MA 330 ASSIGNMENT
INFINITE SERIES INVOLVING UNIT FRACTIONS
DUE WEDNESDAY, NOVEMBER 19

There are many examples showing that infinite sums of reciprocals of the elements in a *subset* of the integers are interesting. One example is the great theorem of JTG Chapter 9, while another is the sum by Leibniz described at the beginning of JTG Chapter 8. Problems 1 and 2 introduce more examples of this type.

Problem 1:

Sylvester's sequence is defined by the recursive formula $s_0 = 2$ and $s_{j+1} = 1 + s_0 s_1 \cdots s_j$. This should remind you of Fermat numbers, which in an earlier homework problem you proved satisfy a similar recursion with a "2" instead of a "1" added on.

On the wikipedia page for Sylvester's sequence, the following claims are made:

- For $j \geq 1$, the sequence satisfies $s_j = s_{j-1}(s_{j-1} - 1) + 1$.
- The sum of the reciprocals of the Sylvester sequence is

$$\sum_{j=0}^{\infty} \frac{1}{s_j} = 1\,.$$

On the wikipedia page, no proof is given for the recurrence. A proof is given for the infinite sum, but it is not presented with all the details worked out.

Prove these two claims, including all missing details for the given infinite sum proof.

Problem 2:

We know that

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots,$$

and thus e is an infinite sum of reciprocals of integers. In this problem, we will prove that e is irrational by using this series expansion.

Supposing that $e = \frac{m}{n}$ for some non-zero integers m and n, derive a contradiction by proceeding through the following steps.

(1) Explain why, if e is a fraction as written above, $n!e$ is an integer.

(2) On the other hand, explain why (using the series expansion above)

$$n!e = \text{ some integer } + \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots$$

(3) Explain why

$$\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots < \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = 1\,.$$

(4) Explain why this gives a contradiction, and conclude that e is irrational.