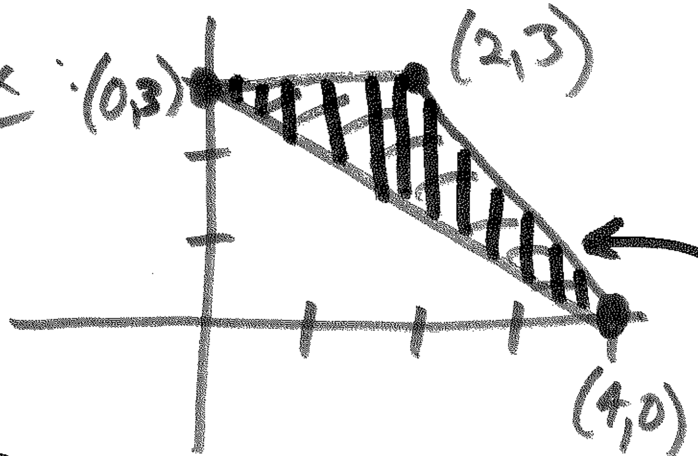
__11/30/16__ __9AM__

1. See Canvas announcement about handout for Monday.

2. fill out TCE's!!! Currently have 12-19% of responses

3. __Q__: What is area of triangle with vertices $(0,3)$, $(2,3)$, and $(4,0)$?

__Today__: §6.1, area between curves

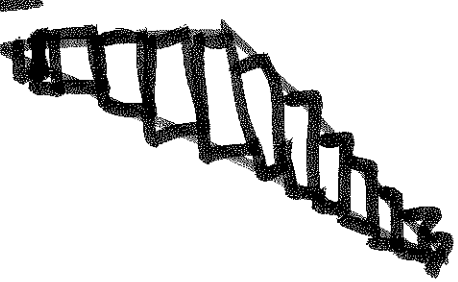
Ex: 

$A = \frac{1}{2} b \cdot h = \frac{1}{2} \cdot 2 \cdot 3 = 3.$

infinitely many vertical lines.

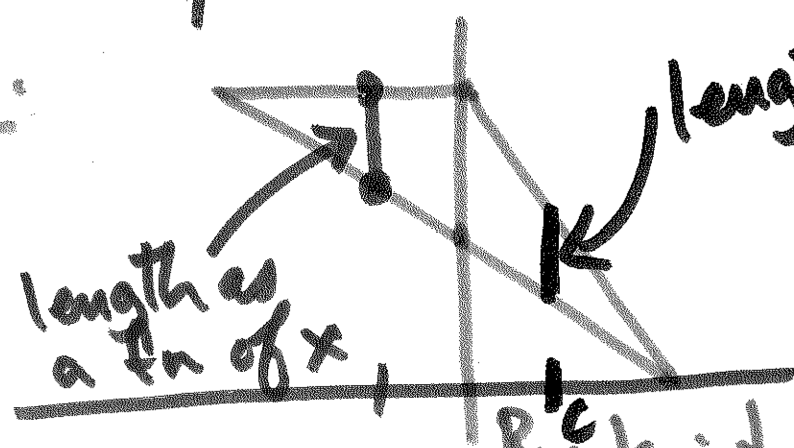
Add up lengths (to get area).

Alt. Riem. Sum



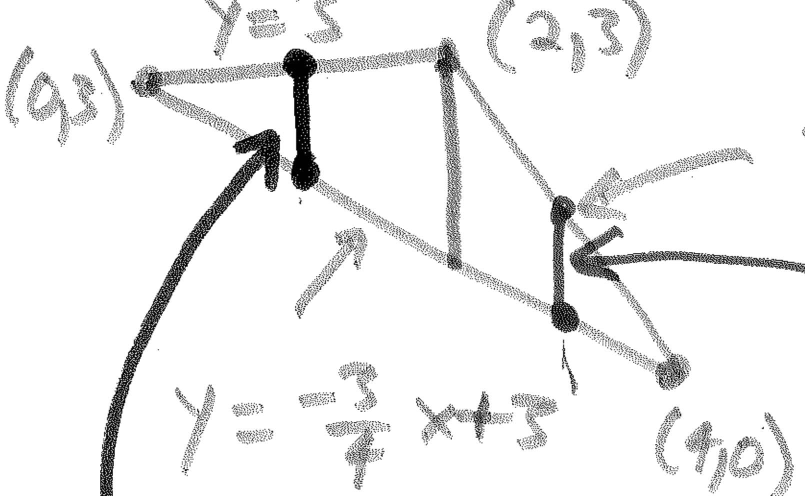
Think of lines as infinitely thin rectangles w/ width dx.

Strategy:



Break into 2 pieces

Add two halves separately.



(By point-slope)

$3 - \left(\frac{-3}{4}x + 3\right)$ is length.

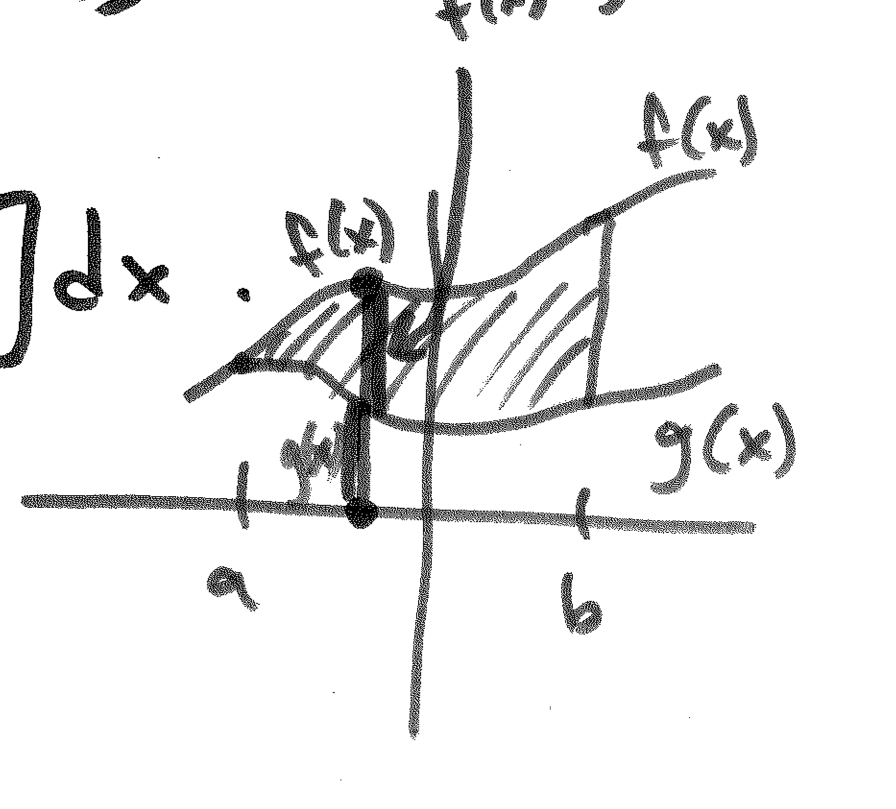
$\left(\frac{-3}{2}x + 6\right) - \left(\frac{-3}{4}x + 3\right)$ is length.

Area is $\displaystyle\int_0^2 \left(3 - \left(\frac{-3}{4}x + 3\right)\right)dx + \int_2^4 \left(\frac{-3}{2}x + 6\right) - \left(\frac{-3}{4}x + 3\right)dx$

$$= \int_0^2 \left(\frac{3}{4}x\right)dx + \int_2^4 \left(\frac{-3}{4}x + 3\right)dx = 3.$$

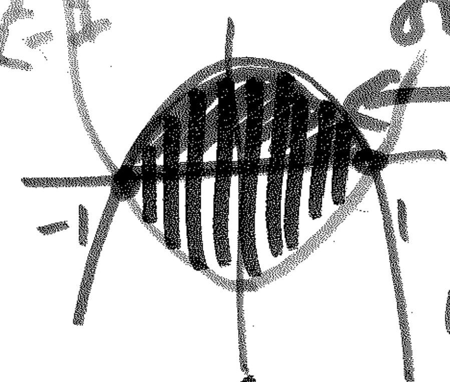
Def$^{\underline{n}}$: The area A bounded by the curves $y=f(x)$, $y=g(x)$, $x=a$, and $x=b$, where f, g are cts and $f(x) \geq g(x)$ on $[a,b]$, is

$$A = \int_a^b [f(x)-g(x)]\,dx.$$



Ex: Find area between $f(x) = x^2 - 1$ and $g(x) = 1 - x^2$ on $x = -1$ to $x = 1$.

Note: If we say "enclosed by" then we will leave off the bounds.



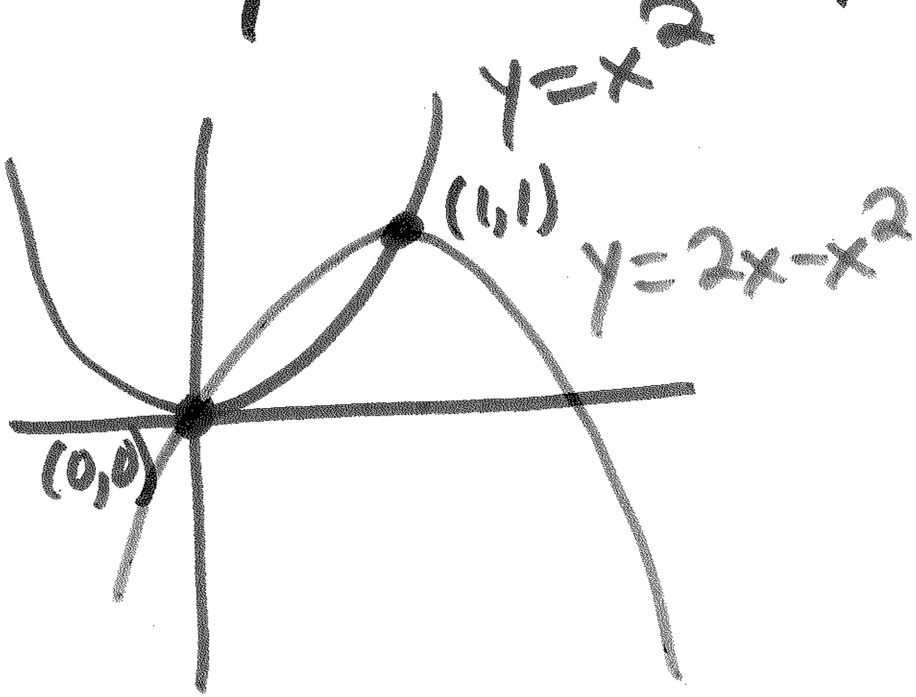
$$\int_{-1}^{1} -x^2+1 \, dx.$$

Use an area integral:

$$\int_{-1}^{1} \Big(\underset{\text{upper}}{(1-x^2)} - \underset{\text{lower}}{(x^2-1)} \Big) dx = \int_{-1}^{1} -2x^2+2 \, dx$$

$$= 2\int_{-1}^{1} \overbrace{-x^2+1}^{g(x)} \, dx = \frac{8}{3}$$

Ex: Find area enclosed between $y=x^2$ and $y=2x-x^2$.



To find intersection pts, set $x^2=2x-x^2$ and solve. $\Rightarrow$

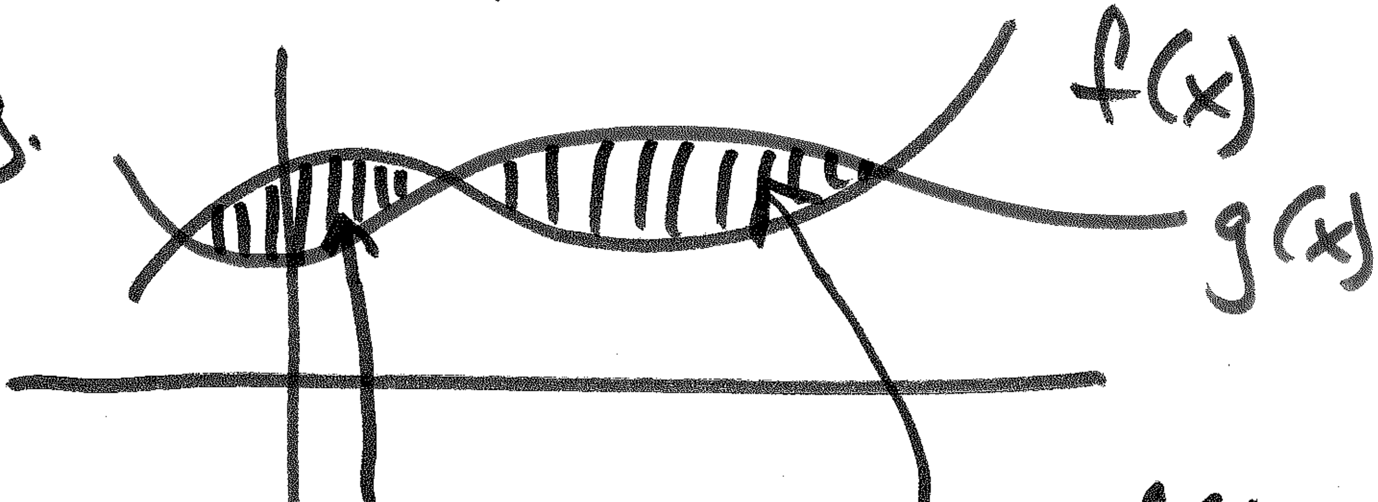
$$0=2x-2x^2$$
$$=2x(1-x)$$

So, intersection pts are $x=0$, $x=1$.

$$\int_0^1 (2x-x^2)-x^2\,dx=\int_0^1 2x-2x^2\,dx=\frac{1}{3}.$$

Q: What happens if $f(x) \ngeq g(x)$ sometimes?

e.g.



$f(x)-g(x)$ is length

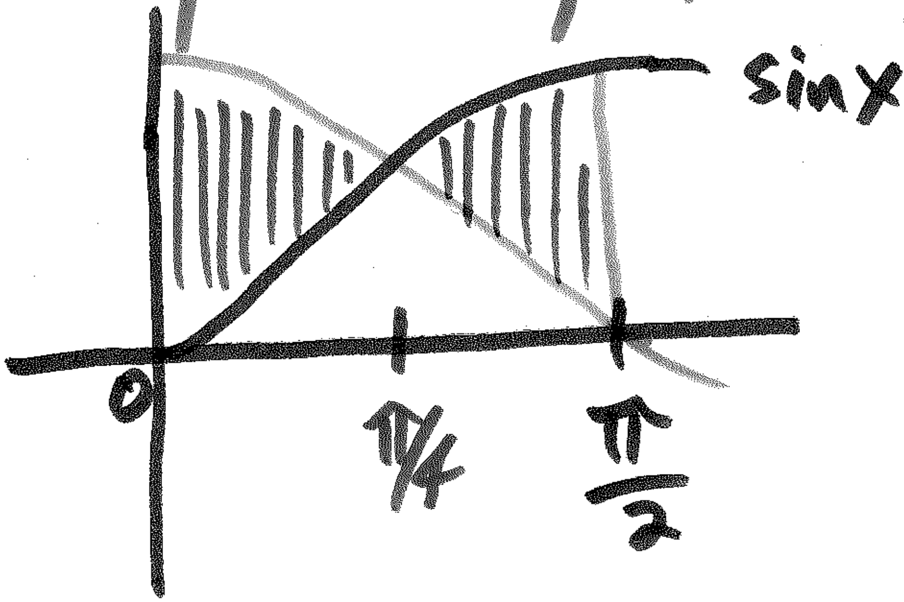
$g(x)-f(x)$ is length

These are both $|f(x)-g(x)|$.

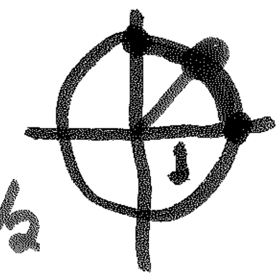
Def$^{\underline{n}}$: The total positive area between $f(x)$, $g(x)$, $x=a$, $x=b$, is

$$\int_a^b |f(x)-g(x)|\,dx.$$

Ex: Find area bounded by $y = \sin x$, $y = \cos x$, $x = 0$, $x = \frac{\pi}{2}$.



Pt of intersection for x in $[0, \frac{\pi}{2}]$ is soln to $\sin x = \cos x$, i.e. $x = \frac{\pi}{4}$.

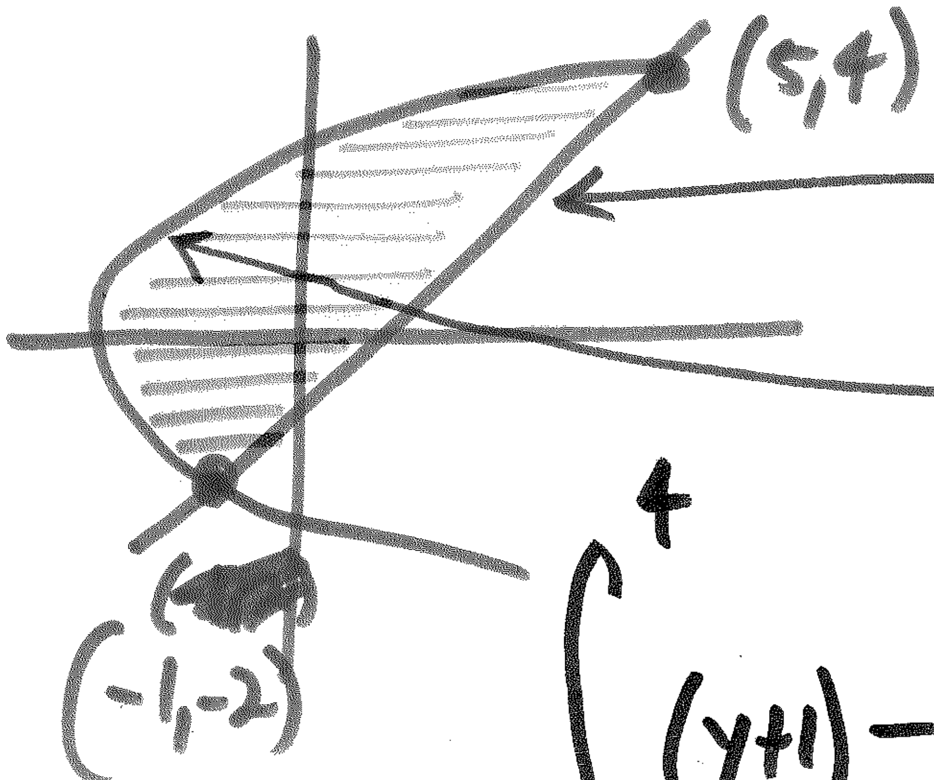


Area is $\displaystyle\int_0^{\pi/2} |\sin x - \cos x|\,dx = \int_0^{\pi/4} \cos x - \sin x \, dx + \int_{\pi/4}^{\pi/2} \sin x - \cos x \, dx$

$$= \Big[\sin x + \cos x\Big]_0^{\pi/4} + \Big[-\cos x - \sin x\Big]_{\pi/4}^{\pi/2} = 2\sqrt{2} - 2.$$

Ex: Use y-axis to integrate sometimes.

Find the area enclosed by $y = x - 1$ and $y^2 = 2x + 6$.



$x = y + 1$

$x = \dfrac{y^2 - 6}{2} = \dfrac{y^2}{2} - 3$

$$\int_{-2}^{4} \underbrace{(y+1)}_{\substack{\uparrow \\ \text{upper on} \\ x\text{-axis}}} - \underbrace{\left(\frac{y^2}{2} - 3\right)}_{\substack{\uparrow \\ \text{lower on} \\ x\text{-axis.}}} \, dy = 18$$

12/2/16 9 AM

$\boxed{0}$ Sign attendance

$\boxed{1}$ Fill out TCEs!!!

Response rates are 17-25%

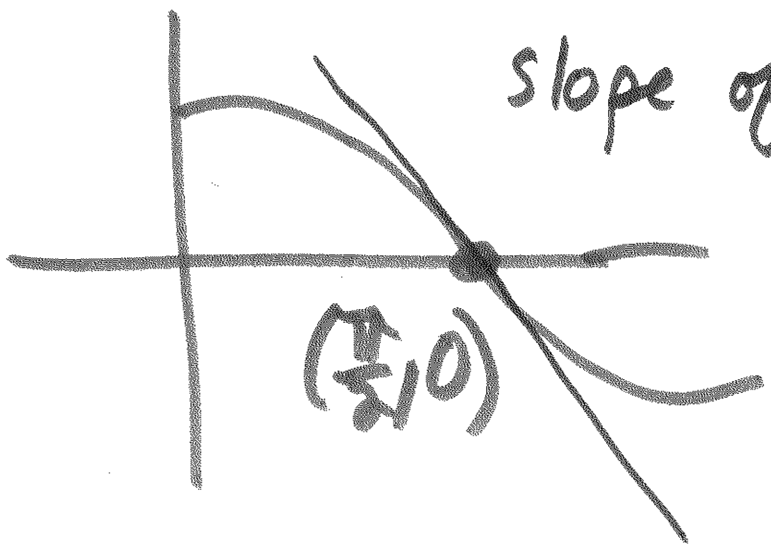
$\boxed{2}$ Q: When $x \approx \frac{\pi}{2}$, why is

$$\cos x \approx \frac{\pi}{2} - x \text{ ?}$$

Today: §3.10, linear approximation

Maybe also start handout on higher order approx.

$\frac{\pi}{2} - x$ is the tangent line to $\cos x$ at $(\frac{\pi}{2}, 0)$.



slope of tangent line is derivative of $\cos x$ at $\frac{\pi}{2}$.

$$\frac{d}{dx}\cos x \Big|_{\frac{\pi}{2}} = -\sin(\tfrac{\pi}{2}) = -1.$$

Point-slope $\Rightarrow$ $y = -1 \cdot (x - \frac{\pi}{2})$

$= \frac{\pi}{2} - x$. is eqn of tangent line.

Ex: If $x \approx 0$, what is a good line approximating $\sin x$?

slope of tangent to $\sin x$ at $(0,0)$ is

$$\frac{d}{dx}(\sin x)\Big|_{x=0} = \cos(0) = 1.$$

So, tangent line is $y = x$.

NOTE: $\sin x = x$ if x small is physics convention.

should be $\approx$.

$\underline{\text{Def}^{\underline{n}}}$: The <u>linear</u> <u>approximation</u> of $f(x)$ at $x=a$ is

$$f(x) \approx f(a) + f'(a)(x-a).$$

why? $y - f(a) = f'(a)(x-a)$ is eqn for tangent line to f at $(a, f(a))$.

$$\Rightarrow y = \underbrace{f'(a)(x-a) + f(a)}_{\text{This is approx of } f.}$$

<u>NOTE</u>: $L(x) = f(a) + f'(a)(x-a)$ is common notation.

<u>Application</u>: Approximate $\sqrt{2}$.

<u>Idea</u>: choose a good function.

$f(x) = \sqrt{1+x}$. use pt $(0,1)$ on f.

<u>$f'(0)$ is</u>: $f'(x) = \dfrac{1}{2\sqrt{1+x}}$ by chain rule.

$$\text{So, } f'(0) = \frac{1}{2}.$$

$$\text{So, } f(x) \approx \underbrace{1}_{f(0)} + \underbrace{\tfrac{1}{2}}_{f'(0)}\overbrace{x}^{(x-0)}.$$

$$\sqrt{1+1} \approx 1 + \frac{1}{2} = \frac{3}{2} = 1.5.$$

Q: $\sqrt{2} \approx 1.5$, how far off are we?

(we assume calculator is an oracle w/ correct answer.)

Def$^{\underline{n}}$: Given an approximation $f(a) \approx A$,

$$\text{error} = f(a) - A \qquad \left(\text{use a calc. for } f(a)\right)$$

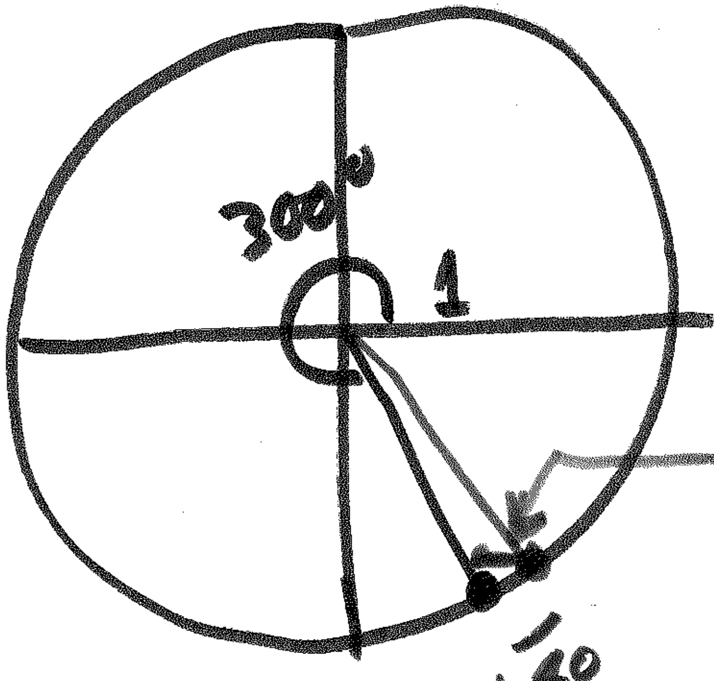
and

$$\text{percentage error} = \left| \frac{f(a) - A}{f(a)} \right| \times 100.$$

For $\sqrt{2}$, $\text{err} = \sqrt{2} - 1.5 = -0.0857864\ldots$

$\text{per. err} = \left| (\sqrt{2} - 1.5)/\sqrt{2} \right| \times 100 = 6.066\ldots\%$.

Ex: Approximate $\cos(304^\circ) - \cos(300^\circ)$.



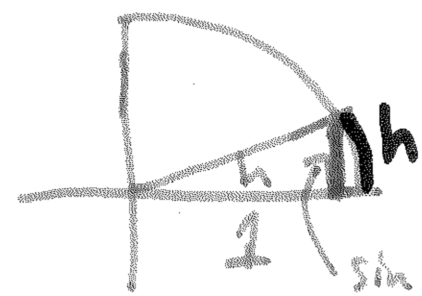
approximate hor. displacement between those two points.

Step 1: Always convert to radians!

reason: $\frac{d}{dx}\sin x = \cos x$ only holds when x is in radians.

Recall: Proof of $\frac{d}{dx}\sin x = \cos x$ required

$$\lim_{h\to 0} \frac{\sin h}{h} = 1$$



Q for Monday:

Is $\frac{d}{dx} \sin x = \cos x$

true only for x in radians?

Back to $\cos 304° - \cos 300°$.

Converting to radians, $300° = \frac{5\pi}{3}$ rad.

$$304° - 300° = 4° = \frac{\pi}{45} \text{ rad.}$$

We should approximate $\cos(304°)$ using a linear approx of $\cos(x)$ at $x = 300°$.

Set $f(x) = \cos(x)$, find Lin. approx at pt $\left(\frac{5\pi}{3}, \frac{1}{2}\right)$.

$f'(x) = -\sin x$,

so $f'\left(\frac{5\pi}{3}\right) = -\sin\left(\frac{5\pi}{3}\right) = \frac{\sqrt{3}}{2}$.

For $x \approx \frac{5\pi}{3}$, $\cos(x) \approx \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{5\pi}{3}\right)$.

$304^\circ = \frac{5\pi}{3} + \frac{\pi}{45}$ rad, thus

$$\cos(304^\circ) \approx \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{5\pi}{3} + \frac{\pi}{45} - \frac{5\pi}{3}\right)$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\pi}{45}.$$

So, $\cos(304^\circ) - \cos(300^\circ) =$

$$\cos\left(\frac{5\pi}{3}+\frac{\pi}{45}\right) - \cos\left(\frac{5\pi}{3}\right) \approx$$

$$\frac{1}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\pi}{45} - \frac{1}{2} = \frac{\sqrt{3}}{2}\cdot\frac{\pi}{45}.$$

$$= 0.06045\ldots\ldots$$

12/5/16 9AM

1. Webwork due this week
2. Final Exam Comments → SEE CANVAS!!
3. Fill out TCE's! Current response rates 27 - 47%
4. Today: Handout on Taylor polynomials & approximations

Linear Approx: $f(x) \approx f(a) + f'(a)(x-a)$

when $x \approx a$, f is diff.

Q: what if we approximated f by a poly?

e.g. cubic. Suppose

$$f(x) \approx b + c(x-a) + d(x-a)^2 + e(x-a)^3$$

for some constants b, c, d, e and $x \approx a$.

$$f(a) \approx b + c(a-a) + d(a-a)^2 + e(a-a)^3$$
$$= b.$$

So, $f(x) \approx f(a)$ + higher order terms.

Taken first derivatives:

$$f'(x) \approx c + 2d(x-a) + 3e(x-a)^2$$

So, $f'(a) \approx c + 2d(a-a) + 3e(a-a)^2$

$$= c$$

So, $c \approx f'(a)$.

Second derivatives:

$$f''(x) \approx 2d + 3 \cdot 2 \cdot e(x-a)$$

So, $f''(a) \approx 2d + 3 \cdot 2 \cdot e(a-a)$

$$= 2d.$$

$$\Rightarrow \frac{f''(a)}{2} \approx d.$$

Third derivatives:

$$f^{(3)}(x) \approx 3\cdot 2\cdot e$$

$$\Rightarrow \quad \frac{f^{(3)}(a)}{3\cdot 2} \approx e.$$

Assumption:

$f(x) \approx$
$b +$
$c(x-a)^1 +$
$d(x-a)^2 +$
$e(x-a)^3.$

Def$^{\underline{n}}$: Let $f(x)$ be infinitely differentiable, on an open interval I containing a.
The n^{th} Taylor polynomial for $f(x)$ at a is

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3\cdot 2}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n\cdot(n-1)(n-2)\cdots 3\cdot 2}(x-a)^n.$$

Ex: Approximate $\sqrt{2}$ using quad. approx to $\sqrt{1+x}$ at $x=0$.

Compute two derivatives:

$$f'(x) = \frac{1}{2\sqrt{1+x}} \qquad f''(x) = \frac{-1}{4(1+x)^{3/2}}.$$

So, when $x \approx 0$,

$$\sqrt{1+x} \approx f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2$$

$$= 1 + \frac{1}{2}x + \left(\frac{-1}{8}\right)x^2$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8}.$$

So, $\sqrt{2} = \sqrt{1+1} \approx 1 + \frac{1}{2} - \frac{1}{8} = 1.375$.

<u>Ex</u>: Find a degree 7 approximation to $\sin(x)$ at $x = 0$.

$f(0) = 0$

$f'(x) = \cos(x)$. $f'(0) = 1$

$f''(x) = -\sin(x)$ $f''(0) = 0$

$f^{(3)}(x) = -\cos(x)$ $f^{(3)}(0) = -1$

$f^{(4)}(x) = \sin(x)$ $f^{(4)}(0) = 0$

$5^{th}, 6^{th}, 7^{th}$ der's are repeats.

$f^{(5)}(0) = 1$

$f^{(6)}(0) = 0$

$f^{(7)}(0) = -1$.

So, $\sin x \approx$

$$x + \frac{-1}{3 \cdot 2} x^3 + \frac{1}{5 \cdot 4 \cdot 3 \cdot 2} x^5 + \frac{-1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} x^7.$$

<u>Thm</u>: If $f(x)$ is

- any polynomial
- e^x
- $\sin(x)$ or $\cos(x)$

Then $\lim_{n\to\infty} T_n(x) = f(x)$ for any choice of a and any x.

If $f(x) = (1+x)^k$ for k a real #,

Then $\lim_{n\to\infty} T_n(x) = f(x)$ for $a=0$ and any x in $(-1,1]$.

__Ex:__ Approximate $\displaystyle\int_{-1/2}^{1/2} e^{-x^2}\,dx$.

$x \longmapsto -x^2 \longmapsto e^{-x^2}$ means we can find Taylor poly for e^x and plug $-x^2$ into it.

when $a=0$, $\frac{d^n}{dx^n} e^x = e^x$, so n^{th} der of e^x at $a=0$ is $e^0=1$.

So, $e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{3\cdot 2} + \frac{x^4}{4\cdot 3\cdot 2}$ when $x \approx 0$.

So, $e^{-x^2} \approx 1 + (-x^2) + \frac{(-x^2)^2}{2} + \frac{(-x^2)^3}{3\cdot 2} + \frac{(-x^2)^4}{4\cdot 3\cdot 2}$

$$= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24}.$$

So $$\int_{-1/2}^{1/2} e^{-x^2}\,dx \approx \int_{-1/2}^{1/2} 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24}\,dx$$

$$= \frac{1785491}{1935360} \approx 0.922562727\,3478\ldots.$$

Wolfram alpha value of $\int_{-1/2}^{1/2} e^{-x^2}\,dx$ is

0.9225620128...

<u>Ex</u>: $f(x) = \frac{1}{1-x} = (1-x)^{-1}$.

What is $T_n(x)$ at $a=0$?

$f'(x) = -1 \cdot -1(1-x)^{-2} = (1-x)^{-2}$.

$f''(x) = -1 \cdot -2(1-x)^{-3} = 2(1-x)^{-3}$.

$f'''(x) = -1 \cdot -3 \cdot 2 \cdot (1-x)^{-4} = 3 \cdot 2 \cdot (1-x)^{-4}$.

$f^{(4)}(x) = -1 \cdot -4 \cdot 3 \cdot 2 \cdot (1-x)^{-5} = 4 \cdot 3 \cdot 2 \cdot (1-x)^{-5}$.

$\vdots$

$f^{(n)}(x) = n \cdot (n-1) \cdot (n-2) \cdots 4 \cdot 3 \cdot 2\,(1-x)^{-(n+1)}$.

So, $f^{(n)}(0) = n \cdot (n-1) \cdot \cdots \cdot 3 \cdot 2$.

Thus, when x in $(-1,1]$, we have

$$\frac{1}{1-x} \approx 1 + x + x^2 + x^3 + \cdots + x^n.$$

(finite geometric series)