MA 330 ASSIGNMENT # 11

Solutions to problems may be handwritten.

Problem 1:
We know that

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots ,$$

and thus e is an infinite sum of reciprocals of integers. In this problem, we will prove that e is irrational by using this series expansion.

Supposing that $e = \frac{m}{n}$ for some non-zero integers m and n, derive a contradiction by proceeding through the following steps.

(1) Explain why, if e is a fraction as written above, $n!e$ is an integer.

(2) On the other hand, explain why (using the series expansion above)

$$n!e = \text{ some integer } + \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots$$

(3) Explain why

$$\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots < \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = 1\,.$$

(4) Explain why this gives a contradiction, and conclude that e is irrational.

Problem 2:
On page 216 of Journey Through Genius, it is shown that

$$\frac{\sin(x)}{x} = \left[1 - \frac{x^2}{\pi^2}\right]\left[1 - \frac{x^2}{4\pi^2}\right]\left[1 - \frac{x^2}{9\pi^2}\right]\left[1 - \frac{x^2}{16\pi^2}\right]\cdots$$

By evaluating this at $x = \frac{\pi}{2}$, derive the following infinite product representation for $\frac{2}{\pi}$:

$$\frac{2}{\pi} = \frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{3 \cdot 5}{4 \cdot 4} \cdot \frac{5 \cdot 7}{6 \cdot 6} \cdot \frac{7 \cdot 9}{8 \cdot 8} \cdots$$

NOTE: This expression was first discovered by Wallis in 1655. You can find a "Calculus I" level proof of this using integration of powers of $\sin(x)$ on the wikipedia page for *Wallis product.*