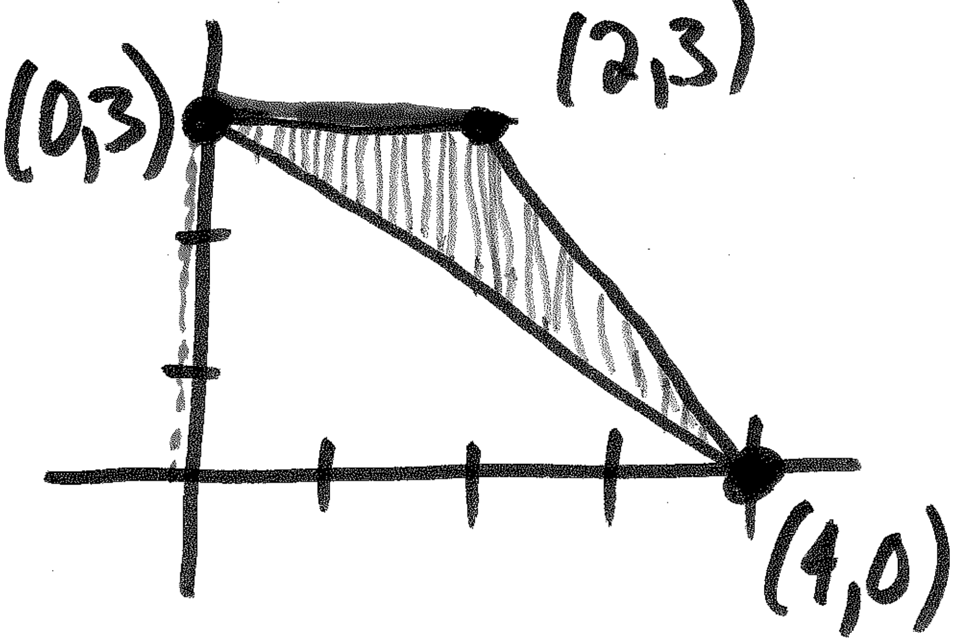
11/30/16 12PM

1. See Canvas announcement about handout for Monday.

2. fill out TCE's!!! Currently have 9-18% of responses.

3. Q: What is area of triangle with vertices $(0,3)$, $(2,3)$, and $(4,0)$?

Today: §6.1, area between curves



$$A = \frac{1}{2} \cdot \underline{b} \cdot \underline{h}$$

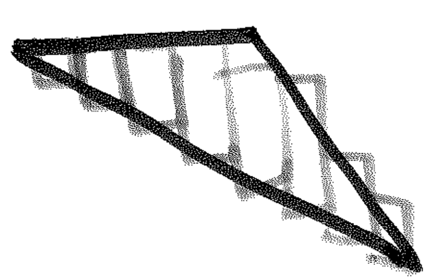
$$= \frac{1}{2} \cdot 2 \cdot 3 = 3.$$

Idea: each of vertical cross-sections is an "infinitely thin" rectangle w/ area $dx \cdot$ length of line.

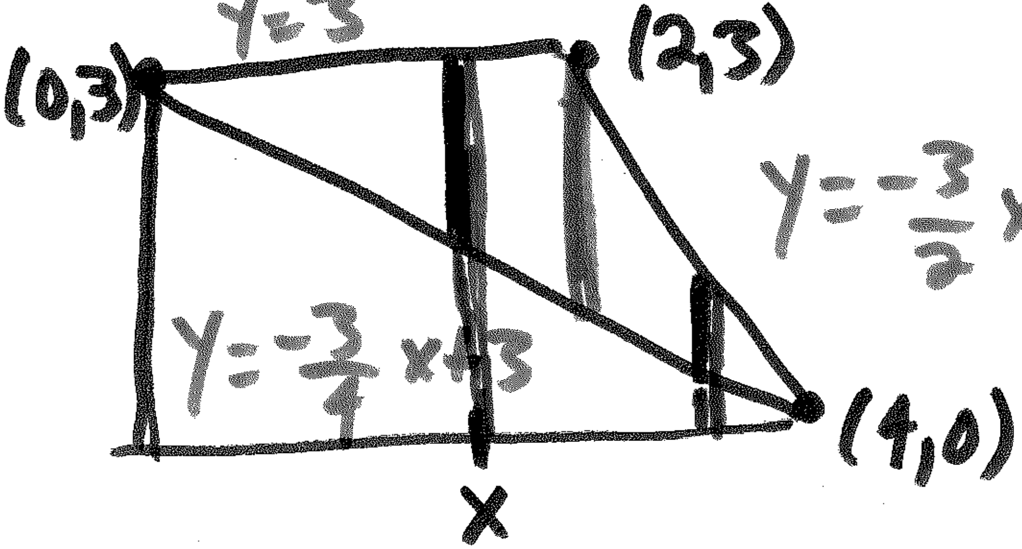
infinitely thin width

height.

NOTE: Riemann Sums gives picture:



Idea is area should be sum of (line length) $\cdot\, dx$ over all cross-sections.



(by pt-slope)

line length =
red line length
− blue line length.

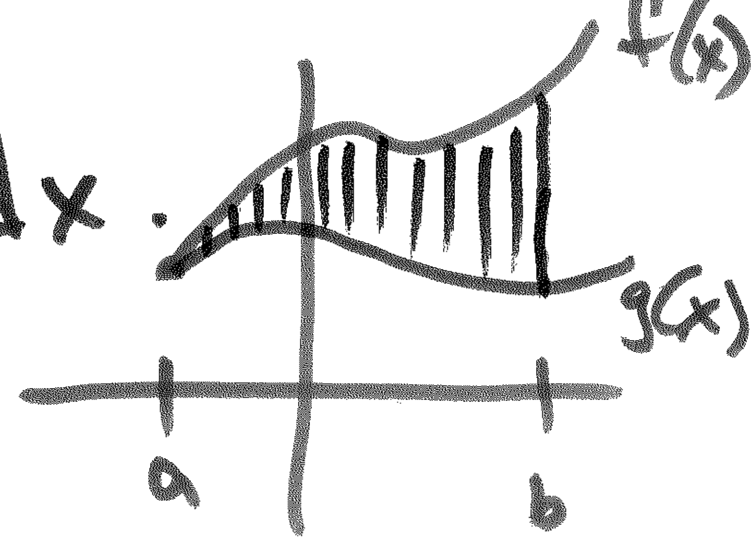
$$\text{Area} = \int_0^2 \text{red} - \text{blue}\, dx + \int_2^4 \text{red} - \text{blue}\, dx$$

$$= \int_0^2 3 - \left(\frac{-3}{4}x + 3\right) dx + \int_2^4 \left(-\frac{3}{2}x + 6\right) - \left(\frac{-3}{4}x + 3\right) dx$$

$$= \int_0^2 \frac{3}{4}x \, dx + \int_2^4 -\frac{3}{4}x + 3 \, dx = 3$$

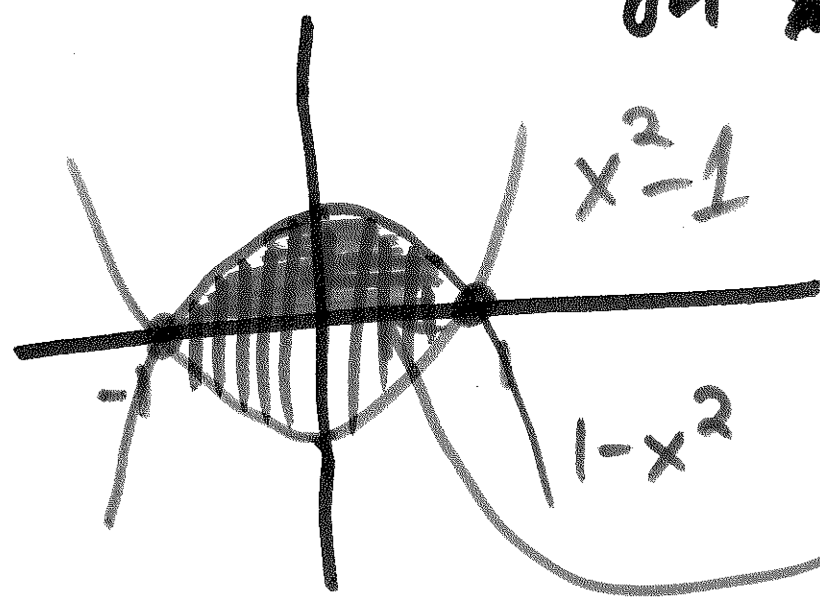
Defⁿ: The area A bounded by the curves $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$, where $f(x) \geq g(x)$ for x in $[a, b]$, is

$$A = \int_a^b f(x) - g(x) \, dx.$$



Ex: Find area between $f(x) = x^2 - 1$ and $g(x) = 1 - x^2$ on $[-1, 1]$.

Comment: If we ask for area enclosed by f & g, we will omit interval. You find it by solving $x^2 - 1 = 1 - x^2$.

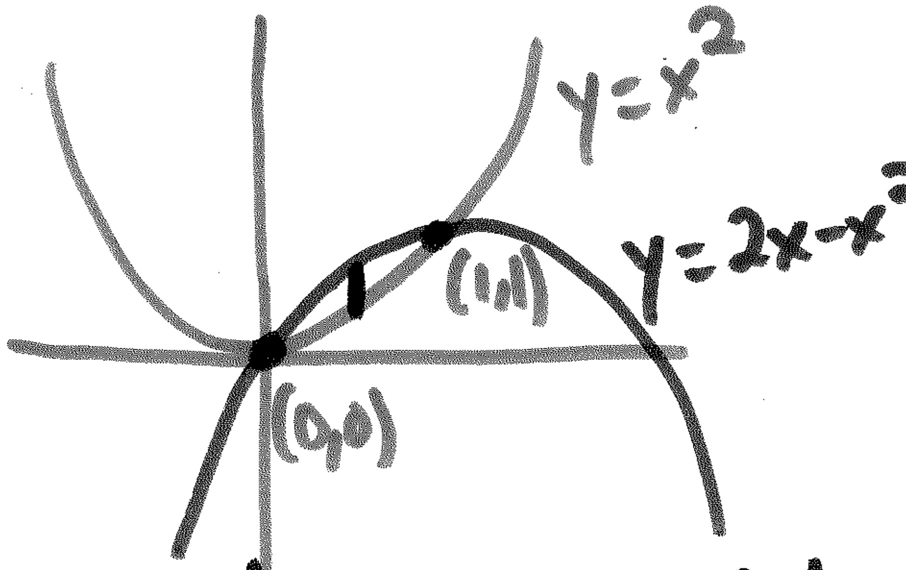


$$\int_{-1}^{1} 1-x^2\,dx$$

$$\text{Area} = \int_{-1}^{1} (1-x^2) - (x^2-1)\,dx = \int_{-1}^{1} 2-2x^2\,dx$$
$$= 2\int_{-1}^{1} 1-x^2\,dx$$

$$= 2\left[x - \frac{x^3}{3}\right]_{-1}^{1} = \frac{8}{3}.$$

Ex: Find area enclosed between $y = x^2$ and $y = 2x - x^2$.



solve $x^2 = 2x - x^2$

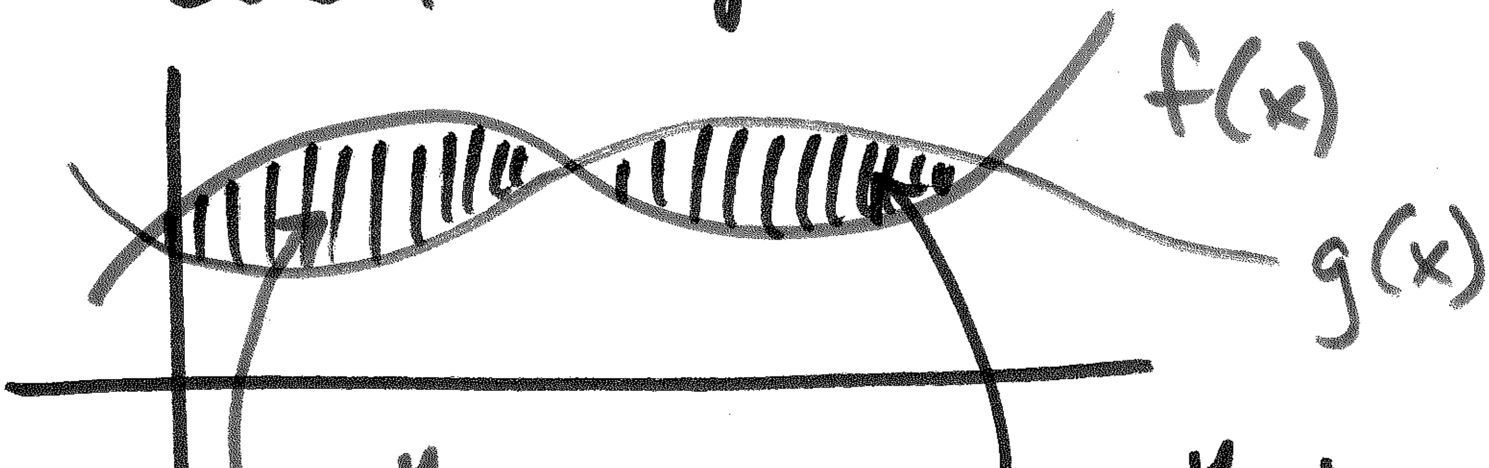
$\Rightarrow \; 0 = 2x - 2x^2$

$= 2x(1-x)$

so $x=0, x=1$ are coords for int. pts.

$$\int_0^1 (2x-x^2) - x^2\,dx = \int_0^1 2x - 2x^2\,dx = \frac{1}{3} \text{ is area.}$$

Q: What happens if $f(x) \geq g(x)$ doesn't always hold?



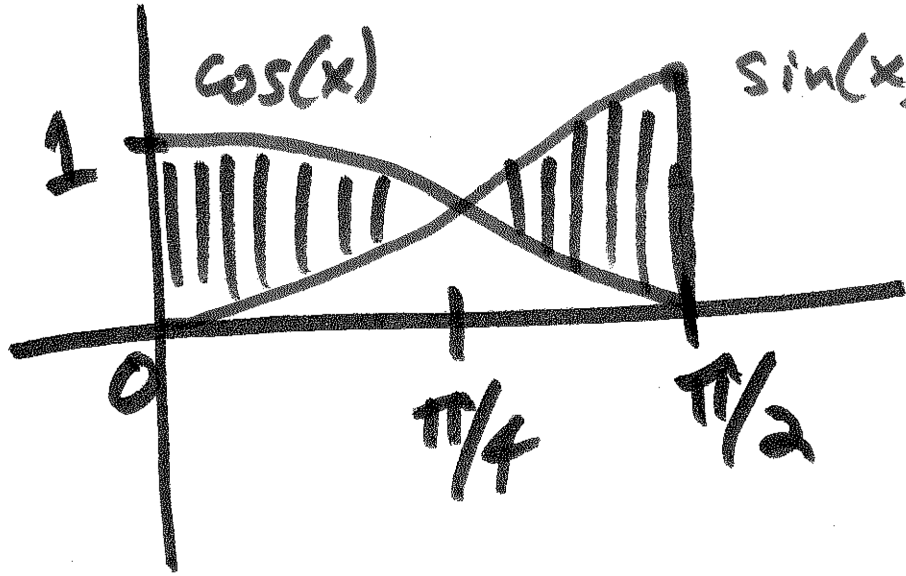
length is $f(x)-g(x)$

length is $g(x)-f(x)$.

for these functions w/ given graphs, both are equal to $|f(x)-g(x)|$.

Def^{n}: The total positive area between $f, g, x=a, x=b$; is $\int_a^b |f(x)-g(x)|\,dx$.

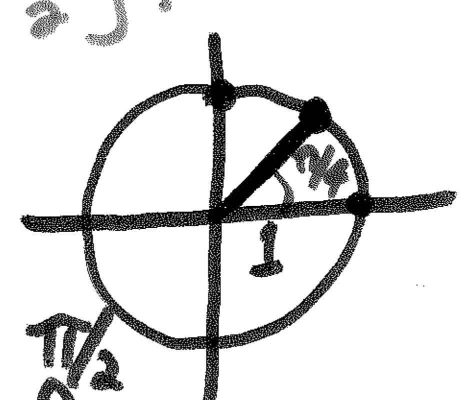
Ex: Find area bounded by $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \frac{\pi}{2}$.



Find pt of intersection:

Need $\cos x = \sin x$ for x in $[0, \frac{\pi}{2}]$.

$x = \frac{\pi}{4}$ gives this pt.

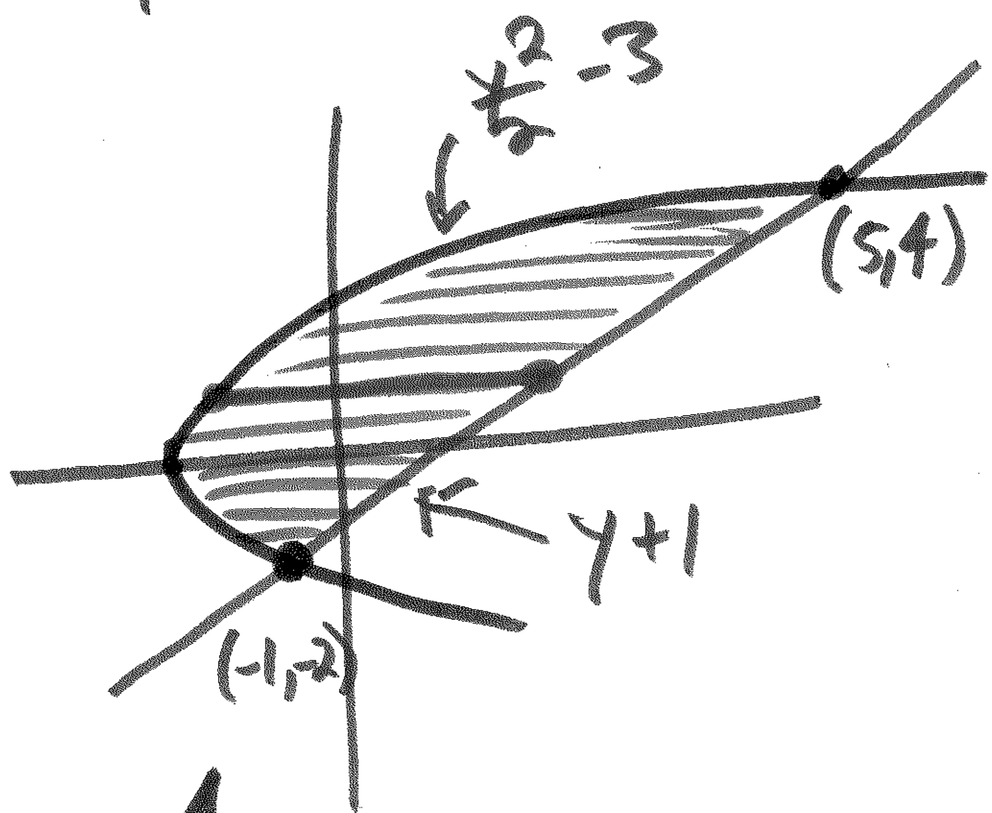


$$\int_0^{\pi/2} |\sin x - \cos x|\, dx = \int_0^{\pi/4} \cos x - \sin x\, dx + \int_{\pi/4}^{\pi/2} \sin(x) - \cos x\, dx$$

$$= \Big[\sin x + \cos x\Big]_0^{\pi/4} + \Big[-\cos x - \sin x\Big]_{\pi/4}^{\pi/2} = 2\sqrt{2} - 2.$$

Ex: Use y-axis to integrate.

Find area enclosed by $y = x-1$ and $y^2 = 2x+6$.



rewrite:

$x = y+1$ and

$x = \frac{y^2-6}{2} = \frac{y^2}{2} - 3$

$$\int_{-2}^{4} (y+1) - \left(\frac{y^2}{2} - 3\right) dy = \int_{-2}^{4} -\frac{y^2}{2} + y + 4 = 18.$$

12/2/16 12PM

0 Sign attendance!

1 Fill out TCE's!!! Response rates are 20-36%

2 Q: When $x \approx \frac{\pi}{2}$, why is

$$\cos x \approx \frac{\pi}{2} - x \,?$$

Today: §3.10, linear approximation
Maybe also start handout
on higher order approx

$\frac{\pi}{2} - x$ is tangent line to $\cos(x)$ at $x = \frac{\pi}{2}$.

$$\left.\begin{aligned} f(x) &= \cos(x), \\ f'(x) &= -\sin(x). \end{aligned}\right\} \Rightarrow \text{at } \tfrac{\pi}{2},\ f'\left(\tfrac{\pi}{2}\right) = -1.$$

So, eqn of tangent line at $\left(\frac{\pi}{2}, 0\right)$ is

$$y - 0 = -1\left(x - \frac{\pi}{2}\right)$$

$$\Rightarrow \quad y = \frac{\pi}{2} - x.$$

<u>Aside:</u> Similar computation shows $\sin x \approx x$ when $x \approx 0$.

Physicists write $\sin x \underset{\uparrow \approx}{=} x$ if x is small.

__Def$^{\underline{n}}$__: The __linear approximation__ of $f(x)$ at a is

$$f(x) \approx \underbrace{f(a) + f'(a)(x-a)}_{\text{Frequently written as } L(x).}$$

__why?__ This is pt-slope ~~form for~~ ~~tangent~~ line:

at $(a, f(a))$, slope $f'(a)$, tangent line is

$$y - f(a) = f'(a)(x-a)$$

$$\Rightarrow \quad y = f(a) + f'(a)(x-a).$$

$\approx f(x)$. near a.

Ex: Approximate $\sqrt{2}$.

Step 1: Find a good function to use.

$f(x) = \sqrt{1+x}$. Compute $f'(x) = \dfrac{1}{2\sqrt{1+x}}$ by chain rule.

Consider $(0, 1)$ to take tangent line. $f'(0) = \dfrac{1}{2}$, thus

$$L(x) = f(0) + f'(0)(x-0)$$
$$= 1 + \frac{1}{2}x.$$

Thus, $\underbrace{\sqrt{2}}_{\sqrt{1+1} \leftarrow x=1} \approx 1 + \frac{1}{2} \cdot \underset{\uparrow\, x=1}{1} = \frac{3}{2} = 1.5.$

Q: How good is an approximation?

(We will use calculators as an oracle to query accurate value).

Defn: Given an approximation $f(a) \approx A$, we have the following:

$$\text{error} = f(a) - A$$ ← in MA113, use a calculator for $f(a)$.

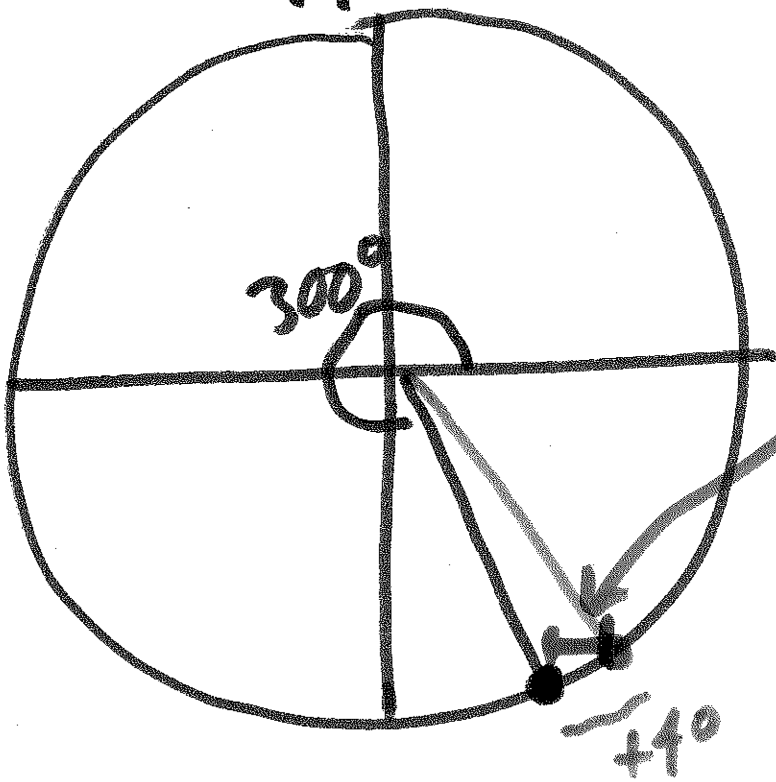
$$\text{percentage error} = \left| \frac{f(a) - A}{f(a)} \right| \times 100 \%$$

For $\sqrt{2}$, we had $\sqrt{2} \approx 1.5$.

$$\text{error} = \sqrt{2} - 1.5 = -0.0857864\ldots$$

$$\text{perc. Error} = \left| \frac{\sqrt{2} - 1.5}{\sqrt{2}} \right| \times 100 = 6.066\ldots\%$$

<u>Ex</u>: Approximate $\cos(304°) - \cos(300°)$.



we want to approx. this horizontal displacement

<u>Step 1</u>: Convert to radians.

<u>Why radians?</u> If x is in degrees, then

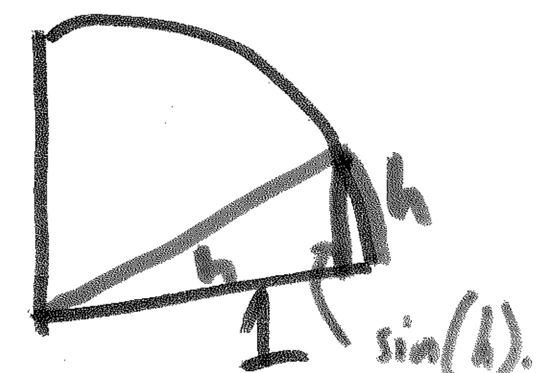
$$(\sin x)' = \frac{\pi}{180} \cos x.$$

Further, if you want $\frac{d}{dx} \sin x = \cos x$,

you must use radians.

In our proof that $\frac{d}{dx} \sin x = \cos x$,

we needed $\lim\limits_{h \to 0} \frac{\sin h}{h} = 1$.



So, convert to radians.
Down with degrees!!!

Convert $304°$ to radians...

a mess.

Convert $300°$ to radians: $\frac{5\pi}{3}$ rad.

$$304° - 300° = 4° = \frac{\pi}{45} \text{ rad}.$$

So, $304° = \frac{5\pi}{3} + \frac{\pi}{45}$ rad.

Use the point $(300°, \cos(300°)) = \left(\frac{5\pi}{3}, \frac{1}{2}\right)$ for our tangent line.

Slope of line is $\frac{d}{dx}\cos x \Big|_{x=\frac{5\pi}{3}} = -\sin\left(\frac{5\pi}{3}\right) = \sqrt{3}/2$.

So, $\cos(\underset{\substack{\uparrow \\ \text{rad}}}{x}) \approx \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{5\pi}{3}\right)$.

if $x \approx \frac{5\pi}{3}$.

$$\cos\Big(\underbrace{\frac{5\pi}{3} + \frac{\pi}{45}}_{304^\circ}\Big) \approx \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{5\pi}{3} + \frac{\pi}{45} - \frac{5\pi}{3}\right)$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\pi}{45}.$$

So, $\cos(304^\circ) - \cos(300^\circ) \approx$

$$\frac{1}{2} + \frac{\sqrt{3}}{2}\frac{\pi}{45} - \frac{1}{2} = \frac{\sqrt{3}}{2} \cdot \frac{\pi}{45} = 0.06045....$$

We can do better!

Suppose $f(x) \approx b + c(x-a) + d(x-a)^2$

when $x \approx a$. $f'(x) = c + 2d(x-a)$

Q: what should b, c, d be?

$b \approx f(a)$.

$f(a) \approx b + c(a-a) + d(a-a)^2 = b$.

$f'(a) \approx c + 2d(a-a) = c$. So, $c \approx f'(a)$

$f''(a) \approx 2d$ So, $d \approx \dfrac{f''(a)}{2}$.

Quadratic Approx: for $x \approx a$,

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

12/5/16 12PM

1. Webwork due this week
2. Final Exam Comments → SEE CANVAS!
3. Fill out TCE's! Current response rates 32-55%
4. <u>Today</u>: Handout on Taylor Polynomials + approximations

linear approx: if $x \approx a$, then

$$f(x) \approx f(a) + f'(a)(x-a).$$

Q: what if want to approximate $f(x)$ by a polynomial?

suppose $f(x) \approx b + c(x-a) + d(x-a)^2 + e(x-a)^3$ when $x \approx a$, for some b, c, d, e. ← what are they?

$$f(a) \approx b + c(a-a) + d(a-a)^2 + e(a-a)^3$$
$$= b. \quad \text{So, Set } b = f(a).$$

Take a derivative:

$$f'(x) \approx c + 2d(x-a) + 3e(x-a)^2.$$

Set $x=a$, get $f'(a) \approx c + 2d(a-a) + 3e(a-a)^2$

$$= c.$$

So, set $c = f'(a)$.

Take another derivative:

$$f''(x) \approx 2d + 3 \cdot 2 \cdot e(x-a).$$

Set $x=a$, get $f''(a) \approx 2d + 3 \cdot 2 \cdot e(a-a)$

$$= 2d.$$

So, set $d = \frac{f''(a)}{2}$.

Take another derivative:

$$f^{(3)}(x) \approx 3 \cdot 2 \cdot e.$$

Set $x=a$, get $f^{(3)}(a) \approx 3 \cdot 2 \cdot e$, So set $e = \frac{f^{(3)}(a)}{3 \cdot 2}$

<u>Def$^{\underline{n}}$</u>: Let $f(x)$ be infinitely diff on an open interval I. Let a be in I. The n^{th} <u>Taylor polynomial</u> for $f(x)$ at a is

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3\cdot 2}(x-a)^3 +$$

$$\frac{f^{(4)}(a)}{4\cdot 3\cdot 2}(x-a)^4 + \cdots + \frac{f^{(n)}(a)}{n(n-1)(n-2)\cdots 4\cdot 3\cdot 2}(x-a)^n.$$

<u>Ex</u>: Approximate $\sqrt{2}$ using quad. approx of $\sqrt{1+x}$ at 0.

$$f(x) = \sqrt{1+x}, \quad f'(x) = \frac{1}{2\sqrt{1+x}}, \quad f''(x) = \frac{-1}{4(1+x)^{3/2}}.$$

$$T_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2.$$

$$\sqrt{1+1} = \sqrt{2} \approx T_2(1) = 1 + \frac{1}{2} - \frac{1}{8} = 1.375.$$

<u>Ex</u>: Find an approximation of e.

Let's find $T_n(x)$ for $a=0$, $f(x)=e^x$.

$f^{(n)}(x) = e^x$. So, $f^{(n)}(0) = e^0 = 1$.

$$T_n(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3\cdot 2}x^3 + \frac{1}{4\cdot 3\cdot 2}x^4 + \frac{1}{5\cdot 4\cdot 3\cdot 2}x^5 + \text{etc} \ldots + \frac{1}{n\cdots 4\cdot 3\cdot 2}x^n.$$

$e^x \approx T_7(x)$, so $e^1 \approx T_7(1)$.

$$\Rightarrow e \approx 1 + 1 + \frac{1}{2} + \frac{1}{3\cdot 2} + \frac{1}{4\cdot 3\cdot 2} + \frac{1}{5\cdot 4\cdot 3\cdot 2} + \frac{1}{6\cdot 5\cdot 4\cdot 3\cdot 2}$$

$$+ \frac{1}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} = \frac{685}{252} \approx 2.71825\ldots$$

<u>Ex</u>: $\sin(x)$ w/ degree 7 approx. at $a=0$.

$f(x) = \sin(x)$.	$f(0) = 0$.	when $x \approx 0$,
$f'(x) = \cos x$.	$f'(0) = 1$	$\sin x \approx x - \frac{1}{3\cdot 2} x^3$
$f''(x) = -\sin x$	$f''(0) = 0$	
$f^{(3)}(x) = -\cos x$	$f'''(0) = -1$	$+ \frac{1}{5\cdot 4\cdot 3\cdot 2} x^5 +$
$f^{(4)}(x) = \sin x$.	$f^{(4)}(0) = 0$	
Things repeat.	$f^{(5)}(0) = 1$	$\frac{-1}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} x^7$.
	$f^{(6)}(0) = 0$ $f^{(7)}(0) = -1$.	

__Thm:__ If $f(x)$ is

- any polynomial
- e^x
- $\sin(x)$ or $\cos(x)$

then $\lim_{n\to\infty} T_n(x) = f(x)$ for any choice of a, x real #'s.

If $f(x) = (1+x)^k$, k any real #, then

$\lim_{n\to\infty} T_n(x) = f(x)$ when $a = 0$ and x in $(-1, 1]$.

Ex: Approximate

$$\int_{-1/2}^{1/2} e^{-x^2}\,dx.$$

$$x \longmapsto -x^2 \longmapsto e^{-x^2}.$$

plug $-x^2$ into Taylor poly for e^x.

$e^{-x^2} \approx T_n(-x^2)$ where $T_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3 \cdot 2} + \cdots$ etc.

Use $n=4$. $\displaystyle\int_{-1/2}^{1/2} e^{-x^2}\,dx \approx \int_{-1/2}^{1/2} T_4(-x^2)\,dx =$

$$\int_{-1/2}^{1/2} 1 + (-x^2) + \frac{(-x^2)^2}{2} + \frac{(-x^2)^3}{3\cdot 2} + \frac{(-x^2)^4}{4\cdot 3\cdot 2}\,dx$$

$$= \int_{-1/2}^{1/2} 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24}\,dx = \frac{1785941}{1935360}$$

$$\approx 0.\underbrace{922562}7\ldots$$

Wolframalpha says $\int_{-1/2}^{1/2} e^{-x^2}dx = 0.\underbrace{922562}3101\ldots$

<u>Exercise:</u> $\frac{1}{1-x} = f(x) \Rightarrow (1-x)^{-1}.$

$x \longmapsto -x \longmapsto (1+(-x))^{-1}.$ So, $T_n(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots + x^n.$