11/16/16 12PM

$\boxed{1}$ Sign attendance

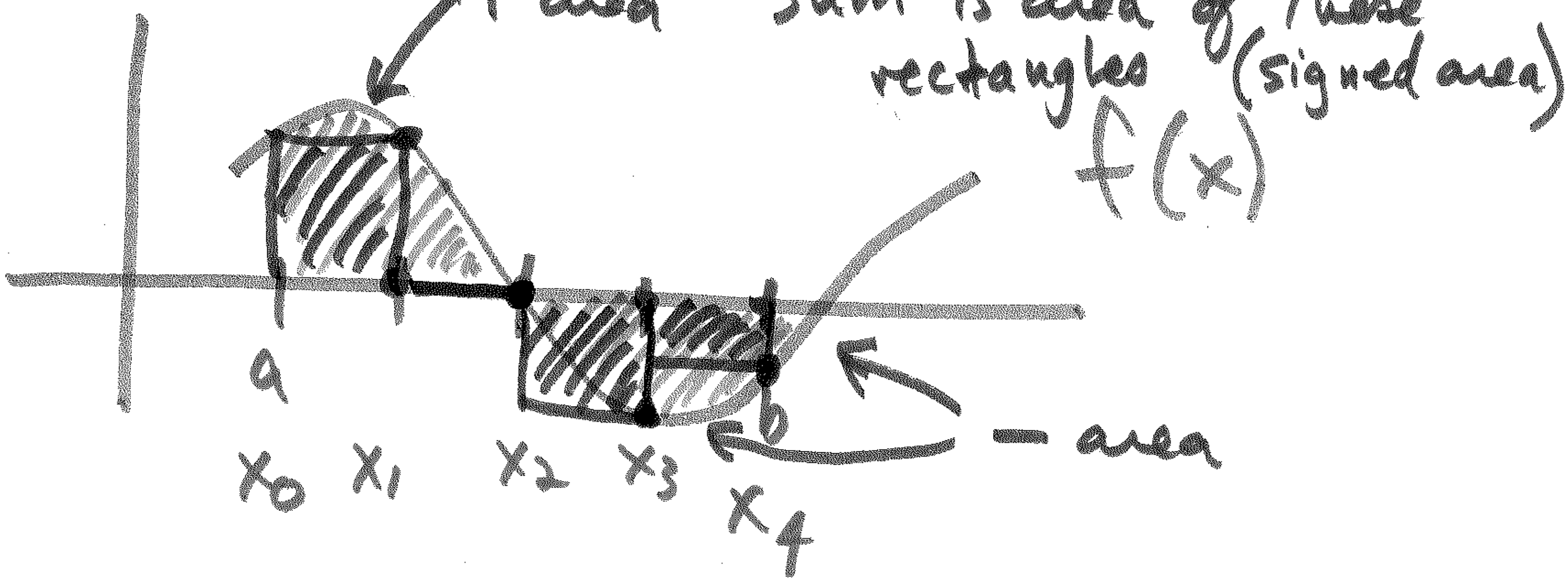
$\boxed{2}$ Webwork due MONDAY

$\boxed{3}$ Exams might not be returned until Tues. (Grades will be posted to Canvas).

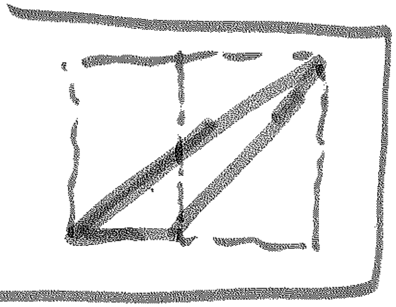
(4) Q: Why does

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} \sum_{i=1}^{n} f(\overbrace{x_0 + i\Delta x}^{x_i})\,\Delta x$$

equal the area btwn x-axis and graph of f? Draw a picture to remind yourself.



Q: what is area of △? $\frac{1}{2} \cdot b \cdot h$



Area of rectangle is $b \cdot h$.

We define area by the limit of the Riemann Sums.

F.T.C. says:

1. $\int_a^b f(t)\,dt$ has properties that we can study; these are related to derivatives, motivated by area.

2. We need a way to actually compute $\int_a^b f(t)\,dt$.

F.T.C. Pt 1: If f is cts on $[a,b]$, then

$$g(x) = \int_a^x f(t)\,dt \quad \text{for } a \le x \le b,$$

$g(x)$ is cts on $[a,b]$, diff. on (a,b),

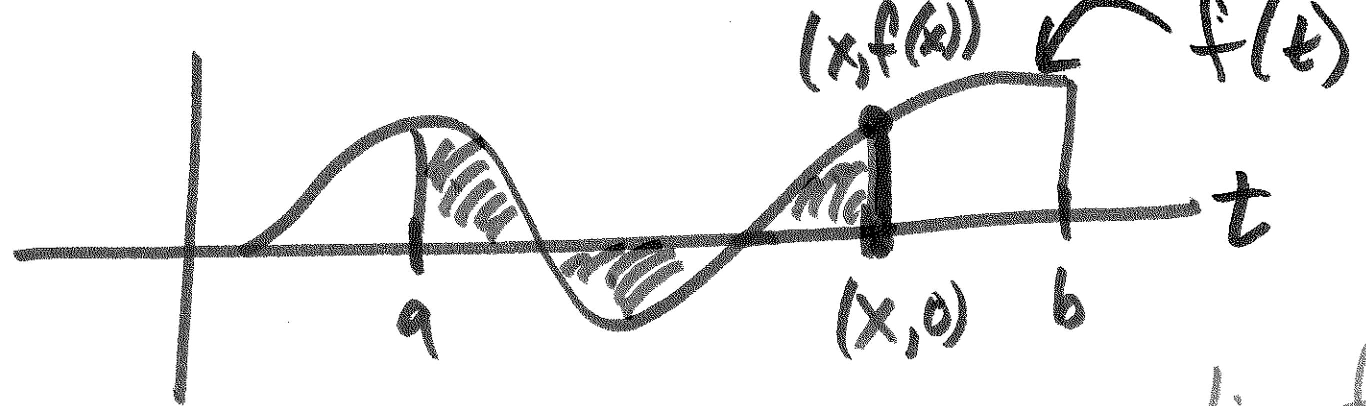
and $g'(x) = f(x)$.

F.T.C Pt 2: If f is cts on $[a,b]$, then for any antiderivative F w/ $F' = f$,

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

NOTE: If $b<a$, Then $\int_a^b f(t)dt = -\int_b^a f(t)dt$.

why is $g'(x) = f(x)$? $\quad g(x) = \text{area (w/ signs)}$



as x increases, line from (x,0) to (x,f(x)) is sweeping to right, adding area.

Length of line, f(x), is infinitesimal change in area.

(Proof in book)

__Ex__: what is $\frac{d}{dx}$ of $\int_1^x \sqrt{2t^3}\,dt$? $x > 0$

__A__: $\sqrt{2x^3}$.

__Ex__: what is $\frac{d}{dx}\int_0^{x^2} \cos(t)\,dt$ for $x > 0$?

Watch for chain rule!

$$x \longmapsto u(x) = x^2 \longmapsto g(u) = \int_0^u \cos(t)\,dt$$

$$\frac{dg}{dx} = \frac{dg}{du}\cdot\frac{du}{dx} \Rightarrow \cos(u)\cdot 2x = \underbrace{\cos(x^2)\cdot 2x}_{\text{is } \frac{dg}{dx}}.$$

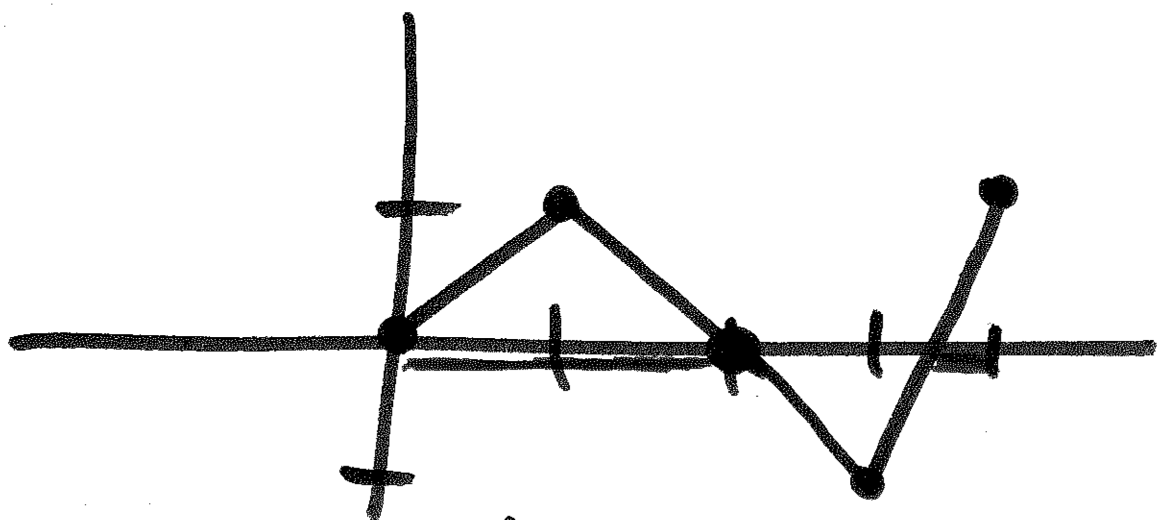
Ex: Find derivative d/dx of $\displaystyle\int_{x^2}^{x^3} t^2\,dt$ for $x>0$.

Technique: integral $= \underbrace{\displaystyle\int_0^{x^3} t^2\,dt}_{\text{chain rule}} - \underbrace{\displaystyle\int_0^{x^2} t^2\,dt}_{\text{chain rule}}$

$$\frac{d}{dx}\int_0^{x^3} t^2\,dt = 3x^2\cdot(x^3)^2. \quad \Bigg| \quad \frac{d}{dx}\int_0^{x^2} t^2\,dt = 2x\cdot(x^2)^2$$

So, desired soln is $3x^8 - 2\cdot x^5$.

Ex: Say $f(t)$ is:



on $[0,4]$

when is $\int_0^x f(t)\,dt$ increasing/dec?

Inc: derivative is >0, i.e. $f(x)>0$.

$(0,2) \cup (3.5,4)$.

Dec: derivative is <0, if $f(x)<0$.

$(2,3.5)$

11/18/16 12PM

$\boxed{1}$ Sign attendance

$\boxed{2}$ Webwork D1 due next wk

$\boxed{3}$ Find antiderivative for $3x^8 - 2x^5$.

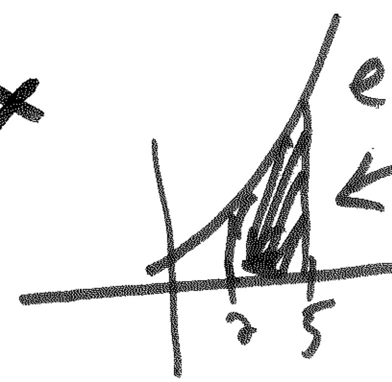
A: $\frac{3x^9}{9} - \frac{2x^6}{6}\ (+c)$

F.T.C. Pt 2: If $F' = f$ on $[a,b]$, f cts

Then $\int_a^b f(t)\,dt = F(b) - F(a)$.

Ex: $\int_2^5 e^x\,dx = F(5) - F(2) = e^5 - e^2$

$f(x) = e^x$, $F(x) = e^x$



Notation:

$F(x)\Big|_a^b = F(b) - F(a)$

$F(x)\Big]_a^b = F(b) - F(a)$

$\Big[F(x)\Big]_a^b = F(b) - F(a)$

Ex: $\int_0^2 x^2\,dx = \left[\underbrace{\frac{x^3}{3} - 7}\right]_0^2$ antider of x^2.

$$= \left(\frac{2^3}{3} - 7\right) - \left(\frac{0^3}{3} - 7\right)$$

$$= \frac{8}{3} - \cancel{7} + \cancel{7} = \frac{8}{3}.$$

Exercise: check that

$$\frac{8}{3} = \lim_{n\to\infty} \sum_{i=1}^{n} \Big(\underbrace{i \cdot \frac{2}{n}}_{x_0 + i\Delta x}\Big)^2 \cdot \underbrace{\frac{2}{n}}_{\Delta x}.$$

Why is F.T.C. Pt 2 true?

Q: If f cts, why is there an F I can use?

A: F.T.C. Pt 1 says

$$g(x) = \int_a^x f(t)\,dt$$

satisfies $g'(x) = f(x)$.

So, an antider. exists.

Q: why $F(b)-F(a)$?

A: For $g(x)$, this is clear:

$$\int_a^b f(t)\,dt = \int_a^b f(t)\,dt - \underbrace{\int_a^a f(t)\,dt}_{=0} = g(b)-g(a).$$

Q: Why is this true for any antider. $F(x)$?

A: We must have $F(x) = g(x) + C$ since $F' = g'$.

So, $F(b) - F(a) = \big(g(b) + c\big) - \big(g(a) + c\big)$

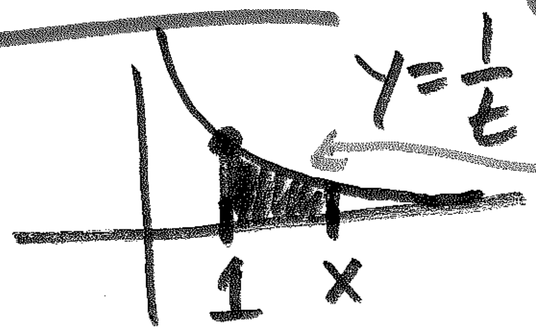
$= g(b) + \cancel{c} - g(a) - \cancel{c} = g(b) - g(a)$.

$= \int_a^b f(t)\,dt$

Ex: $\displaystyle\int_3^6 \frac{1}{x}\,dx = \Big[\ln(x)\Big]_3^6 = \ln(6) - \ln(3)$

$= \ln\left(\frac{6}{3}\right) = \ln(2)$.

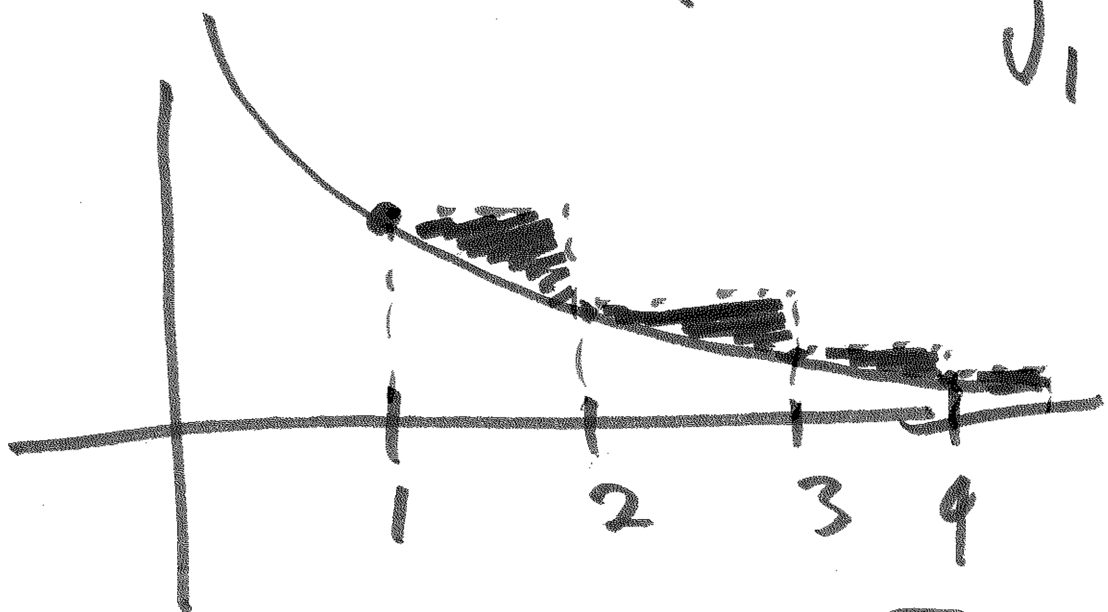
Remark: Define $\displaystyle \ln(x) = \int_1^x \frac{1}{t}\,dt$.



This $\underset{=}{\text{is}}$ $\ln(x)$ (area)

This leads to other insights..

Remark: $\ln(n) = \int_1^n \frac{1}{t}\,dt.$



Compare L_n for $\frac{1}{t}$ to $\ln(n)$.

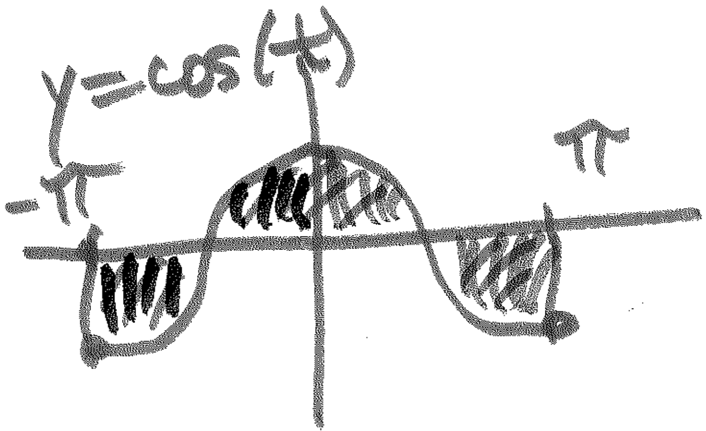
Look @ difference in area.

i.e. how bad is the overestimate?

$$\lim_{n\to\infty}\left[\sum_{i=1}^{n-1}\frac{1}{i} - \ln(n)\right] = \gamma \leftarrow \text{"gamma"}$$

Euler's constant.

Ex: $\displaystyle\int_{-\pi}^{\pi} \cos(t)\,dt$ = (should be 0, since areas cancel + & −.)



$$= \Big[\sin(t)\Big]_{-\pi}^{\pi}$$

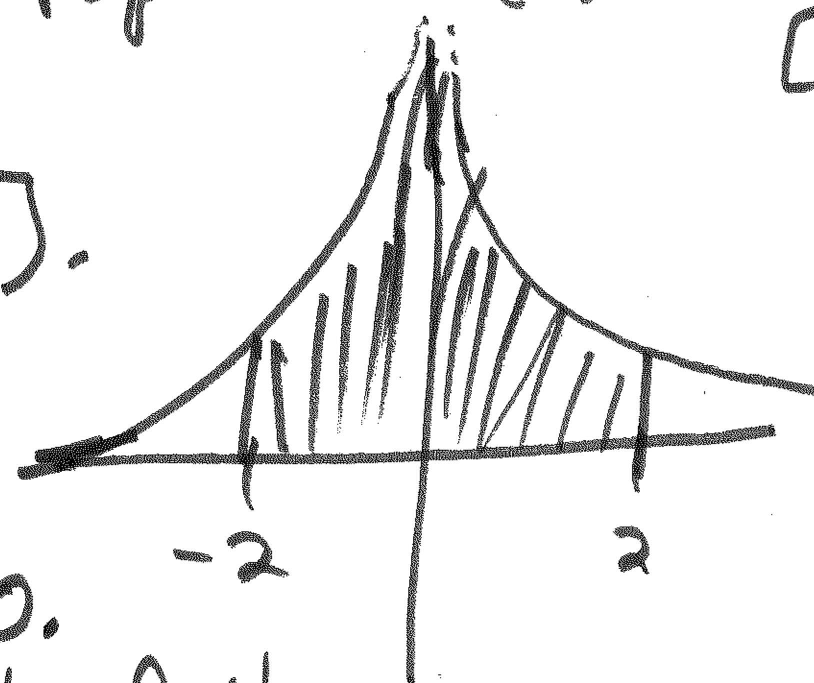
$$= \sin(\pi) - \sin(-\pi)$$

$$= 0 - 0 = \underline{\underline{0}}.$$

__Ex:__ $\displaystyle\int_{-2}^{2} \frac{1}{x^2}\,dx = \infty$. (You'll learn this in MA114).

F.T.C. Pt 2 requires $f(t)$ is cts on $[a,b]$.

$\frac{1}{x^2}$ on $[-2,2]$.



$\frac{1}{x^2}$ not defined at 0.

So, continuity fails.

So, F.T.C. Pt 2 __cannot be applied__.

<u>Ex</u>: $\int_0^4 e^{x+1}\,dx = \int_0^4 e^x \cdot e\,dx = e\int_0^4 e^x\,dx$

$= e\left[e^x\right]_0^4 = e^5 - e$

<u>WARNING</u>:

- $\int_0^4 e^{2x+1}\,dx$ works differently.

  This requires substitution.

- $\int_0^4 e^{-x^2}\,dx$ can't be evaluated using F.T.C. Pt 2.

Ex: $\displaystyle\int_{x^2}^{x^3} t^2\,dt = \left[\frac{t^3}{3}\right]_{x^2}^{x^3} = \frac{x^9}{3} - \frac{x^6}{3}.$

11/21/16 12PM

1 Sign Attendance

2 Webwork D1 This week

3 Evaluate $\int_0^3 \left(x^2 + \frac{x^5}{4}\right) dx = \left[\frac{x^3}{3} + \frac{x^6}{4 \cdot 6}\right]_0^3$

$$= \frac{3^3}{3} + \frac{3^6}{4 \cdot 6} - \left(\frac{0^3}{3} + \frac{0^6}{4 \cdot 6}\right)$$

$$= \frac{3^3}{3} + \frac{3^6}{4 \cdot 6}.$$

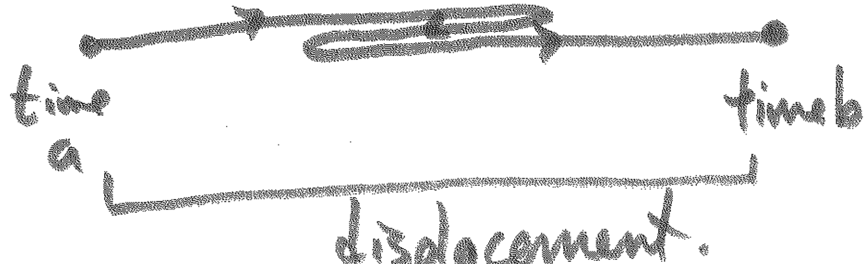
(*) Integrals measure net change.
(§ 5.4)

E.g. ① If $V(t)$ = volume of water at time t,
$V'(t)$ = rate of flow of water in/out of tank,

$$\int_a^b V'(t)\,dt = \underbrace{V(b) - V(a)}_{\text{net change in volume}}.$$

② If $s(t)$ = position + $v(t)$ = velocity of a particle, then

$$\int_a^b v(t)\,dt = s(b) - s(a) = \underline{\underline{\text{displacement}}}.$$



③ Total distance traveled is

$\int_a^b |v(t)|\,dt$. $|v(t)|$ measures speed without direction.

<u>Ex</u>: A particle moves in a line w/ velocity
$v(t) = t^2 - t - 6$ m/s.

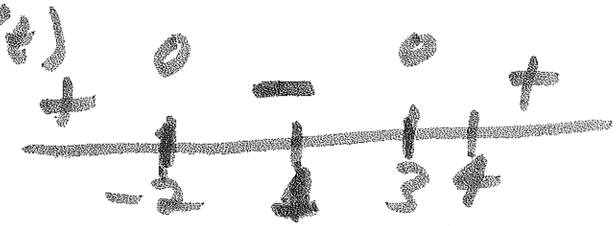
ⓐ Find displacement over $1 \le t \le 4$.

$$\int_1^4 t^2 - t - 6\,dt = \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4 = \frac{-9}{2}\,\text{m}.$$

(b) Find total distance traveled over $1 \le t \le 4$.

Check sign on velocity.

$v(t) = t^2 - t - 6 = (t-3)(t+2)$.



Thus, $v(t) \le 0$ on $[1,3]$, $v(t) \ge 0$ on $[3,4]$

$$\int_1^4 |v(t)|\,dt = \int_1^3 -v(t)\,dt + \int_3^4 v(t)\,dt$$

$$= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 6t\right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4$$

$$= \frac{61}{6} \text{ m.}$$

<u>Ex</u>: Water flows into an empty reservoir at a rate of $r(t) = 4{,}912 + 16t$ Liters/hr.

How much water is there after 8 hours?

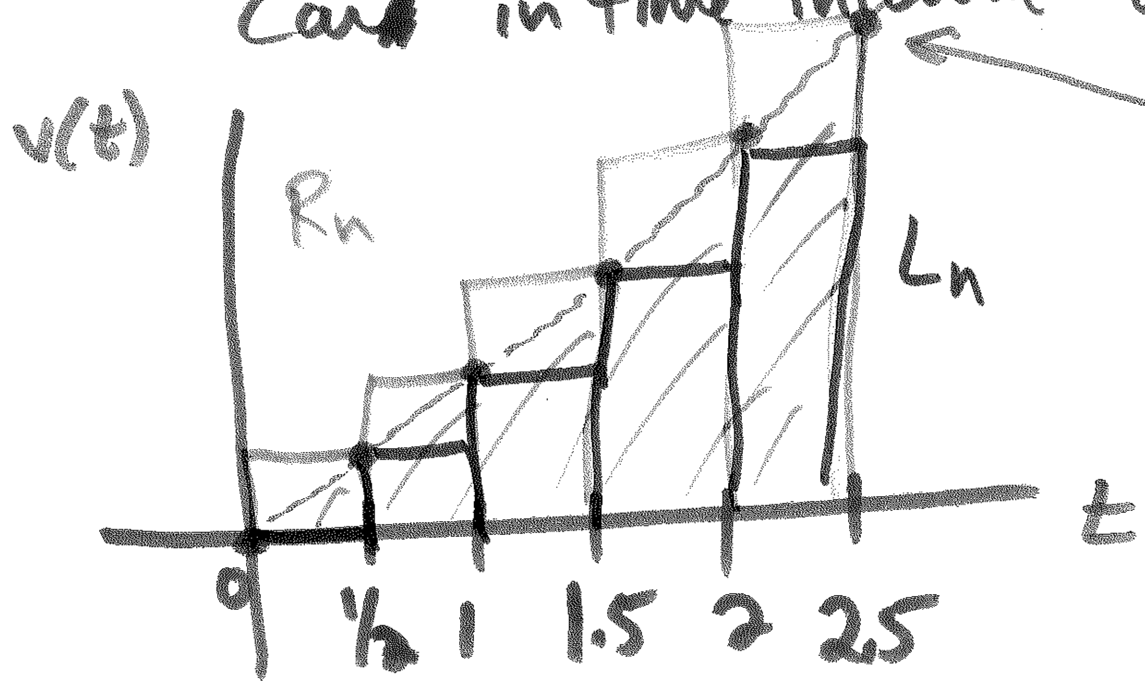
$$\int_0^8 4{,}912 + 16t \, dt = \left[4912t + \frac{16t^2}{2}\right]_0^8$$

$$= 4912 \cdot 8 + 8(8^2) = 39{,}808 \text{ liters.}$$

__Ex:__ The velocity of a car is recorded as:

t sec	0	1/2	1	1 1/2	2	2 1/2
v(t) m/s	0	1	3	7	17	26

__Problem:__ Estimate the displacement of the car in time interval $0 \le t \le 2.5$ sec.



guess at velocity curve

displacement is $\int_0^{2.5} v(t)\,dt$.

we don't know this!

$$R_n = \frac{1}{2}(1+3+7+17+26) = 27\,m$$

$$L_n = \frac{1}{2}(0+1+3+7+17) = 14\,m$$

A third estimate: $\dfrac{R_n + L_n}{2} = 20.5\,m$

11/28/16 12PM

$\boxed{1}$ Webwork D2, D3, D4 due this week.
Quiz on Thurs.

$\boxed{2}$ Today: §5.5, substitution

$\boxed{3}$ w/ neighbors: Evaluate (if you can...)

$$\int e^{3x}\,dx$$

composition!

$x \longmapsto 3x \longmapsto e^{3x}$

Also: Fill out your TCE's!!!

See your email

Many integration techniques are "reversing" derivative rules.

e.g. Chain rule $\longleftrightarrow$ Substitution

CalcII: Product Rule $\longleftrightarrow$ Integration by parts

All proofs today: see book!

Sub. Rule: If f is cts and g is diff, then

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(u)\,du$$

where $u = g(x)$.

NOTE: End result needs to be a fn of x....

<u>Ex</u>: $\int 2x e^{x^2}\,dx = \int e^{\overbrace{x^2}^{u}} \cdot \underbrace{2x\,dx}_{du} = \int e^u\,du = *$

Look for composed functions!

$x \mapsto \underbrace{x^2}_{} \mapsto e^{x^2}$

This is my u.

$u = x^2$

$\frac{du}{dx} = 2x \Rightarrow \underbrace{du = 2x\,dx}$

<u>useful notation:</u>

Usually $\frac{du}{dx} = \frac{d}{dx}u$, not a fraction.

$* = e^u + C = e^{x^2} + C.$

<u>Check:</u> $\frac{d}{dx}(e^{x^2}+C) = 2x \cdot e^{x^2}$ by chain rule.

So, correct.

Ex: $\frac{1}{2}\int 2xe^{x^2}\,dx \underset{\uparrow}{=} \frac{1}{2}e^{x^2} + C$

by previous example

$u = x^2$

$du = 2x\,dx$

Ex: $\frac{1}{4}\int e^{\overbrace{4x+1}^{u}} \underbrace{4\,dx}_{du} = \frac{1}{4}\int e^u\,du = \frac{1}{4}e^u + C$

$= \frac{1}{4}e^{4x+1} + C$

$u = 4x+1$

$du = 4 \cdot dx$

Ex: $$\frac{1}{3}\int \sqrt{\underbrace{3x+2}_{u}}\,\underbrace{3\,dx}_{du} = \frac{1}{3}\int \sqrt{u}\,du$$

$$= \frac{1}{3}\frac{u^{3/2}}{3/2} + C$$

$$= \frac{2}{9}(3x+2)^{3/2} + C.$$

$u = 3x+2$

$du = 3\,dx$

$x \mapsto \boxed{3x+2} \mapsto \sqrt{3x+2}$

Ex: $$\int \sqrt{1+x^2}\cdot x^5\,dx = \frac{1}{2}\int \sqrt{\underbrace{1+x^2}_{u}}\cdot \underbrace{x^4}_{(x^2)^2}\cdot \underbrace{2x\,dx}_{du}$$

$u = 1+x^2 \Rightarrow du = 2x\,dx$

$x^2 = u-1 \to (x^2)^2 = (u-1)^2$

$x \mapsto 1+x^2 \mapsto \sqrt{1+x^2}$

$$= \frac{1}{2}\int \sqrt{u}\cdot(u-1)^2\,du = \frac{1}{2}\int u^{1/2}(u^2-2u+1)\,du$$

$$= \frac{1}{2}\int \left(u^{5/2} - 2u^{3/2} + u^{1/2}\right) du = \frac{1}{7}(1+x^2)^{7/2} - \frac{2}{5}(1+x^2)^{5/2} + \frac{1}{3}(1+x^2)^{3/2} + C.$$

<u>Ex</u>: $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \underbrace{\frac{1}{\cos x}}_{u}\cdot \underbrace{-\sin x\,dx}_{du}$

$x \mapsto \underbrace{\cos x}_{u} \longmapsto \frac{1}{\cos x}$ $\quad u = \cos x$, $du = -\sin x\,dx$

$$= -\int \frac{1}{u}\,du = -\ln|u| + C$$

$$= -\ln|\cos x| + C = \ln|\cos x|^{-1} + C = \ln|\sec x| + C.$$

Rule for Definite integrals:

$$\int_a^b f(g(x))\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Ex: $\int_0^3 e^{5x}\,dx$. Method 1: $\int e^{5x}\,dx = \frac{1}{5}e^{5x} + C$.

FTC, Pt 2 $\Rightarrow \int_0^3 e^{5x}\,dx = \left[\frac{1}{5}e^{5x}\right]_0^3$

$= \frac{1}{5}e^{15} - \frac{1}{5}e^{0} = \frac{1}{5}(e^{15}-1)$

does not use $g(a)$ + $g(b)$ change!

$u = 5x$

$du = 5\,dx$

$0 \longmapsto 5 \cdot 0 = 0 = g(0)$

$3 \longmapsto 5 \cdot 3 = 15 = g(3)$

$$\int_0^3 e^{5x}\,dx = \frac{1}{5}\int_0^{15} e^u\,du = \left[\frac{1}{5}e^u\right]_0^{15}$$

$$= \frac{1}{5}\left(e^{15} - e^0\right)$$

Ex: $$\frac{1}{\pi}\int_{\pi}^{2\pi} \sin(\pi x + 1)\,\pi\,dx = \frac{1}{\pi}\int_{\pi^2+1}^{2\pi^2+1} \sin(u)\,du =$$

$u = \pi x + 1$

$du = \pi\,dx$

$\pi \longmapsto \pi^2 + 1$

$2\pi \longmapsto 2\pi^2 + 1$

$$x \longmapsto \underbrace{\pi x + 1}_{u} \longmapsto \sin(\pi x + 1)$$

$$\left[\frac{1}{\pi} \cdot -\cos(u)\right]_{\pi^2+1}^{2\pi^2+1}$$

$$= \frac{-\cos(2\pi^2+1) + \cos(\pi^2+1)}{\pi}$$

WARNING: Substitution often fails!

e.g. $\int e^{-x^2}\,dx$ cannot be expressed using $+, -, \times, \div$, $\sqrt[n]{\ }$, trig, polys, exp, logs

BUT... $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

e.g. $\int_2^x \frac{1}{\ln(t)}\,dt \sim$ number of primes between 2 and x.

Prime # Theorem