickly find the area of lattice polygons by applying Pick's Theorem.

Resources and Materials needed: Guided Investigation Packet (included below), dot paper, straight-edge (optional), and writing utensils.

Description of Plan: Motivate students by asking them to find the area of a complicated, non-convex lattice polygon. The students should approach this task by breaking the region into rectangles and triangles. They may also find the area of a larger rectangle and subtract regions outside the polygon. Then ask them to count the number of dots (or gird points) in the interior and on the boundary. Ask which task was easier. (For instance, which task would they rather have on their next test or quiz?) Define the term "Lattice Polygon" as a polygon all of whose vertices lie at lattice points. If the students are familiar with working with lattices (dot paper, geoboards, etc) emphasize that "lattice points" are just the dots in the lattice (grid). If students are more familiar with coordinate geometry, emphasize that lattice points are those with integer (whole number) coordinates. Give a few examples and non-examples and have the class identify the examples.

Form students into groups and pass out materials. Read students the brief sketch of Pick's life. [The bio-sketch is entirely optional. I included it as part of an on-going effort to show that the math students learn about in school was developed by real people and has human stories behind it. It is safe to omit this to save a few minutes.]

Have groups complete the data gathering tasks and answer the reflection questions as a group. Each student should write short summary of the groups reflections. After providing sufficient time for discussion, bring the whole class together to discuss after each question and introduce the next Question.

The investigation is structured to guide students through a formalized version of the initial investigation a mathematician might do to approach a new problem. Although the students are presented with the Questions and reflection prompts, students should be encouraged, during each group discussion period, to make suggestions of what the next Question should be before it is revealed to the class and likewise to suggest activities that might help answer the Question.

One important point in both mathematical investigations and scientific ones is that the activity or experiment does not always answer the question. Activity 5 will not produce an answer to Question 5 right away. It is very likely that many or even all students will not find a correct to answer Questions 5-8. You may wish to point out to them that the first guy (Pick) to answer question 8 got a paper published and is still remembered for it more than 120 years later. And he didn't discover the formula until he was 40!

As an additional way to speed up the investigation, you may want to dissolve the groups and work as a class after Activity 5. This should feel natural since you will have to coordinate the whole class at this point to get all of the data in one place.

Lesson Source: Inspired by NCTM Student Math Notes, November 2005 "Area Problems? 'Pick' a Way to Solve Them"

Instructional Mode: Group Investigation

Date Given: 23, 25 September 2008 **Estimated Time:** 90-120 minutes

Compiled at 18:39 on August 26, 2009.

Who was Pick?

Georg Alexander Pick was an Austrian Jew born in 1859. He was home-schooled by his father until he was 11. He then entered the Leopoldstaedter Communal Gymnasium (A Gymnasium is like a High School.) He graduated from the Gymnasium in 1875 at 16 years old and enrolled at the University of Vienna (Universität Wien). He published his first paper the next year at just 17. He earned his Ph.D. in 1880.

After he finished his degree, Pick was became an assistant to Ernest Mach at the Karl-Ferdinand University in Prague. This is the same mach from which we get "Mach numbers" like "Mach 5" which is 5 times the speed of sound (and the name of the car from speed racer). Mach was a very prominent physicist at the time.

Pick spent the rest of his career in Prague except for one year he spend studying with Felix Klein in Leipzig, Germany. In 1899 he published an 8 page paper titled "Geometrisches zur Zahlenlehre" (Geometric results for number theory) that contained the theorem he is best known for today. "Pick's Theorem" is what you are about to investigate.

In 1910, Pick was on the committee that considered Albert Einstein's application to join the faculty at the German University in Prague. Pick's support was a strong factor in Einstein's appointment as chair of mathematical physics in 1911. The two men were close friends in the following years and Pick introduced Einstein to the many people in the scientific and musical communities in Prague. Both men had passionate interests in music and were themselves musicians.

After Pick retired in 1927 he was named professor emeritus and returned to Vienna, the town of his birth. However, in 1938 he returned to Prague after the Anschluss (The annexation of Austria by Germany – the event at the end of "The Sound of Music") on 12 March when German troops marched into Austria. Pick had been elected as a member of the Czech Academy of Sciences and Arts, but after the Nazis took over Prague, Pick was excluded from the Academy. The Nazis set up a camp at Theresienstadt in Nordboehmen on 24 November 1941 to house elderly, privileged, and famous Jews. Of around 144,000 Jews sent to Theresienstadt about a quarter died there and around 60% were sent on to Auschwitz or other death camps. Pick was sent to Theresienstadt on 13 July 1942 and he died there two weeks later aged 82.

Instructions:

- Read **The Question** and the **Activity**.
- If there are choices to be made about the **Activity**, write a brief description of how your group chooses to do the **Activity**. Explain how you think this will help you address **The Question**.
- Do the **Activity**. Be sure you number the pictures, tables, graphs, and any other work that you have on your dot paper.
- Read and discuss the **Reflection** questions with your group. Write down a summary of your group's answers in complete sentences. Add any additional observations, questions, or reflections your group has about **The Question**.
- Wait for the instructor to begin the class discussion. Do not peek ahead!

1. - **The Question**: Is the area of a lattice polygon determined entirely by the number of interior points?
   - **Activity**: Everyone construct several different lattice polygons that have no interior lattice points.
   - **Reflection**: Is it possible that the area of a polygon is determined only by the number of interior lattice points? If no, why not? If yes, do you think it is true?

2.
   - **The Question**: Is the area of a lattice polygon determined entirely by the number of boundary points?
   - **Activity**: Everyone draw several different lattice polygons with some fixed number of lattice points on the boundary.
   - **Reflection**: Is it possible that the area of a lattice polygon is determined entirely by the number of lattice points on the boundary? If no, why not? If yes, do you think that the area really is determined by the number of boundary points? look back at your previous **Reflection**. What do you conjecture (conjecture = educated guess that you think is probably correct) about the relationship between the number of lattice points and the area of a lattice polygon?

3. - **The Question**: Is the area of a lattice polygon entirely determined by the total number of lattice points in or on the polygon?
   - **Activity**: ??? (Come up with an activity to help answer this question.)
   - **Reflection**: What is the answer to **The Question**. How did you come up with that answer?

4.
- **The Question**: If we know the number of lattice points on the boundary, and we know the number of lattice points in the interior, is that enough information to determine the area?
- **Activity**: ??? (You come up with the activity!)
- **Reflection**: What do you think the answer is? Can you be certain about your answer?

In his 1899 paper, Pick showed that the area of a lattice polygon is actually determined these two numbers (The number of interior lattice points and the number boundary lattice points.)

5. - **The Question**: What is the relationship between the area of a lattice polygon and the number of interior lattice points and the number of boundary lattice points.
   - **Activity**: We don't have enough data yet. Number the members in your group starting from 3. If you have four people, they should be numbered 3, 4, 5, and 6. Number the groups starting from 0. Everyone construct a polygon with your group number of interior lattice points, and your personal number of boundary lattice points. Find the area. Once you are done, we will share our results.
   - **Reflection**: Now you have lots of data. What should you do with it? What would be a useful way to display the data? How should you deal with the fact that you have three variables instead of just two?

6. - **The Question**: What is the relationship between the area of a lattice polygon and the number of boundary points?
   - **Activity**: Look at all the data you have with the number of interior points the same fixed number. Use the method you choose in the previous **Reflection** to display that data.
   - **Reflection**: Do you think you chose a good way to display the data? Why or why not? If you didn't like the method you chose, what method do you think you should have used? Do you see a pattern in the data? What kind of relationship is there between the area and number of boundary points. How does the area change as you increase the number of boundary points but keep the number of interior points the same? Write an equation that describes the relationship you see and carefully describe what it means in words. Be sure to say specifically what kinds of polygons you think it applies to.

7. - **The Question**: What is the relationship between the area of a lattice polygon and the number of interior points?
   - **Activity**: Look at all the data you have with the number of boundary points the same fixed number. Use the method you decided was best in the previous **Reflection** to display that data.
   - **Reflection**: Do you see a pattern in the data? What kind of relationship is there between the area and number of boundary points. How does the area change as you increase the number of interior points but keep the number of boundary points fixed? Write an equation that describes the relationship you see and carefully describe what it means in words. Be sure to say specifically what kinds of polygons you think it applies to.

8. - **The Question**: How can you find the area of a lattice polygon by counting dots?
   - **Activity**: Pick a way to display *all* of your data. Do it, or explain why you can't and what it would take to make it possible to display all of that data.
   - **Reflection**: Look at the data and the two equations from the last two **Reflections**. Write a single equation that describes the relationship between the three variables. State exactly when you think this equation is true. Explain in words how this equation lets you find the area of a lattice polygon.

Pick's Theorem: *For any polygon whose vertices are lattice points, then the area of the polygon is*

$$A = i + b/2 - 1$$

where i is the number of lattice points in the interior of the polygon and b is the number of lattice points on the boundary of the polygon.

(Does your equation match Pick's?)

Further Reflection: Do you think there is a similar formula in three dimensions for finding the volume of polyhedra by counting lattice points? How could you show that there is no such formula?