<u>Topolo-what?</u>
Lesson Plan
****This lesson was presented at the NSF conference at Washington, D.C. on March 29, 2009****

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Grade/Course: 10th /Honors Geometry
KY Standards:
MA-HS-G-S-FS2 Students will explore geometries other than Euclidean geometry in which the parallel postulate is not true.
MA-HS-3.1.12 Students will apply the concepts of congruence and similarity to solve real-world and mathematical problems.

Objectives:
- The students will gain knowledge about an advanced area of mathematics called topology.
- Students will have a better understanding of how three-dimensional figures are related.
- Students will understand the differences between homeomorphisms and quotient maps.
- Students will understand the definition of equivalence class and be able to put items into appropriate equivalence classes.
- Students will be able to list several examples of sets.
- Students will discover the meaning of non-orientable independent of instruction.

Resources/Materials needed:
- Construction Paper
- Tape
- Möbius band
- Self-constructed cylinder which transforms to cone/hour-glass
- Markers
- Scissors
- Topolo-what activity sheet (attached)

Motivation

Students in Mrs. Callahan's classes seemed to be getting bored with the material they were working on. They were beginning a unit on three-dimensional figures, so Mrs. Callahan asked if I could find some activity for them to do which would motivate them. Because I had wanted to incorporate topology into the classroom, I used this opportunity to present the work I had been doing for the past semester and a half.

I wanted to present topology in a friendly way so that it would interest students, and not make them feel as if the material was beyond them. I also hoped that by introducing students to an upper level math topic in a friendly way, they would see that a mathematics degree is something they can aspire to. Dissolving the mystery behind what goes on in upper level mathematics helped the students stay motivated in their own studies, and helped them realize that math is not a subject to be afraid of. Math is not a class to be viewed as something to "get finished" with.

Prior Knowledge

The only prior knowledge required is that students have a basic understanding of the definitions/illustrations of three-dimensional figures. Because this is a basic introduction to topology, the students need only have an attention span and a willing mind.

Outline of Lesson

I. The lesson begins with an explanation that while topology is an advanced topic, not usually discussed in the high school classroom, the teacher feels as though the students are ready for this mature information. This will help to build confidence and intrigue.

II. The first phase of this lesson is a Powerpoint presentation. I am not generally a fan of Powerpoint, but topology involves many interesting illustrations that I would not do justice to by drawing on the board. The Powerpoint is treated as a "Topology Treasure Hunt". The students are notified in the first slide what terms they should be looking for throughout the presentation. They should write these definitions/examples down so that they can complete the activity sheet.

III. (A description of what should be explained on each slide is attached as well. More or less information can be added to the script as time allows.)

IV. The Powerpoint should take a little less than half of the class. Or, if you are lucky enough to have a 90 minute class, the presentation should take no more than 30 minutes.

V. The students will then complete the activity sheet. There are two main exercises.

a. The students are prompted to classify the English upper-case alphabet into homeomorphism equivalence classes. (This activity was inspired by a preliminary exam question in my program which asked students to do the same for the homotopy equivalence classes.) While the students complete this first activity, remind students that there is no cutting, shrinking or gluing allowed! Only stretching.

b. The students then complete a series of activities with Möbius strips. The ultimate goal of this activity is for students to answer the homework question I had this semester: How many twists can be put into a band to make it homeomorphic to a Möbius band? The activity is very straightforward, and along the way students discover what a non-orientable figure is. (Note: I did not find this activity from any outside source, but that doesn't mean that it doesn't exist! There are several other interesting websites detailing Möbius strip activities, one is listed below. We did not have time to use these activities.)

VI. The teacher should circulate during the entire activity making sure that students do not get confused. I found that after asking students one letter which is homeomorphic to A, then explaining that R is such a letter because the circle is homeomorphic to the triangle, students really took off on their own. There were some great conversations coming from this!

VII. At the end of the lesson, the teacher should review the "Treasure Hunt" terms. The teacher should ask for some basic examples of quotient spaces, equivalence classes and homeomorphic spaces.

It is important throughout the lesson to remind students that these are VERY basic intuitive definitions of these topics. For example, defining a topology in a graduate class will take at least 50 minutes, while in this presentation it took less than one minute. This is illustrated by the nine topologies on a three-point space.

It is also important that throughout the lesson the students are reminded of how well they are doing with the material. I explained to the students that at the end of the Powerpoint, they learned what I learned in one semester of a 500 level graduate class. I also explained that by the end of the Powerpoint that they had answered nearly ten difficult homework questions that I had. In the activity sheet, they answered one preliminary exam question and one very recent homework question.

Suggestion for follow up:
My students loved the lesson so much, that the next day we have that doesn't need to be filled with curriculum, we are having another "Topology-Day". They ask me nearly every day now what I am learning in this class. My suggestion for not taking up too much other class time with this material –because that is not what was intended – is to occasionally insert topological facts into their already planned curriculum. So, when they talk about volumes/surface areas of 3-D figures, one might ask for them to give an example of a space which is homeomorphic to the figure they are working with.

This can also be followed up with in other classes such as algebra by incorporating coding theory into a lesson on matrices.

Lesson Source
I created this lesson independent of outside sources. Any definitions/explanations were created by me.

The homeomorphism equivalence class alphabet activity was modeled after the preliminary exam question about homotopy equivalence classes. When searching afterward for examples like this online, I did find one at http://en.wikipedia.org/wiki/Topology.

Several of the images in the Powerpoint were taken by doing an image search on www.google.com for the specific figure I wanted. For example, to find the Klein Bottle pictures I searched "Klein Bottle" and took from these two websites:
http://qlink.queensu.ca/~4lb11/interesting.html
http://post-fordism.com/topo.htm
The bridges of Konigsburg was found by searching "bridges of konigsburg": http://www.nationmaster.com/encyclopedia/Seven-Bridges-of-K%C3%B6nigsberg
The algebraic topology cartoon was found at:
http://brownsharpie.courtneygibbons.org/?p=434
—Much thanks to Courtney Gibbons for several other interesting mathy-comics as well!—

The rest can be found similarly.

The nine topologies of the three-point space and the quotient space of the sphere/disc were taken directly from Munrkes Second Edition Topology.

Here is a link for some interesting Möbius strip activities which we did not have time to use.
http://mathssquad.questacon.edu.au/mobius_strip.html

Mode of Instruction
This lesson is part lecture and part hands-on activity.

Date of Implementation/ Estimated Time

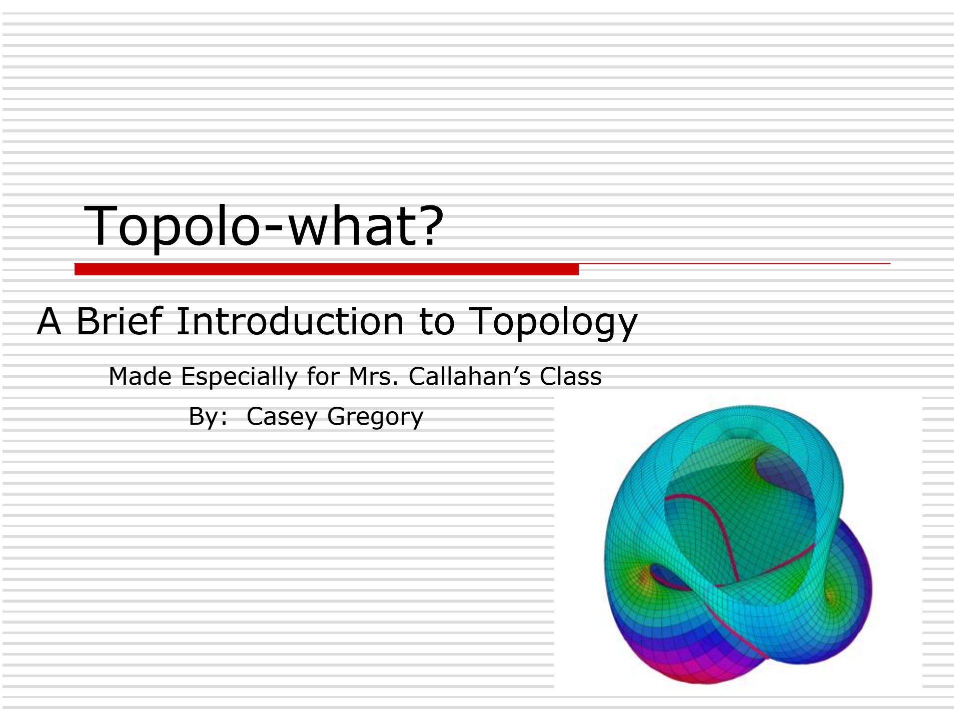
Thursday, March 12, 2009/ 50 minutes

Date Submitted to Algebra^3

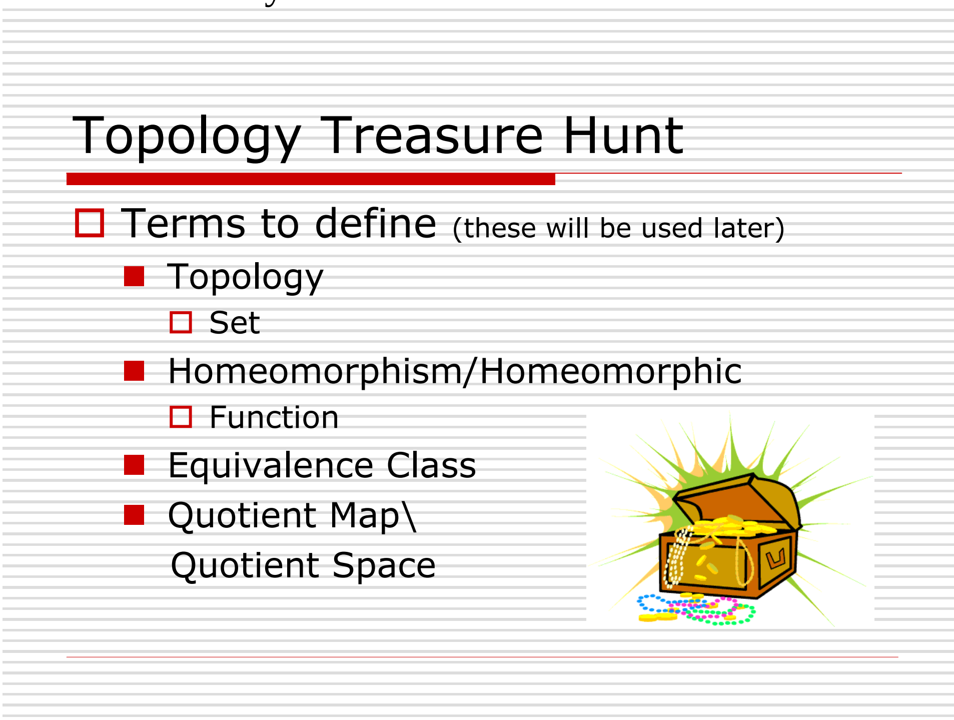
Thursday, March 26, 2009

attachment: Topolo-what? slides/mini-script
Topolo-what? activity sheet

Slides/Script



This is a topic you may not ever learn about unless you major in mathematics. There are some very technical topics which have been summarized for you to understand.



Throughout this presentation, you should look for these terms. They will be in bold. When you see them, write down their definitions and one or two examples. It will help you later!

Math is like a puzzle…

- But sometimes puzzles can be taken a little too far…

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Topology was inspired by a puzzle!

Perhaps ask students what their favorite type of mathematical puzzle is.

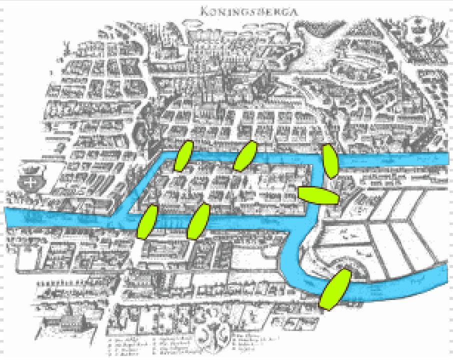
Euler = Genius

- Leonhard Euler 1707-1783
  - Many contributions to math, physics and astronomy.
    *V-E+F=2
  - Solved the difficult puzzle which inspired the founding of topology! (In 1736)

Talk about Euler’s main contributions (if time).

Which puzzle?

- Seven Bridges of Königsberg



Can you begin at one point, cross each bridge exactly once, and end up where you started?

Such a difficult problem that the idea Eulerian circuit was named after Euler!

Describe the puzzle. My students had a hard time understanding that there is no possible way to complete the task explained above. We had a nice discussion of how sometimes it is harder to prove that a solution doesn’t exist. We talked about how this puzzle was the foundation of Graph Theory. We also got to talk a tiny bit about Eulerian circuits because one of my students wanted to know what other types of these puzzles cannot be solved. I explained that anything that is NOT an Eulerian circuit would not be able to be solved.

Topology is all about **connections**

- Topology **connects** many areas of math: Geometry, Set Theory, Algebra, Analysis (Calculus)…
- Topology studies the way spaces are **connected**
- There is even a topological property called **connectedness**!

Self-explanatory.

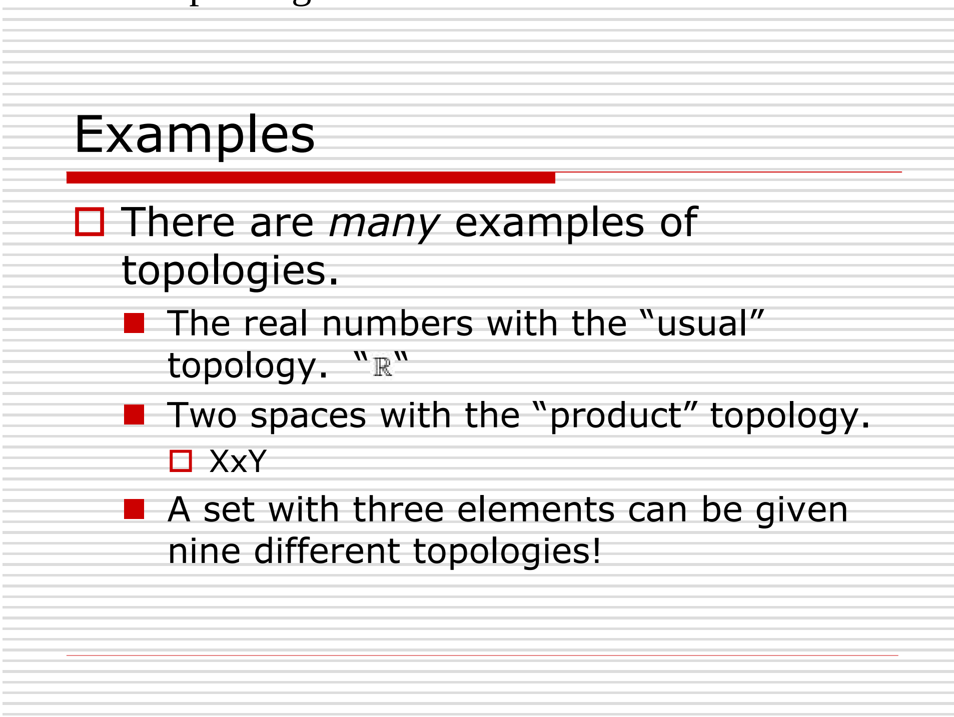
Topology IS:

- A **set** is a collection of objects.
  - Eg: {a,b,c}, {2,4,6,8,…}
    - Can you think of an example of a set?
- A **topology** is *basically* a collection of sets.
- When we talk about a **topological space**, we think about the set of objects in it and the topology that it has.

We had a great discussion of sets. High school students are responsible for an understanding of basic set theory, so we spent a long time giving examples of sets. I heard the examples of the odd integers, all numbers, the people in our room, cars in a parking lot, etc. I also explained how topologies are actually much more difficult to define.

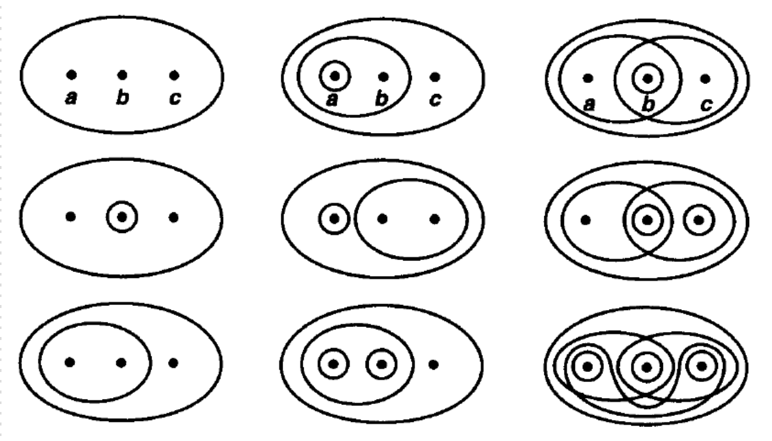
They involve finite intersections and infinite unions, etc. This was just to give them an idea of how complicated it gets.

The idea of a set of sets was hard to get across to students, but the one student who really grasped it explained it as a set of the "sets of colors" of cars in a parking lot. This was clear to most students afterward.



These examples were just to give students a flavor of the language we use in my classes. We did not go into definitions of any of these examples.

and here they are…



The students really liked looking at these, but we did not discuss what did and did not make them topologies.

Examples

- We will mostly talk about shapes in two dimensions, and objects in three (and even four) dimensions.
  - Circles "S^1"
  - Polygons (triangles, pentagons…)
  - Cylinders, cones, cubes
  - Möbius strips
  - Klein bottles
  - Coffee cups?

I introduced the symbol S^1 and talked a little bit about what spheres in higher dimensions could be called. I showed them the cylinder and Möbius strip I constructed.

Homeomorphisms are…

- **Functions** (maps) between topological spaces that preserve important properties.
- If there is a **homeomorphism** between two spaces we say they are **homeomorphic.**
- If two spaces are homeomorphic, they are *topologically equal* via stretching. (Without cutting or gluing)

We discussed what functions are. (They take one element from a specific space and assign it to an element in another specific space.) We did not explain that homeomorphisms are continuous functions, because they have not learned what continuous functions are yet. I really stressed the idea that there is no cutting/gluing/shrinking allowed. This helped the students later in the presentation, and during their activity sheet.

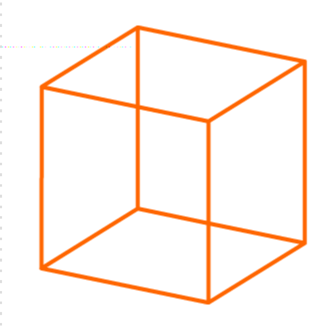
Examples

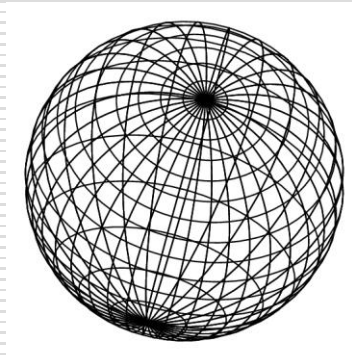
- The triangle is homeomorphic to the circle
- The square is homeomorphic to the circle
- A pentagon is homeomorphic to the circle
- Any n-gon is homeomorphic to the circle!

I asked students to explain to me how you could make these shapes into one another. Because the students had just written down that we could "stretch" when talking about homeomorphisms, several students described stretching these n-gons into circles until the edges were smooth. I described that actually writing down the function that does this is somewhat difficult. I explained that I would be able to do this at the end if there was time, but I really wanted to move on.

More examples

- The cube is homeomorphic to the sphere.

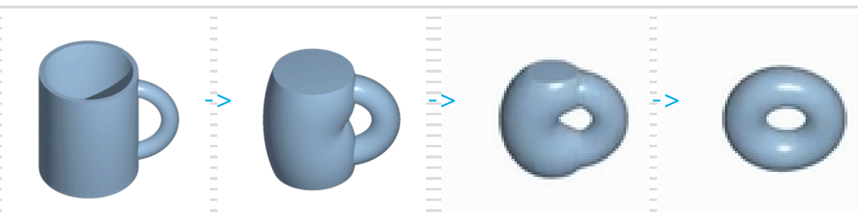




This was a fun example to explain, because while students could easily imagine stretching a square into a circle, they couldn't think of how to stretch the cube to the sphere. I explained getting inside the cube, pushing outward on all the edges/faces/vertices until everything was smooth –like you are in a hamster ball. They really got it after this.

More examples

- A coffee cup is homeomorphic to the donut.



They liked this example, but then again who doesn't.

Equivalence Classes are

- A collection of items that are homeomorphic to each other.
- Can you put each of these items into correct **equivalence classes**?

*coffee cup *circle *square *cube
*sphere *donut *triangle *n-gon

[circle]={circle, square, triangle, n-gon}
[coffee cup]={coffee cup, donut}
[sphere]={sphere, cube}

The sphere/cube can't be with the circle/square because the sphere/cube are 3-D.

One of the students actually started this discussion. He explained that this definition would be easy to remember because everyone in our physical classroom had several things in common. We all knew Mrs. Callahan, we were all in Powell County, and we all have had a Geometry class before. These connections are exactly what topology is all about.

Exercise

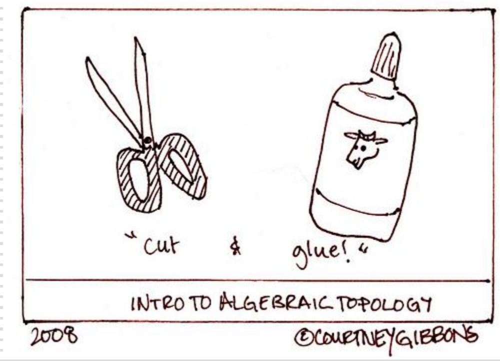
- Can you put the first 9 numbers into homeomorphism equivalence classes?
- {1,2,3,4,5,6,7,8,9}
- Answer:
  - [1]={1,2,3,5,7}
  - [4]={4,6,9}
  - [8]={8}

You can see that the top of the 4 is like a triangle, which is homeomorphic to the circle, the top of the 9.

This was a good exercise, because I was able to give some foreshadowing about what their alphabet activity was going to be like.

More interesting things

- Algebraic Topology



I explained that throughout first semester topology (point-set), I wasn't allowed to cut or glue. Then, we were! Now we are moving past first semester, and into more advanced topics. This helped students end the definition of homeomorphism and begin quotient spaces.

Quotient Maps

- A **quotient map** is also a function that relates two spaces, but we are allowed to shrink and glue now.
- The two spaces related are not "equal", but they have some similar properties.
- The resulting space is called a **quotient space.**

I explain here again that the precise definition is much more involved.
I explain that the examples will help them understand better.

Examples of Quotient Spaces

-  rectangle with right and left ends glued together becomes a band. (The band is the quotient space.)

I held up an example of a band. (This will later be called the 0 band.)

More quotient spaces

- A tube (like an empty toilet paper roll) with the two end circles glued together makes an empty donut.

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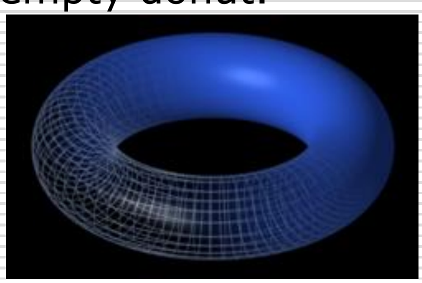
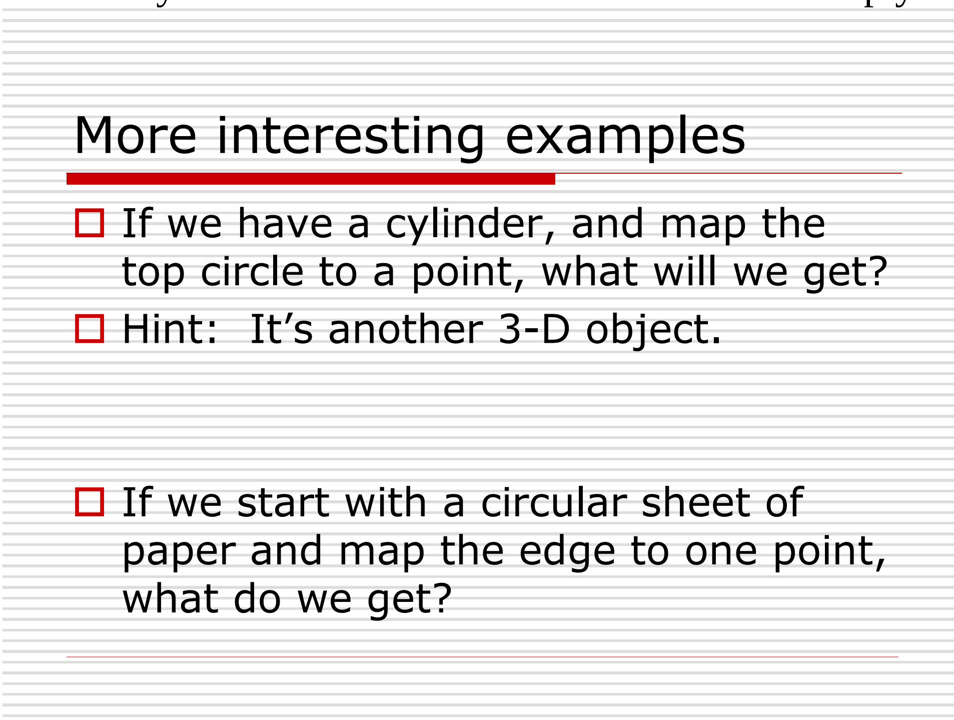


This is called a “torus”.

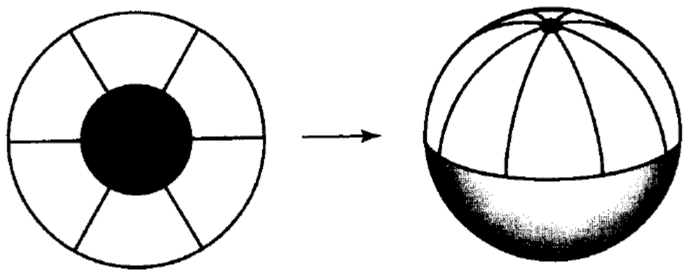
I asked them if they had ever heard of a type of math where some of the most common figures were donuts and toilet paper rolls. They really liked this example, but it was a bit harder for them to imagine. Eventually most of the students could visualize an empty donut.



This was where I used my pipe-cleaner cylinder. I asked students to visualize it first, and several students realized that mapping the top circle to a point would create a cone! Some students said that it would create a disc, but we then talked about how to get a disc from a cylinder by mapping the top circle onto the bottom circle (as well as the sides down onto the bottom circle).

The second example on this page was harder to explain, but some really beneficial pieces came out of it. To explain it, you must stress that the circular sheet is rubbery (as everything in topology is!) and that you can stretch it but must maintain that all of the points inside the circular sheet of paper not be shrunk down. I also described it after the picture was revealed as a parents dream wrapping paper would be topological wrapping paper. That way, they could wrap a basketball with no problem.

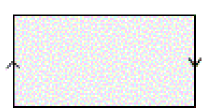
An empty ball! (Basketball)



More quotient spaces

- Möbius bands

Instead of gluing the left and right ends of the rectangle together, twist them first.



Line up the arrows when gluing.

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(We will make these shortly.)

Self-explanatory.

Klein bottles

- Mobius band with edges glued together.

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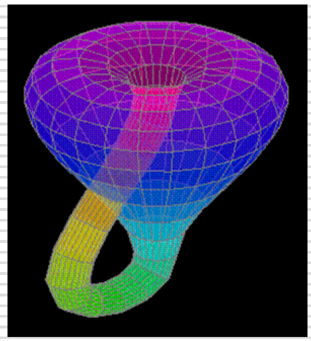
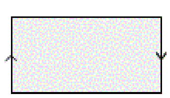


I explained that people actually manufacture Klein bottles. The students thought it was really interesting that there is no inside or outside to these bottles. Later they will learn that this means they are non-orientable.

So…

A Klein bottle is like a quotient space of a quotient space!



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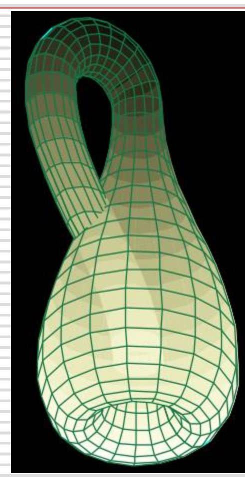
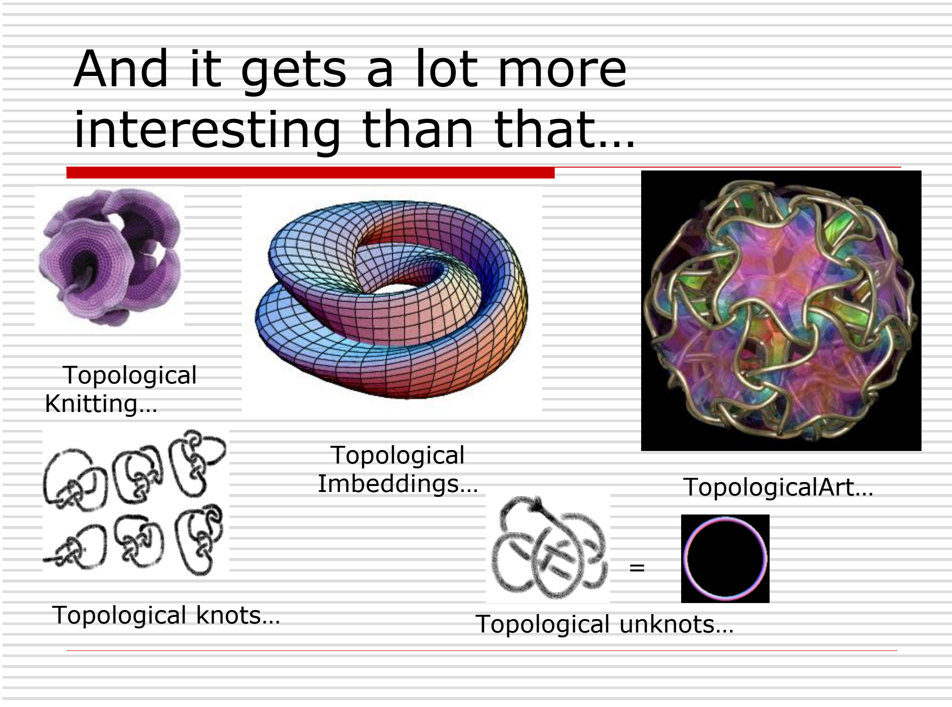
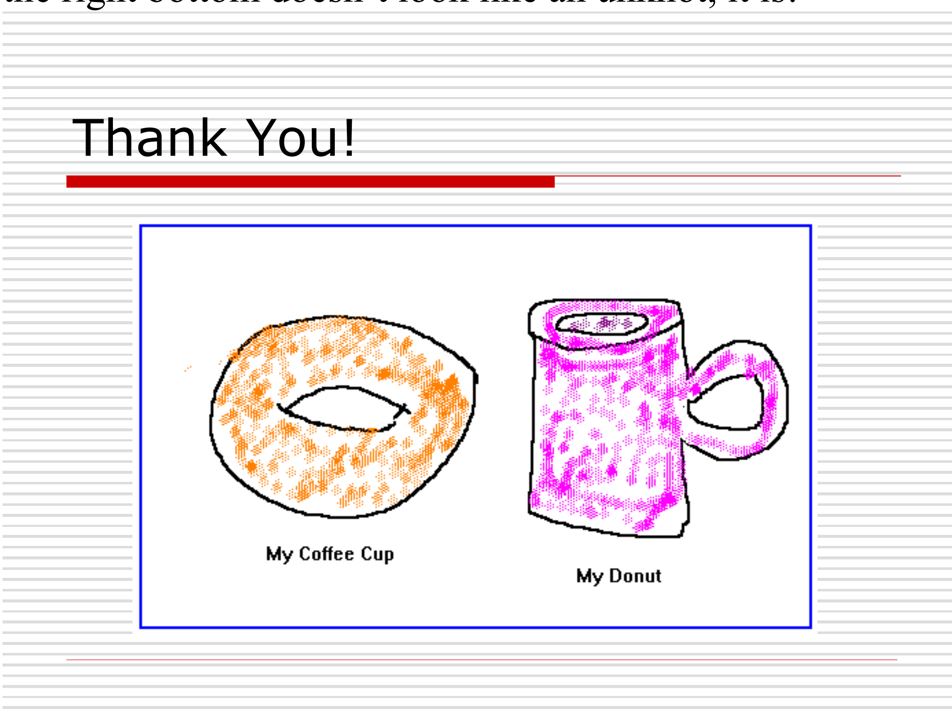


This is like a super-quotient space.



Talked a little bit about knot theory, and how even though the figure on the right bottom doesn’t look like an unknot, it is!



Note that I did not draw this, but thought it was very cute.

Names: ___________________________________

Topolo-what? Activity List

Today you and your partner learned about a topic called topology. The two main concepts talked about were homeomorphisms and quotient maps. To show that you understand the difference, please complete the following:

1. Classify the English alphabet into homeomorphism equivalence classes.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

Hint: There are 8, two containing only one letter, four containing only two letters, one containing four letters and the final one contains twelve.

2. Make the quotient space of a Möbius strip, and determine its homeomorphism equivalence classes. (By completing the following instructions)

A.) You will need 4 long strips of paper for you and your partner, a pair of scissors, some tape and a pen/pencil or marker. Take those back to your seat.

B.) Make one strip (like a headband) with no twists. Label it with a 0.
Make one strip with one twist (a Möbius band). Label it with a 1.
Make one strip with two twists. Label it with a 2.
Make one strip with three twists. Label it with a 3.
(Make sure you put tape on both sides of the strip so that it is secure.)

C.) Now, take your 0 band and on one side start to draw a line down the middle. Go all the way around until you come back to where you started. Do the same on the 1, 2, and 3 bands.

D.)What do you notice about your line on the different bands. (Does it show up on only one side of the band or does it go on both sides?)

On the 0 band it goes on ____________ side(s).
On the 1 band it goes on ____________ side(s).
On the 2 band it goes on ____________ side(s).
On the 3 band it goes on ____________ side(s).

E.) A non-orientable surface is an object on which you can't pick out the top, bottom, inside, outside, left side or right side. A Möbius strip is a non-orientable surface.

F.) Which other band(s) are non-orientable?

G.) Since homeomorphisms between two spaces mean that they are equal, and have all the same properties (and non-orientable is a property), which of the bands you made are homeomorphic to the Möbius band? (The Möbius band is the 1 band).

H.) Can you guess what other numbered bands would be homeomorphic to the Möbius band? (ie: the band with 4, 5, 6, 7, 8, 9, ... twists)

I.) Test some bands with this many twists if you have time.

5. Now, take your 0 band and cut along the line you drew. Do the same for the 1 and 2 bands. What happened that was different? What do you think will happen when you cut the 3 band along the line?

Questions to turn in:

2. Give an example of a set.

3. What is a homeomorphism?

4. Give an example of two homeomorphic spaces.

5. What is a quotient map?

6. Give your favorite example of a quotient space.

7. Can you think of two spaces that are homeomorphic that we didn't talk about today?

8. Can you think of a quotient space that we didn't talk about today?

9. Did you like this activity?

Things you should have learned today:

3. Topology is about the relationships the elements in the same space have with each other.
4. Topology is about the relationships the objects in different spaces have with each other.
5. Topology is everywhere. (Circles, spheres, coffee cups)
6. Topology is pretty. (Knitting, art...)
7. Topology is cool.

Thanks for participating!