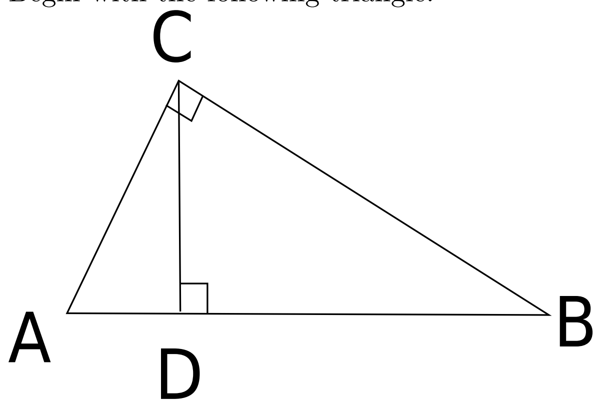
MA 330 ASSIGNMENT
THE PYTHAGOREAN THEOREM AND INFINITUDE OF PRIMES
DUE FRIDAY, SEPTEMBER 19

Answers to problems may be handwritten.

Problem 1: The following is an incorrect proof of the Pythagorean theorem. Give a detailed explanation of why this proof is not valid.

Begin with the following triangle:



Suppose the Pythagorean theorem is true. Then $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$, $\overline{BC}^2 = \overline{CD}^2 + \overline{BD}^2$, and $\overline{AC}^2 = \overline{AD}^2 + \overline{CD}^2$. Combining these three, we get that

$$\overline{AB}^2 = \overline{AD}^2 + 2\overline{CD}^2 + \overline{BD}^2 .$$

But, $\overline{CD}^2 = \overline{AD} \cdot \overline{BD}$. Thus, we may substitute and take square roots to obtain

$$\overline{AB} = \overline{AD} + \overline{BD}$$

which is true. Thus, the Pythagorean theorem holds.

Problem 2:

As you are familiar with, in modern mathematics we often do not begin geometry courses with the Euclidean plane. We instead use the Cartesian plane, which we define as follows. The *Cartesian plane*, denoted $\mathbb{R} \times \mathbb{R}$ or $\mathbb{R}^2$, consists of all ordered pairs (x, y) where x and y are real numbers. We then define the *distance between two points* (x_1, y_1) and (x_2, y_2) to be

$$\operatorname{dist}((x_1, y_1), (x_2, y_2)) := \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} .$$

We further define a *right triangle in the Cartesian plane* to be three points $(0, 0)$, (x_1, y_1), and (x_2, y_2) satisfying $x_1x_2 + y_1y_2 = 0$. Using only the properties of the Cartesian plane just given, prove that the Pythagorean theorem holds for any right triangle in the Cartesian plane.

Problem 3:

In this exercise, we will see another proof that there are infinitely many primes. Let $F_n = 2^{2^n} + 1$ for $n = 0, 1, 2, \ldots$. These values are called the *Fermat numbers*; F_5 will be the subject of one of the later great theorems in JTG. In the following steps, will prove that any two Fermat numbers are relatively prime.

(1) Use induction on n to prove that

$$\prod_{k=0}^{n-1} F_k = F_n - 2\,.$$

(If you haven't seen it before, $\prod_{k=0}^{n-1} F_k$ means $F_0 F_1 F_2 \cdots F_{n-1}$.)

(2) Show that if there exists k and n such that m is a common divisor of F_k and F_n, then m must be equal to 1 or 2.

(3) Explain why in this case m cannot be equal to 2, and therefore $m = 1$, hence all pairs F_k and F_n must be relatively prime.

(4) Explain why this implies that there are infinitely many primes.