11/16/16 9AM

$\boxed{1}$ Sign attendance

$\boxed{2}$ Webwork due MONDAY

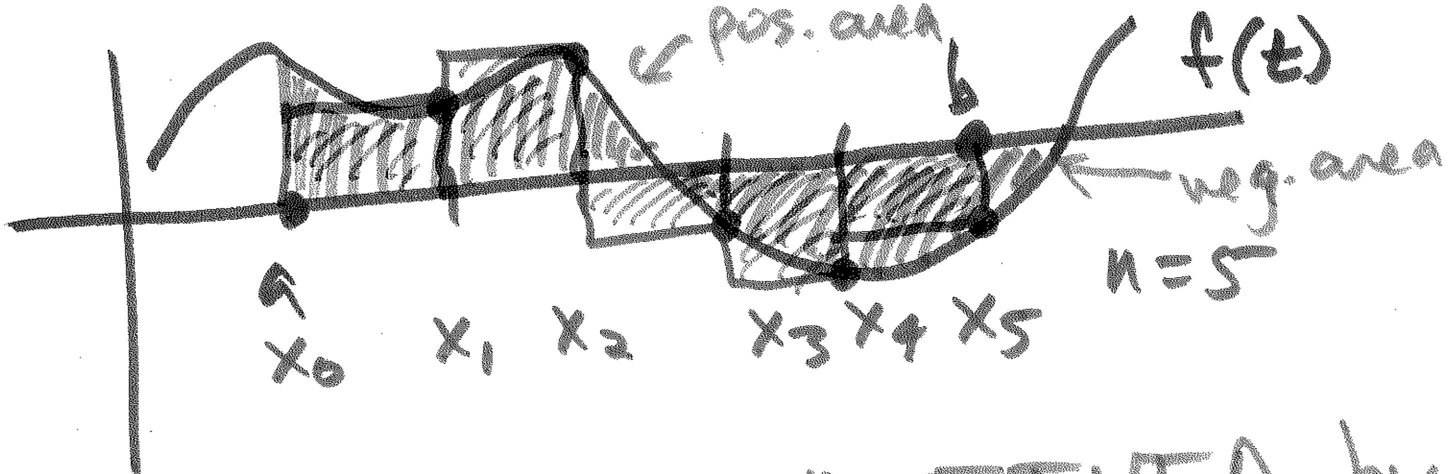
$\boxed{3}$ Exams might not be returned until Tues.
(Grades will be posted to canvas)

$\boxed{4}$ Q: [redacted] why does

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} \sum_{i=1}^{n} f(\overbrace{x_0 + i\Delta x}^{x_i})\,\Delta x$$

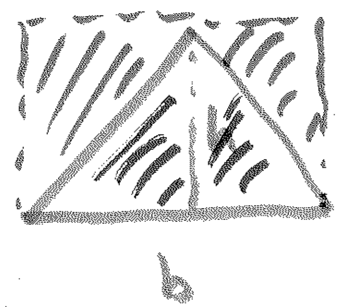
equal the area btwn x-axis + graph of f?

Draw a picture to remind yourself...



NOTE: AREA is DEFINED by this limit.

Area of Δ:



$A = \frac{1}{2} b \cdot h$

GOAL of Fund. Thm of Calc:

1. Tell us properties about $\int_a^b f(t)\,dt$ defined as a limit.
2. Tell us how to compute this.

__F.T.C. Part 1:__ If f is cts on $[a,b]$,

Then $g(x) = \int_a^x f(t)\,dt$ for $a \le x \le b$ is

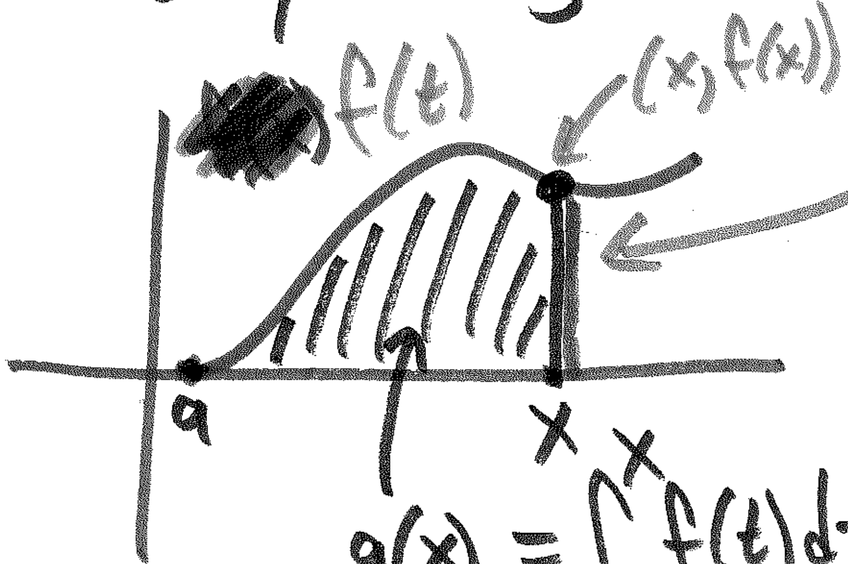
- cts on $[a,b]$, and diff. on (a,b)

with $g'(x) = f(x)$.

__F.T.C. Part 2:__ If f is cts on $[a,b]$, then if f has an antiderivative F s.t. $F' = f$,

Then $\int_a^b f(t)\,dt = F(b) - F(a)$.

Why is $g'(x) = f(x)$?



$$g(x) = \int_a^x f(t)\,dt$$

This line segment is the "infinitesimal" area being added to $g(x)$ at x.
The length of this line segment <u>is</u> $f(x)$.

See proof in textbook.

Ex: What is the derivative of

$\int_1^x \sqrt{2t^3}\,dt$? (with respect to x, i.e. apply $\frac{d}{dx}$.)

replace t w/x.

Derivative is $\sqrt{2x^3}$.

Ex: $g(x) = \int_0^{x^2} \cos(t)\,dt$? What is $g'(x)$?

This is a chain rule!

$u(x) = x^2$

$g(u) = \int_0^u \cos(t)\,dt$

$g(x) = g(u(x))$.

Chain rule:

$$\frac{d}{dx}\int_0^{x^2}\cos(t)\,dt = \frac{dg}{du}\cdot\frac{du}{dx}$$ using notation above.

- $\dfrac{du}{dx} = 2x$
- $\dfrac{dg}{du} = \dfrac{d}{du}\displaystyle\int_0^{u}\cos(t)\,dt \overset{\text{FTC 1}}{=} \cos(u)$.

So, $g'(x) = \cos(u)\cdot 2x = \cos(x^2)\cdot 2x$.

<u>Ex</u>: Find $\frac{d}{dx}$ for $\int_{x^2}^{x^3} t^2\,dt$, $x > 0$.

<u>Key idea</u>: Rewrite

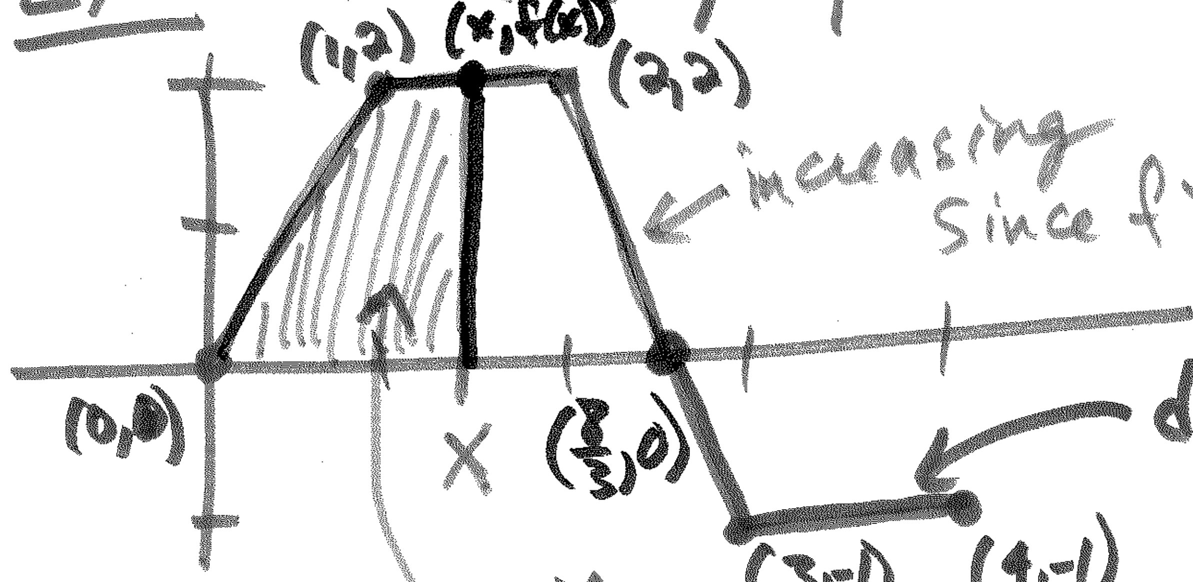
$$\int_{x^2}^{x^3} t^2\,dt = \int_0^{x^3 \leftarrow \text{u in chain}} t^2\,dt - \int_0^{x^2 \leftarrow \text{u in chain}} t^2\,dt.$$

use chain rule on each of these.

<u>Derivative is</u>: $3x^2 \cdot (x^3)^2 - 2x \cdot (x^2)^2$

$$= 3 \cdot x^8 - 2 \cdot x^5.$$

Ex: For $f(t)$ graphed as follows: $f(t)$ on $[0,4]$.



decreasing since $f<0$

when is $\int_0^x f(t)\,dt$ increasing / decreasing?

$$\frac{d}{dx}\int_0^x f(t)\,dt = f(x) \text{ by F.T.C. 1.}$$

increasing on $\left(0, \frac{8}{3}\right)$

decreasing on $\left(\frac{8}{3}, 4\right)$

11/18/16 9AM

1. Sign attendance
2. Webwork D1 due next wk
3. Find antiderivative for $3x^8 - 2x^5$.

$$\int 3x^8 - 2x^5\,dx = \frac{3x^9}{9} - \frac{2x^6}{6} + C$$

$$= \frac{x^9}{3} - \frac{x^6}{3} + C.$$

FTC Pt 2: If $F' = f$, on $[a,b]$, then

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

Why is this true?

~~FF~~ FTC Pt 1 says if f is cts, then f has an antiderivative!

$$g(x) = \int_a^x f(t)\,dt \text{ satisfies } g' = f.$$

We also know that

$$g(b) - g(a) = \int_a^b f(t)\,dt - \overbrace{\int_a^a f(t)\,dt}^{=0} = \int_a^b f(t)\,dt.$$

This is what we want! An antider. with $\int_a^b f(t)\,dt = g(b) - g(a)$.

If $F'(x) = f(x)$, then $F(x) = g(x) + C$.

So, $g(x) = F(x) - C$.

Thus,

$$\int_a^b f(t)\,dt = g(b) - g(a) = [F(b) - C] - [F(a) - C]$$

$$= F(b) - \cancel{C} - F(a) + \cancel{C}$$

$$= F(b) - F(a).$$

So, this works for <u>any</u> antider.

Ex: $\int_2^5 e^x \, dx = e^5 - e^2$

Since $\frac{d}{dx} e^x = e^x$. So, $F(x) = e^x$, we have $F(5) - F(2)$.

Notation: $F(x)\Big|_a^b := F(b) - F(a)$.

$$F(x)\Big]_a^b := F(b) - F(a).$$

$$\Big[F(x)\Big]_a^b := F(b) - F(a).$$

Ex: $$\int_0^2 x^2\,dx = \left[\frac{x^3}{3} + 10^{100}\right]_0^2$$
$$= \left[\frac{2^3}{3} + \cancel{10^{100}}\right] - \left[\frac{0^3}{3} + \cancel{10^{100}}\right]$$
$$= \frac{8}{3}.$$

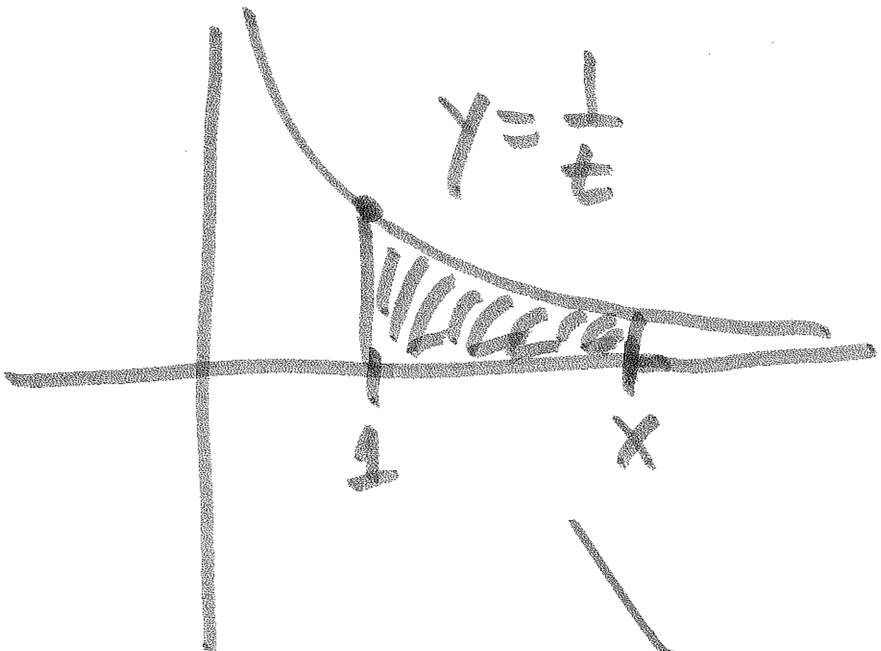
Exercise: Compute $$\lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\left(0 + i\frac{2}{n}\right)}_{x_0 + i\Delta x}^2 \cdot \underbrace{\frac{2}{n}}_{\Delta x}$$

For this example, You should get $\frac{8}{3}$.

Ex: $$\int_3^6 \frac{1}{x}\,dx = \Big[\ln(x)\Big]_3^6 = \ln(6) - \ln(3) = \ln\left(\frac{6}{3}\right) = \ln(2).$$

Logs Correctly.

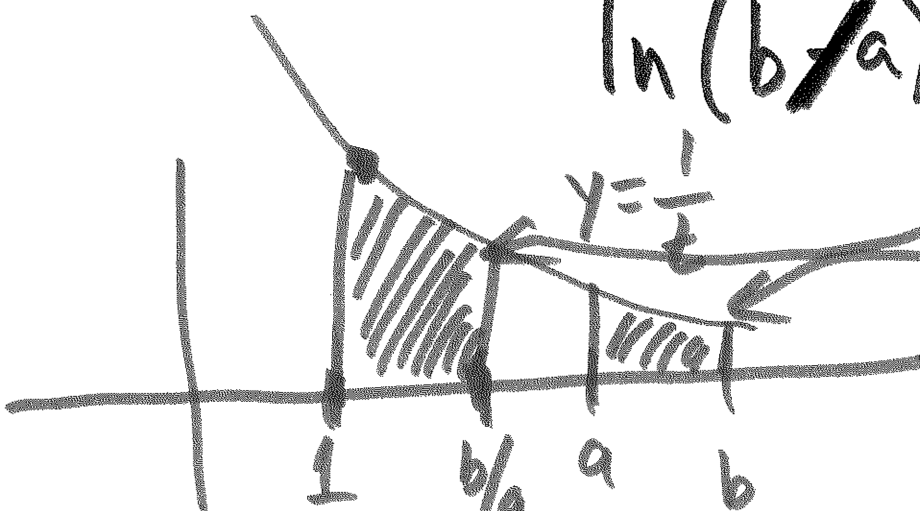
Define $\ln(x) = \int_1^x \frac{1}{t}\,dt$.



If $1 < a < b$,

why is $\underbrace{\int_1^b \frac{1}{t}\,dt}\;\; \underbrace{\int_1^a \frac{1}{t}\,dt}$

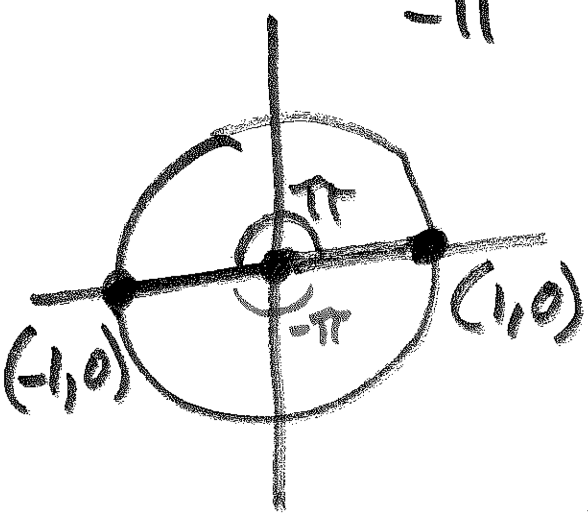
$\ln(b/a) = \ln(b) - \ln(a)$?

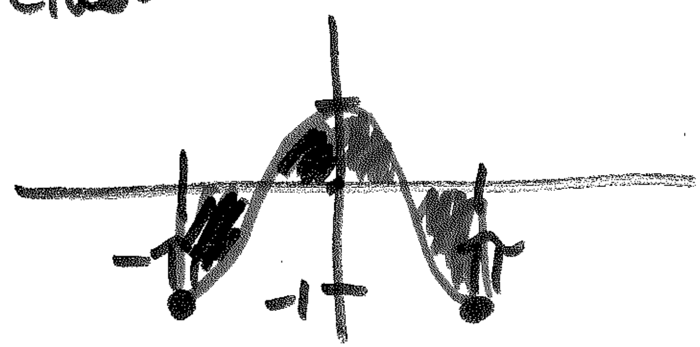


These two areas are equal.
(not drawn to scale)

Ex: $\int_{-\pi}^{\pi} \cos(x)\,dx = 0$ b/cause areas cancel.

Guess the answer:





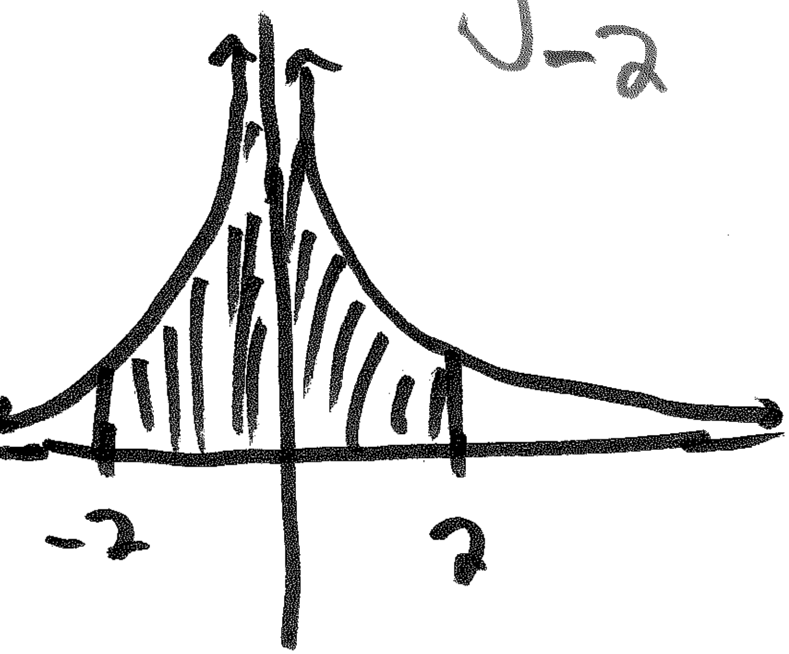
$$\int_{-\pi}^{\pi} \cos(x)\,dx = \Big[\sin(x)\Big]_{-\pi}^{\pi} = \sin(\pi) - \sin(-\pi)$$

$$= 0 - (-0)$$

$$= 0 + 0 = 0.$$

Ex: $\displaystyle\int_{-2}^{2} \frac{1}{x^2}\,dx$ = **Problem!**

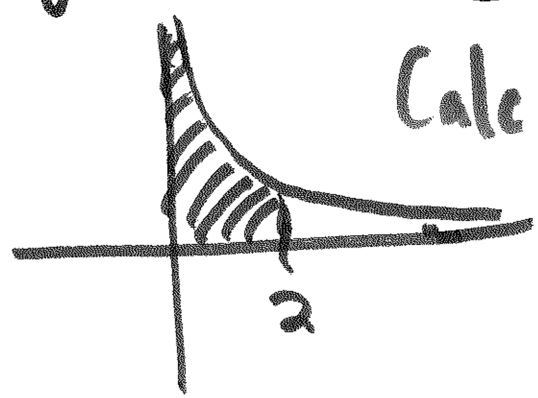
Calc II $\Rightarrow \infty$



$\frac{1}{x^2}$ not defined at 0.

So, FTC doesn't apply, since $f(t)$ must be defined on $[a,b]$.

Side Comment: $\displaystyle\int_{0}^{2} \frac{1}{x^{1/2}}\,dx$



Calc II $\Rightarrow$ finite

Ex: $\displaystyle\int_0^4 e^{x+1}\,dx = \int_0^4 e^x \cdot e\,dx$

$\displaystyle = e\int_0^4 e^x\,dx = e \cdot \left[e^x\right]\Big|_0^4 = e^5 - e.$

Compare to: $\displaystyle\int_0^4 e^{2x+1}\,dx$ requires one to "undo" a chain rule.

We will deal w/ this after Thanksgiving break using substitution.

__Ex__: Find $\displaystyle\int_{x^2}^{x^3} t^2\,dt$.

Last class we showed using F.T.C. 1 that

$$\frac{d}{dx}\int_{x^2}^{x^3} t^2\,dt = 3x^8 - 2x^5.$$

$\updownarrow$ correct $\frac{d}{dx}$ + $\int$ combo.

Using Pt 2 + get

$$\int_{x^2}^{x^3} t^2\,dt = \left[\frac{t^3}{3}\right]_{x^2}^{x^3} = \frac{x^9}{3} - \frac{x^6}{3}.$$

11/21/16 9AM

1. Sign Attendance
2. Webwork D1 this week
3. Evaluate $\displaystyle\int_0^3 \left(x^2 + \frac{x^5}{4}\right) dx = \left[\frac{x^3}{3} + \frac{x^6}{4\cdot 6}\right]_0^3$

$$= \left(\frac{3^3}{3} + \frac{3^6}{24}\right) - \left(\frac{0^3}{3} + \frac{0^6}{24}\right)$$

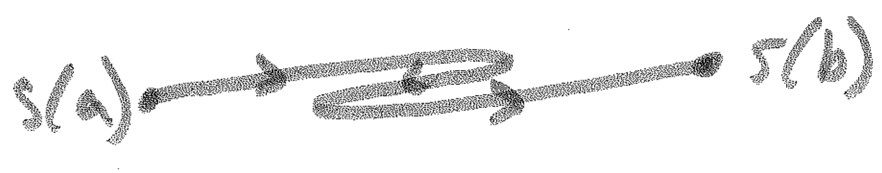
$$= \frac{3^3}{3} + \frac{3^6}{24}.$$

§5.4: Net Change

(*) Integrals measure total change.

eg. ① If $V(t)$ = volume of water at time t,
$V'(t)$ = rate of change of water volume,
then $\displaystyle\int_a^b V'(t)\,dt = \underbrace{V(b) - V(a)}_{\text{change in volume.}}$

② If $s(t)$ = position + $v(t)$ = velocity,
then $\displaystyle\int_a^b v(t)\,dt = s(b) - s(a)$ = $\underline{\text{displacement}}$ between time a + time b.



(3) To calculate <u>total</u> <u>distance</u> traveled,

$$\int_a^b |v(t)|\,dt.$$

<u>Ex</u>: A particle moves in a line w/ velocity
$t^2 - t - 6 = v(t)$ m/s

(a) Find <u>displacement</u> of particle over $1 \le t \le 4$.

$$\int_1^4 v(t)\,dt = \int_1^4 t^2 - t - 6\,dt = \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4 = -\frac{9}{2}\text{ m}$$

(b) Find <u>total distance</u> of particle over $1 \le t \le 4$.

We need to compute $\int_1^4 |v(t)|\,dt$.

$v(t) = t^2 - t - 6 = (t-3)(t+2)$.

So, $v(t) \leq 0$ on $[1,3]$ and $v(t) \geq 0$ on $[3,4]$.

Thus, tot. dist. $= \displaystyle\int_1^3 -v(t)\,dt + \int_3^4 v(t)\,dt$

$$= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 6t\right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4$$

$$= \frac{61}{6}\text{ m}.$$

Ex: Water flows into an empty resevoir at a rate of $r(t) = 4{,}912 + 16t$ liters/hr.

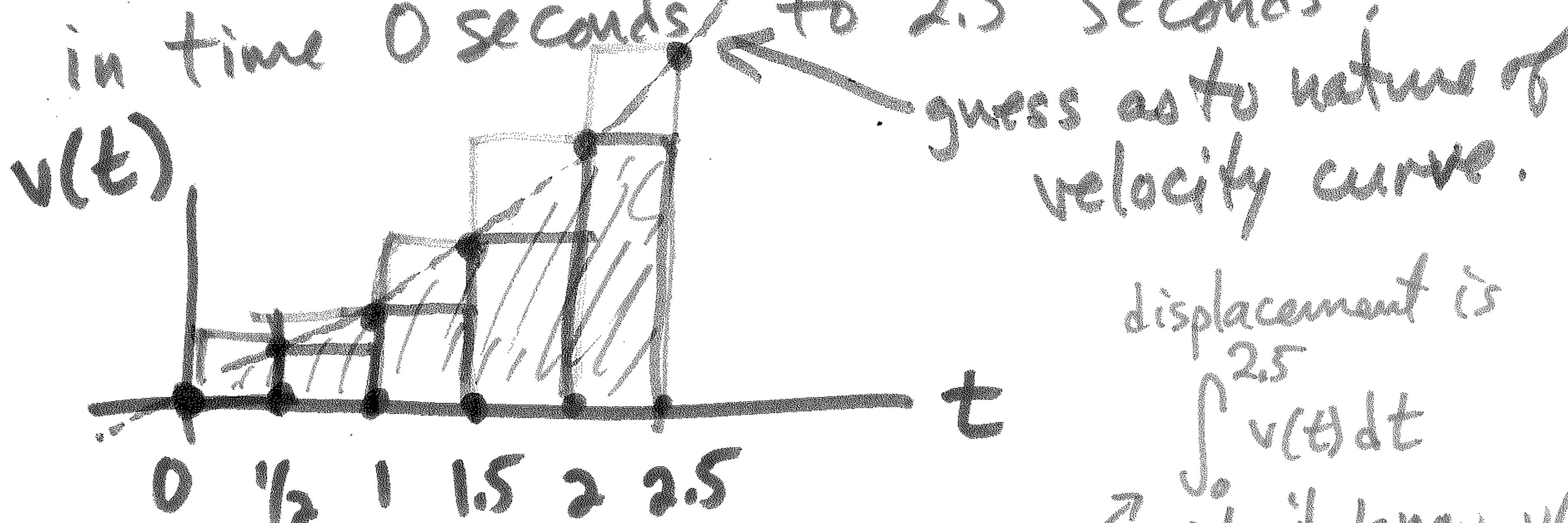
How much water is there after 8 hours?

$$\int_0^8 4{,}912 + 16t\,dt = \left[4{,}912t + \frac{16t^2}{2}\right]_0^8 = 39{,}808\ \ell.$$

Ex: The velocity of a car is recorded as:

t sec	0	0.5	1	1.5	2	2.5
v(t) m/s	0	1	3	7	17	26

Approximately how far has the car moved in time 0 seconds to 2.5 seconds?



displacement is $\int_0^{2.5} v(t)\,dt$

we don't know v(t)...

Use Riemann Sum to approximate.

$$R_n = \frac{1}{2}(1+3+7+17+26) = 27\,m$$

$$L_n = \frac{1}{2}(0+1+3+7+17) = 14\,m$$

Another estimate: $\dfrac{R_n + L_n}{2} = 20.5\,m.$

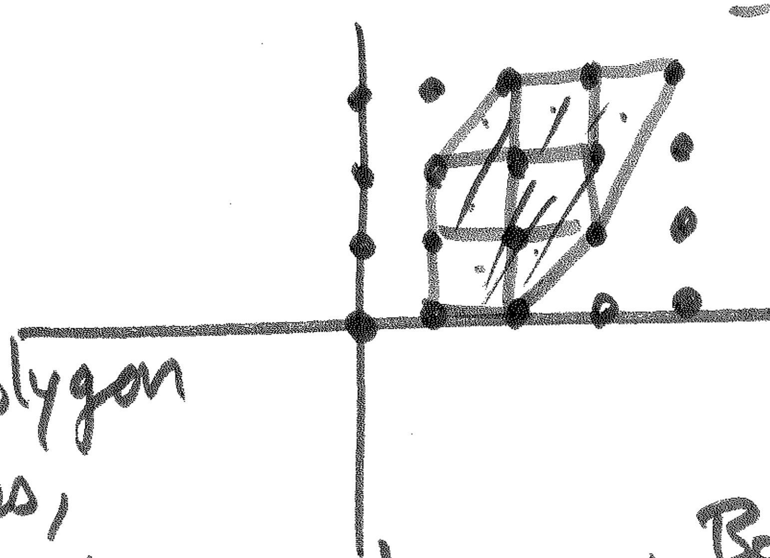
Aside: NOT MA 113:

Pick's Theorem

Area = ?

$$= 3 + \frac{8}{2} - 1$$

$$= 6.$$



For a convex polygon w/ integer vertices,

$$A = \underset{\text{pts}}{\text{Interior}} + \frac{\text{Boundary pts}}{2} - 1$$

11/28/16 9AM

1. Webwork D2, D3, D4 due this week.
   Quiz on Thurs.

2. **Today:** §5.5, substitution

3. w/neighbors: Evaluate (if you can...)

$$\int e^{3x}\,dx = \frac{1}{3}e^{3x} + C$$

Also: Fill out your TCE's !!!
See your email

Many integration techniques are "reversing" derivative techniques.

Chain rule $\longleftrightarrow$ Substitution

calc II $\rightarrow$ Product rule $\longleftrightarrow$ Integration by parts

---

By reversing chain rule, we get:

$$\underline{\star}: \int f(g(x)) \cdot g'(x)\, dx = \int f(u)\, du$$

where $u = g(x)$.

write final antiderivative as a fn of x, not u.

assume f is cts, g is diff

$$\underline{\text{Ex}}: \int 2x e^{x^2}\,dx = \int e^{x^2 \leftarrow u} \cdot \underbrace{2x\,dx}_{du} = \int e^u\,du.$$

$$u = x^2$$

$$\Rightarrow \frac{du}{dx} = 2x.$$

"Typical" shorthand notation:

$$\underbrace{du = 2x\,dx.}_{\text{tells you how to make the substitution.}}$$

This gives $\int e^u\,du = e^u + C$

$$= e^{x^2} + C.$$

$\underline{\text{Check}}$: $\frac{d}{dx}\left(e^{x^2} + C\right) = e^{x^2} \cdot 2x$ by chain rule.

ALWAYS CHECK!!!

__Ex:__ $\frac{1}{2}\int 2xe^{x^2}\,dx = \frac{1}{2}\int 2xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + c$

$u = x^2$

$du = 2x\,dx$

__Ex:__ $\frac{1}{4}\int e^{\overbrace{4x+1}^{u}}\underbrace{4dx}_{du} = \frac{1}{4}\int e^{u}\,du = \frac{1}{4}e^{u} + c$

$= \frac{1}{4}e^{4x+1} + C.$

$u = 4x+1$

$du = 4 \cdot dx$

Ex: $\frac{1}{3}\int \sqrt{3x+2}\,3dx = \frac{1}{3}\int \sqrt{u}\,du = \frac{1}{3}\cdot\frac{u^{3/2}}{3/2}+C$

$= \frac{2}{9}(3x+2)^{3/2}+C$

$u = 3x+2$

$du = 3dx$

Ex: $\int \sqrt{1+x^2}\cdot x^5\,dx = \frac{1}{2}\int \sqrt{\underbrace{1+x^2}_{u}}\; x^4 \cdot \underbrace{2x\,dx}_{du}$

???

$x^2 = u-1 \iff u = 1+x^2$

$du = 2x\,dx$

$= \frac{1}{2}\int \sqrt{\underbrace{1+x^2}_{u}}\,(x^2)^2\cdot\underbrace{2x\,dx}_{du} =$

$u-1$

$$\frac{1}{2}\int \sqrt{u}\cdot(u-1)^2\,du = \frac{1}{2}\int \left(u^{5/2} - 2u^{3/2} + u^{1/2}\right)du$$

$$= \frac{1}{7}(u)^{7/2} - \frac{2}{5}(u)^{5/2} + \frac{1}{3}(u)^{3/2} + C$$

$$= \frac{1}{7}(1+x^2)^{7/2} - \frac{2}{5}(1+x^2)^{5/2} + \frac{1}{3}(1+x^2)^{3/2} + C$$

<u>Ex</u>: $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{1}{\cos x}\cdot -\sin x\,dx$

$u = \cos x$

$du = -\sin x\,dx$

$$= -\int \frac{1}{u}\,du = -\ln|u| + C$$

$$= -\ln|\cos x| + C = \ln|\cos x|^{-1} + C$$

$$= \ln|\sec x| + C.$$

Rule for definite integrals:
(see book for proof)

$$\int_a^b f(g(x))\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

<u>Ex</u>: $\displaystyle\int_0^3 e^{5x}\,dx = \frac{1}{5}\int_0^{15} e^u\,du = \frac{1}{5}\left[e^u\right]_0^{15}$

$\uparrow$

$u = 5x$

$du = 5\,dx$

$0 \longmapsto 5\cdot 0 = 0$

$3 \longmapsto 5\cdot 3 = 15$

$$= \frac{1}{5}\left(e^{15} - e^0\right)$$

$$= \frac{1}{5}\left(e^{15} - 1\right)$$

<u>Alternative</u>: $\displaystyle\int e^{5x}\,dx = \frac{1}{5}e^{5x} + C \Rightarrow \int_0^3 e^{5x}\,dx = \left[\frac{1}{5}e^{5x}\right]_0^3 = \frac{1}{5}\left(e^{15}-1\right)$

<u>Ex</u>: $\displaystyle \frac{1}{\pi}\int_{\pi}^{2\pi} \sin(\pi x+1)\,\pi\,dx \underset{\uparrow}{=} \frac{1}{\pi}\int_{\pi^2+1}^{2\pi^2+1} \sin(u)\,du =$

$u = \pi x + 1$

$du = \pi\,dx$

$\pi \mapsto \pi\cdot\pi+1 = \pi^2+1$

$2\pi \mapsto \pi\cdot 2\pi + 1 = 2\pi^2+1$

$$\frac{1}{\pi}\Big[-\cos u\Big]_{\pi^2+1}^{2\pi^2+1} = \frac{-\cos(2\pi^2+1)+\cos(\pi^2+1)}{\pi}$$

$$\approx 0.059793\ldots$$

WARNING: Not all functions have "elementary" antiderivatives.

$+, -, \times, \div,$, $\sqrt[n]{\ }$, trig, log, exp, rationals w/polys

eg. Gaussian integral

$$\int e^{-x^2}\,dx.$$

But...

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}}$$

eg. $\displaystyle\int_2^x \frac{1}{\ln(t)}\,dt \sim$ number of primes between 2 and x.

Prime # theorem