MA 113 Review Day

12PM

Wesley Hough

① Find $\frac{dy}{dx}$ for $y = \underbrace{e^{3x}}_{f(x)} \cdot \underbrace{2^{\cos(x^2+1)}}_{g(x)}$

$f(x) = e^{3x}$

$f'(x) = 3e^{3x}$

$g(x) = 2^{\cos(x^2+1)}$

$g'(x) = \ln(2) \cdot 2^{\cos(x^2+1)}(-\sin(x^2+1)) \cdot 2x$

$$\frac{dy}{dx} = f'(x)\,g(x) + f(x)\,g'(x)$$

$$= \boxed{3e^{3x}\,2^{\cos(x^2+1)} - (2\ln 2)e^{3x}\,x\,\sin(x^2+1)\,2^{\cos(x^2+1)}}$$

② Find the equation of the tangent line to $y^3 + 2xy + x^3 = 4$ at $(1,1)$.

taking the derivative gives

$$3y^2 \frac{dy}{dx} + \underline{2y} + 2x\frac{dy}{dx} + \underline{3x^2} = 0.$$

$$3y^2 \frac{dy}{dx} + 2x\frac{dy}{dx} = -2y - 3x^2$$

$$\frac{dy}{dx}(3y^2 + 2x) = -2y - 3x^2$$

$$\frac{dy}{dx} = \frac{-2y - 3x^2}{3y^2 + 2x}$$

at $(1,1)$

$$\frac{dy}{dx} = \frac{-2-3}{3+2} = \frac{-5}{5} = -1$$

tangent line

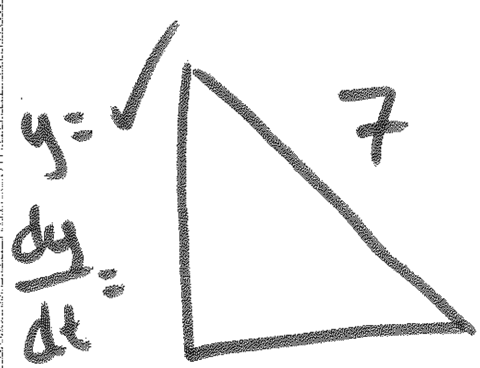
$$y = f(a) + f'(a)(x-a)$$

$$\boxed{y = 1 + -1(x-1) = 2-x}$$

③ A 7-m ladder leans against a vertical wall with bottom 2 m away at $t = 0$. The ladder's bottom is sliding away from the wall at a rate of 0.6 m/s.

What is the velocity of the top of the ladder when $t = 1$?

$$\frac{dy}{dt} = ?$$



$\rightarrow x^2 + y^2 = 7^2$

$(2.6)^2 + y^2 = 49$

$y = \sqrt{49 - (2.6)^2}$

$x = 2.6$ @ $t = 1$

$\frac{dx}{dt} = 0.6$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2(2.6)(0.6) + 2\sqrt{49 - (2.6)^2}\,\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{-\cancel{2}(2.6)(0.6)}{\cancel{2}\sqrt{49 - (2.6)^2}} \text{ m/s}$$

4 Evaluate $\lim_{x \to 0} \dfrac{\sin^3(12x)}{\pi x^3}$.

Recall that $\lim_{x \to 0} \dfrac{\sin(x)}{x} = 1$.

$$\lim_{x \to 0} \frac{1}{\pi} \left(\frac{\sin(12x)}{12x} \right)^3 \cdot 12^3$$

$$= \frac{1}{\pi} \left(\lim_{x \to 0} \frac{\sin(12x)}{12x} \right)^3 \cdot 12^3$$

$$= \frac{12^3}{\pi} \cdot 1^3 = \boxed{\frac{12^3}{\pi}}$$

⑤ Show, using the limit definition of the derivative, that $\frac{d}{dx}x^3 = 3x^2$.

Recall: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

$f(x) = x^3$

$f(x+h) = (x+h)^3 = (x+h)(x+h)(x+h)$

$= x^3 + 3x^2h + 3xh^2 + h^3$

$$\frac{f(x+h) - f(x)}{h} = \frac{\cancel{x^3} + 3x^2\cancel{h} + 3xh^{\cancel{2}} + h^{\cancel{3}\,2} - \cancel{x^3}}{\cancel{h}}$$

$$= 3x^2 + 3xh + h^2$$

$$\boxed{\frac{d}{dx}x^3 = \lim_{h \to 0} 3x^2 + \cancel{3xh} + \cancel{h^2} = 3x^2}$$ ✓

⑥ Find the derivative of

$$\ln\left(\frac{\sqrt[3]{x+1}}{\sqrt{x^2+x+1}}\right).$$

$$= \ln\left(\sqrt[3]{x+1}\right) - \ln\left(\sqrt{x^2+x+1}\right)$$

$$= \ln\left((x+1)^{1/3}\right) - \ln\left((x^2+x+1)^{1/2}\right)$$

$$= \frac{1}{3}\ln(x+1) - \frac{1}{2}\ln(x^2+x+1)$$

derivative

$$\frac{1}{3}\,\frac{1}{x+1}\cdot 1 - \frac{1}{2}\,\frac{1}{x^2+x+1}\,(2x+1)$$

$$= \boxed{\frac{1}{3(x+1)} - \frac{2x+1}{2(x^2+x+1)}}$$

10/17/16 12PM

$\boxed{1}$ Sign Attendance

$\boxed{2}$ Exam 2 tomorrow, see Canvas announcement!

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strategies for dealing w/ stress/ anxiety on exams.

1. You encounter a tough problem + freak out! Augh!

we wouldn't give you this, too hard. → **Ex**: Compute derivative of $e^{\cos(x)} \cdot x^{\sin(x)}$ using limit defn of derivative.

☆ Ignore the problem.

Instead, think about problem type.

Flip to back of previous page + review.

Q: How have we handled problems w/ limit defn of derivative?

- get rid of h
- choose $h \to 0$ or $x \to a$ form
- common denominators w/ fractions
- use limit laws
- split fn using
  - log laws
  - trig ids e.g. $\cos(x+h)$ or $\sin(x+h)$
- mult. by conjugates when $\sqrt{\cdot}$ involved

2. Spend 5 min @ Start of exam making a plan for how you will complete the exam

comfortable: 1, 3, 4, 8, 9, 10, 16 ← do these first!

work: 2, 5, 11, 18

scared: rest.

3. Spend 2 min @ beginning writing down formulas you are afraid you might forget.

4. Write on first page of your exam "I can do well on this exam, go me!"

Ex: Use limit def$^{\underline{n}}$ of derivative to find $f'(x)$ when $f(x) = \frac{2}{x^2}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{2}{(x+h)^2} - \frac{2}{x^2}}{h} \overset{\text{com. den.}}{=}$$

$$\lim_{h\to 0} \frac{\dfrac{2x^2 - 2(x+h)^2}{x^2\cdot(x+h)^2}}{h} = \lim_{h\to 0} \frac{2x^2 - 2(x^2 + 2xh + h^2)}{h\cdot x^2\cdot(x+h)^2}$$

clean expression

cancel $2x^2$'s

$$= \lim_{h\to 0} \frac{-4x\cancel{h} - 2h^{\cancel{2}}}{\cancel{h}\cdot x^2(x+h)^2} = \lim_{h\to 0} \frac{-4x - 2h}{x^2(x+h)^2} = \frac{-4x}{x^2\cdot x^2} = \frac{-4}{x^3}$$

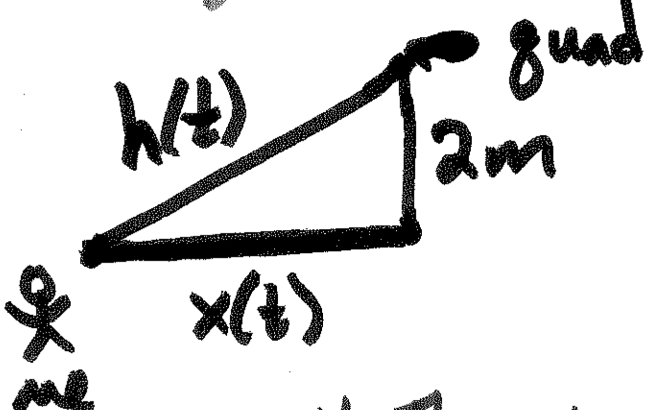
cancel h's

$\boxed{\text{Check for correctness:}}$

$$\frac{d}{dx}\frac{2}{x^2} = \frac{d}{dx}\, 2\cdot x^{-2} = \cancel{-2\cdot 2}\cdot x^{-3} = \frac{-4}{x^3}$$

✓

Ex: Quadcopter is flying away from me horizontally at a constant height of 2m, and constant velocity of 200 m/min. When quad is 4 m from me measured on ground, how fast is the actual distance btwn me + quad changing?



Know: $\frac{dx}{dt} = 200$ m/min

Want: when $x = 4$, find $\frac{dh}{dt}$.

Pyth Thm $\Rightarrow$ $x^2 + 2^2 = h^2$ $\overset{\text{Apply } \frac{d}{dt}}{\Longrightarrow}$ $2x \cdot \frac{dx}{dt} + 0 = 2h \cdot \frac{dh}{dt}$

$\Rightarrow \frac{dh}{dt} = \frac{x}{h} \frac{dx}{dt}$. When $x = 4$, $h = \sqrt{4^2 + 2^2} = \sqrt{20}$.

So, when $x=4$,

$$\frac{dh}{dt} = \frac{4}{\sqrt{20}} \cdot 200 \text{ m/min} .$$

<u>10/19/16</u> 12PM

$\boxed{1}$ Sign attendance

<u>§3.8</u>: <u>Exponential Growth</u>

If a fn $y(t)$ satisfies

differential equation: $\boxed{\dfrac{dy}{dt} = K \cdot y(t)}$, then every soln y

same

Satisfying this is of the form $y = y(0) \cdot e^{kt}$.

<u>Proof → Calc II</u>. <u>Check</u>: $\dfrac{d}{dt}\, y(0) \cdot e^{kt} = k \cdot y(0) e^{kt} = k \cdot y$.

It <u>IS</u> a soln.

If $K>0$, say y models <u>growth</u>; $K<0$ models <u>decay</u>.

Ex: Population Growth:

If $P(t)$ models population & $\frac{dP}{dt} = k \cdot P$, we say k is the relative growth rate.

Q: What is relative growth rate if $P(0) = 2{,}000$ & $P(5) = 10{,}000$?

Note: $\frac{dP}{dt} = kP$ implies $P(t) = P(0) \cdot e^{kt}$.

Know: $P(0) = 2000$, so $P(t) = 2000e^{kt}$.

Know: $P(5) = \underbrace{10000 = 2000 \cdot e^{k \cdot 5}}_{\text{Solve for } k \text{ using log laws}}$

Soln: $k = \frac{\ln(5)}{5}$.

__Ex__: Radioactive decay

Experiments show that mass $m(t)$ of a radioactive substance satisfies $\frac{dm}{dt} = k \cdot m$ (for different k's depend on substance).

__Half-life__ of substance is time t where

$$m(t) = \frac{1}{2} m(0).$$

__NOTE:__ Knowing half-life is equivalent to knowing k, as demonstrated next.

__Q:__ Half-life of radium-226 is 1590 years. If $m(0) = 1\text{kg}$, how much is left after 100 years?

Step 1: Compute k. $\frac{dm}{dt} = km$

$\Rightarrow$ $m(t) = m(0)e^{kt}$. So, half-life says

$$m(1590) = \underbrace{\tfrac{1}{2}\,\text{kg} = 1 \cdot e^{k \cdot 1590}\,\text{kg}}_{\text{solve for } k \text{, get}}$$

$$\boxed{\begin{array}{l} \ln\left(\frac{1}{a}\right) = \\ \ln(a^{-1}) = -\ln(a) \end{array}}$$

$$k = -\ln(2)/1590$$

Step 2: Determine $m(100) = 1 \cdot e^{\frac{-\ln(2)}{1590} \cdot 100}$ kg.

Ex: Newton's Law of Cooling

Say $T(t)$ = temp at time t of object, T_s is constant temp of surroundings.

$$\frac{dT}{dt} = k \cdot (T - T_s) \quad \text{for some } k.$$

Set $y = T - T_s$ $\Rightarrow$ Apply $\frac{d}{dt}$ to both sides

constant!

$$\frac{dy}{dt} = \frac{dT}{dt}.$$

So $\frac{dy}{dt} = k \cdot y.$ $\Rightarrow$ $y = y(0) \cdot e^{kt}.$

<u>Back to T:</u> $T(t) - T_s = (T(0) - T_s)e^{kt}$

$$\Rightarrow \quad T(t) = (T(0) - T_s)e^{kt} + T_s.$$

<u>Q</u>: If temp of air is 20°C, coffee is 98°C, + coffee temp 30 min later is 70°C, what is $T(t)$?

$$T_s = 20°C, \quad T(0) = 98°C$$

So $T(t) = (98-20)e^{kt} + 20$

$= 78e^{kt} + 20$.

To find k, use $\underset{\text{min}}{T(30)} = 70°C$ & get

$70 = 78 \cdot e^{k \cdot 30} + 20$

$\Rightarrow$ solve for k, get $k = \ln(50/78)/30 < 0$ negative!

So, $T(t) = 78e^{t \cdot \ln(50/78)/30} + 20$.

<u>NOTE:</u> Read §3.8 subsection on compound interest.