Wesley Hough

9AM

x^n vs. n^x

MA 113 Review Day

① Find $\frac{dy}{dx}$ for $y = \underbrace{e^{3x}}_{f(x)} \cdot \underbrace{2^{\cos(x^2+1)}}_{g(x)}$

$f(x) = e^{3x}$

$f'(x) = 3e^{3x}$

$g(x) = 2^{\cos(x^2+1)}$

$g'(x) = (\ln 2)\, 2^{\cos(x^2+1)} \left(-\sin(x^2+1)\right)2x$

$$\frac{dy}{dx} = f'(x)\,g(x) + f(x)\,g'(x)$$

$$= 3e^{3x}\, 2^{\cos(x^2+1)} -$$

$$(2\ln 2)\, e^{3x}\, x \sin(x^2+1)\, 2^{\cos(x^2+1)}$$

② Find the equation of the tangent line to $y^3+2xy+x^3=4$ at the point $(1,1)$.

$$y^3 + \underbrace{2xy} + x^3 = 4$$

$$3y^2\frac{dy}{dx} + 2y + 2x\frac{dy}{dx} + 3x^2 = 0$$

$$3y^2\frac{dy}{dx} + 2x\frac{dy}{dx} = -2y - 3x^2$$

$$(3y^2+2x)\frac{dy}{dx} = -2y - 3x^2$$

$$\frac{dy}{dx} = \frac{-2y-3x^2}{3y^2+2x}$$

At $(1,1)$,

$$\frac{dy}{dx} = \frac{-2-3}{3+2} = \frac{-5}{5} = -1$$

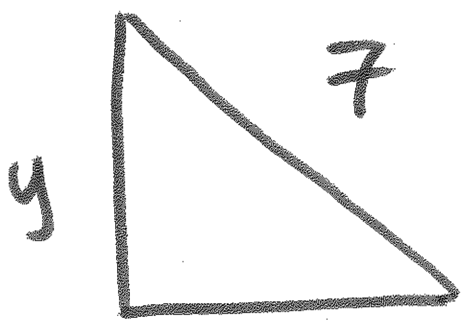
tangent line

$$y = f(a) + f'(a)(x-a)$$

$$\boxed{y = 1 - 1(x-1) = 2-x}$$

③ A 7-m ladder leans up against a wall. The bottom is 2 m away from the wall at $t=0$, and slides away from the wall at 0.6 m/s.

What is the velocity at the top of the ladder at $t=1$? $\frac{dy}{dt}=?$



$\rightarrow x^2+y^2=7^2$

$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$

$x=2\text{m}$ @ $t=0$

$x=2.6\text{m}$ @ $t=1$

$\frac{dx}{dt}=0.6\text{ m/s}$

At $t=1$,

$(2.6)^2+y^2=49$

$\rightarrow y=\sqrt{49-(2.6)^2}$

$\rightarrow 2(2.6)(0.6)+2\left(\sqrt{49-(2.6)^2}\right)\frac{dy}{dt}=0$

$$\boxed{\frac{dy}{dt}=\frac{-\cancel{2}(2.6)(0.6)}{\cancel{2}\sqrt{49-(2.6)^2}}\text{ m/s at } t=1}$$

(4) Evaluate $\displaystyle\lim_{x\to 0} \frac{\sin^3(12x)}{\pi x^3}$

Recall $\boxed{\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1.}$

$$\lim_{x\to 0} \frac{1}{\pi}\left(\frac{\sin(12x)}{12x}\right)^3 \cdot 12^3$$

$$= \frac{12^3}{\pi}\left(\lim_{x\to 0} \frac{\sin 12x}{12x}\right)^3$$

$$= \frac{12^3}{\pi} \cdot 1^3 = \boxed{\frac{12^3}{\pi}}$$

⑤ Show, using the limit definition of the derivative, that $\frac{d}{dx}x^3 = 3x^2$.

Recall $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$f(x) = x^3$

$f(x+h) = (x+h)^3 = (x+h)(x+h)(x+h)$

$= x^3 + 3x^2h + 3xh^2 + h^3$

$\frac{f(x+h) - f(x)}{h} = \frac{\cancel{x^3} + 3x^2\cancel{h} + 3xh^{\cancel{2}} + h^{\cancel{3}\,2} - \cancel{x^3}}{\cancel{h}}$

$= 3x^2 + 3xh + h^2$

$$\boxed{\frac{d}{dx}x^3 = \lim_{\cancel{h \to 0}} 3x^2 + \cancel{3xh} + \cancel{h^2} = 3x^2}$$

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$$y = \ln\left(\frac{\sqrt[3]{x+1}}{\sqrt{x^2+x+1}}\right), \text{ find } y'.$$

$$= \ln\left((x+1)^{1/3}\right) - \ln\left((x^2+x+1)^{1/2}\right)$$

$$= \frac{1}{3}\ln(x+1) - \frac{1}{2}\ln(x^2+x+1)$$

$$y' = \frac{1}{3}\frac{1}{x+1}\cdot 1 - \frac{1}{2}\frac{1}{x^2+x+1}(2x+1)$$

$$\boxed{y' = \frac{1}{3(x+1)} - \frac{2x+1}{2(x^2+x+1)}}$$

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$\boxed{1}$ Exam 2 tomorrow!
See Canvas Announcement.

$\boxed{2}$ Sign attendance

Ex: $\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{d}{dx}(\sin x)\cdot(\cos x)^{-1}$

$= \sec^2 x.$

product + chain rule

How to handle major anxiety w/ hard problem.

1. Ignore specific problem.

   ↳ Instead, focus on problem <u>type</u>.

   <u>Ex</u>: Use limit def$^{\underline{n}}$ to find derivative of $e^{\cos(x)} \cdot x^{\sin(x)}$.

   we wouldn't ask you this, too hard.

   flip page, write down strategies you know.

What are ways to handle limit def$^{\underline{n}}$ derivative calculations?

- Choose $h \to 0$ or $x \to a$ to compute.
- If h-form, try to cancel h.
- Square roots require mult. by conjugate.
- $\frac{1}{x+2}$, similar, need to find common den.
- trig requires $\cos(x+h)$ or similar, need angle addition trig identity.

2. To manage time, take 5 min @ beginning of exam to look over <u>all</u> problems + plan you work.

If you've made a Plan, spending time on hard problems is less stressful.

3. Spend 2 min @ beginning writing down formulas you are afraid of forgetting.

4: <u>Write</u>: "I can do well on this exam, go me!" on front of exam.

$\boxed{1}$ Use limit def$^{\underline{n}}$ of derivative to find $f'(x)$ where $f(x) = \dfrac{2}{x^2}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{2}{(x+h)^2} - \frac{2}{x^2}}{h}$$

cancel

$$= \lim_{h \to 0} \frac{\frac{2x^2 - 2(x+h)^2}{x^2(x+h)^2}}{h} = \lim_{h \to 0} \frac{2x^2 - 2(x^2 + 2xh + h^2)}{h \cdot x^2 (x+h)^2}$$

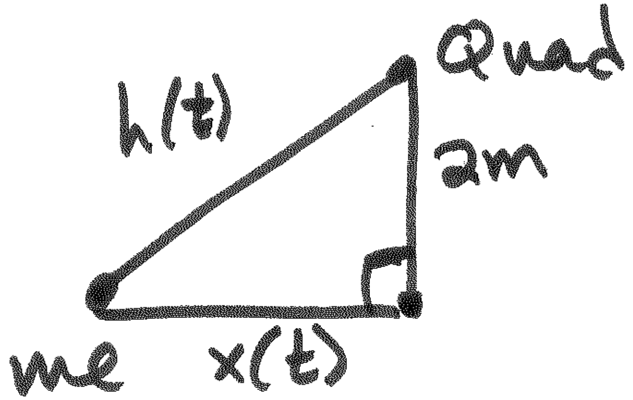
$$= \lim_{h \to 0} \frac{-4xh - 2h^2}{h \cdot x^2 (x+h)^2} = \lim_{h \to 0} \frac{-4x - 2h}{x^2 \cdot (x+h)^2}$$

cancel h's

$$= \frac{-4x}{x^2 \cdot x^2} = \frac{-4}{x^3}.$$

Check: $\frac{d}{dx}\frac{2}{x^2} = \frac{d}{dx} 2\cdot x^{-2} = -2\cdot 2x^{-3}$
$= -4/x^3.$ ✓

Ex: A quad copter is flying directly away from me horizontal to the ground at a height of 2 m, w/ velocity of 200 m/min. When the quad is 4 m from me measured on ground, how fast is the actual distance from me to quadcopter changing?



Givens: $\frac{dx}{dt} = 200$

want $\frac{dh}{dt}$ when $x = 4$.

Know: $x^2 + 2^2 = h^2 \overset{\text{Apply } \frac{d}{dt}}{\Longrightarrow} 2x \cdot \frac{dx}{dt} + 0 = 2h \cdot \frac{dh}{dt}$.

$$\Rightarrow \frac{dh}{dt} = \frac{x}{h} \cdot \frac{dx}{dt}.$$

Determine h when $x = 4$, get $h = \sqrt{4^2 + 2^2}$.

So, $\frac{dh}{dt} = \frac{4}{\sqrt{20}} \cdot 200$ m/min.

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§3.8 Exponential Growth:

"Differential Equation" $\frac{dy}{dt} = k \cdot y$ for some k.

Calc II $\Rightarrow$ only functions y satisfying this are $y = y(0) \cdot e^{kt}$.

Check: $y' = y(0) \cdot e^{kt} \cdot k = k \cdot y(0) \cdot e^{kt}$

$= k \cdot y$.

$\frac{dy}{dt} = k \cdot y$ means rate of change in y is proportional to y.

Ex: Population Growth.

Often, in experimental data, $\frac{dP}{dt}$ for a pop. fn $P(t)$ is proportional to population.

we say $\frac{dP}{dt} = kP$ and k is the relative growth rate.

NOTE: If $k>0$, we usually ~~call~~ say P models growth.

If $k<0$, we say P models decay.

(This is for all exponential models).

Q: What is relative growth rate if $P(0) = 2{,}000$ & $P(5) = 10{,}000$?

If $\frac{dP}{dt} = kP$, then (by our earlier claim) we have

same

$P(t) = P(0) \cdot e^{kt}$ for [redacted] same k.

So, $P(0) = 2{,}000$, $P(5) = 10{,}000$ implies

$P(t) = 2{,}000 \cdot e^{kt}$ and $\underbrace{P(5) = 10{,}000 = 2000 \cdot e^{k \cdot 5}}_{\text{solve for } k \text{, obtain}}$.

$$k = \frac{\ln(5)}{5}.$$

Ex: Radioactive Decay: Experiments show that the mass $m(t)$ of a radioactive substance satisfies

$\frac{dm}{dt} = k \cdot m$, for different k's w/ each substance.

Half-life of substance is the time t where

$$m(t) = \frac{1}{2} m(0).$$

Q: Half-life of radium-226 is 1590 years. If $m(0) = 1$ kg, how much is left after 100 years?

Since $\frac{dm}{dt} = km$, we have $m(t) = m(0) \cdot e^{kt}$.

Given half-life, first compute k, then solve problem.

Step 1: Find k. Use half-life is 1590.

$m(1590) = \frac{1}{2}$ kg, since $m(0) = 1$ kg.

So, $m(t) = 1 \cdot e^{kt} \Rightarrow \frac{1}{2} = e^{k \cdot 1590}$ at $t = 1590$.

Solve for k, get $k = \frac{-\ln(2)}{1590} < 0$ so decay.

Step 2: Find $m(100) = e^{\frac{-\ln(2)}{1590} \cdot 100}$ kg.

Ex: Newton's law of cooling

Object w/ temp $T(t)$ at time t, set out + leave it alone, w/ temp of surroundings T_s, then

$$\frac{dT}{dt} = k(T - T_s) \quad \text{for some } k.$$

NOTE: If $y = T(t) - T_s$, then $\frac{dy}{dt} = \frac{dT}{dt}$.

So, $\frac{dy}{dt} = k \cdot y \implies y = y(0) \cdot e^{kt}$.

Back to T: $T(t) - T_s = (T(0) - T_s)e^{kt}$

So, $T(t) = (T(0) - T_s)e^{kt} + T_s$ $\circledast$ Key tool.

Q: If temp of surroundings is 20°C, coffee is 98°C, w/ temp of coffee 30 min later is 70°C. What is $T(t)$?

$T_s = 20$, $T(0) = 98$.

Formula $\Rightarrow$ $T(t) = 78 \cdot e^{kt} + 20$.

Find k $\Rightarrow$ use $T(30) = 70$ $\Rightarrow$ $70 = 78e^{k\cdot 30} + 20$

$$\Rightarrow k = \frac{\ln(50/78)}{30} < 0.$$

So, $T(t) = 78 e^{\frac{t \ln(59/78)}{30}} + 20.$

<u>NOTE</u>: Read §3.8 about compound interest.