10/26/16 12PM

$\boxed{1}$ Webwork, Quiz, + WA#5 this week!

$\boxed{2}$ With your neighbors:

a) Find critical points of $f(x)=3x^5-5x^3+3$

b) Find the critical points of $f'(x)$.

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§4.3: Derivatives + Shapes of graphs

Theme: §4.1 → abs max/min techniques

§4.2 → MVT → gave us relationship btwn secants + tangents.

today: → How to calculate local max/mins + other qualities of graph of f.

First derivatives + graphs

Increasing/Decreasing test:

- If $f'(x) > 0$ on (a,b), then $f(x)$ is increasing on (a,b).
- If $f'(x) < 0$ on (a,b), then $f(x)$ is decreasing on (a,b).

First derivative test: Suppose c is a critical number of f, and f is cts.

(a) If f' changes from pos to neg at c, then f has a local max at c.

(b) If f' changes from neg to pos at c, then f has a local min at c.

(c) If f' does not change sign at c, then f has neither local max nor local min at c.

Second Derivatives!

Def$^{\underline{n}}$: If near every x in (a,b) the graph of f lies above the tangent line to $(x, f(x))$, then f is called <u>concave up</u> on (a,b). If graph of f is below tangent line for every x, then f is <u>concave down</u> on (a,b).

Concavity Test: (a) If $f''(x) > 0$, for all x in (a,b), then f is concave up on (a,b).

(b) If $f''(x) < 0$ for all x in (a,b), then f is concave down on (a,b).

Def$^{\underline{n}}$: A pt on graph of $f(x)$ is an <u>inflection point</u> if f changes concavity at that point and f is cts.

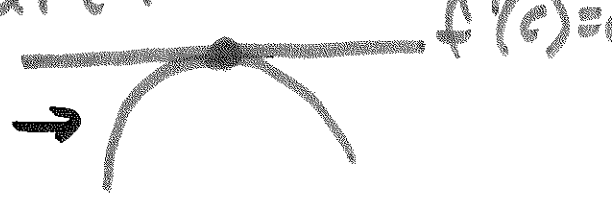
2^{nd} Der. Test: Suppose f'' is cts near c.

(a) If $f'(c)=0$ and $f''(c)>0$, then f has a local min at c.

$f''(c)>0$ means concave up, so f is above horiz. tang. line.

(b) If $f'(c)=0$ and $f''(c)<0$, then f has a local max at c.

similar picture:



$f''(c)<0$ →
$\Rightarrow$ concave down
$\Rightarrow$ graph below tangent.

Ex: For $h(x) = 3x^5 - 5x^3 + 3$, find

- local max/min pts
- intervals of inc/dec.
- intervals of concavity.

Let's try 2^{nd} der. test.

$$h'(x) = 15x^4 - 15x^2 = 15x^2(x^2-1) = 15x^2(x-1)(x+1).$$

$$h''(x) = 60x^3 - 30x = 30x(2x^2-1).$$

Critical numbers are $x = 0, 1, -1$, i.e. where $h'(x) = 0$.

Evaluate $h''(0) = 0$ ⟵ Inconclusive… Need to use 1^{st} der. test.

$h''(1) = 30 > 0$

$h''(-1) = -30 < 0$

By 2^{nd} der. test, $x = 1$ gives a local min,

$x = -1$ gives a local max.

what about $x = 0$? Find signs of $h'(x)$ on $(-\infty, \infty)$.

sign of $h'(x)$: $+$ | $-$ | $-$ | $+$

test pts for $h'(x)$: $h'(-2) = 180$, -1, $h'(-\frac{1}{2}) = -2.8$, 0, $h'(\frac{1}{2}) = -2.8$, 1, $h'(2) = 180$

$h'(x)$ does not change sign at 0,
So no local max/min at $x = 0$.

Intervals of increase: $(-\infty, -1) \cup (1, \infty)$.

" " decrease: $(-1, 0) \cup (0, 1)$

NOTE: A similar analysis ~~show~~ on $h''(x)$ shows
h is concave up when x in $(\frac{-1}{\sqrt{2}}, 0) \cup (\frac{1}{\sqrt{2}}, \infty)$
" " " down " " " $(-\infty, \frac{-1}{\sqrt{2}}) \cup (0, \frac{1}{\sqrt{2}})$.

10/28/16 12PM

$\boxed{1}$ Turn in WA #5

$\boxed{2}$ w/ neighbors, compute

$$\lim_{x \to \infty} \frac{x^2+2}{x^3+x+1}$$ using our technique from before.

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§4.4: L'Hospital's Rule

→ Proof in textbook

Ex: $\lim_{x \to \infty} \frac{x^2}{e^x}$ $x^2 \to \infty$, $e^x \to \infty$

Hmm.... What to do?

If this were....

$$\lim_{x \to \infty} \frac{x^2+2}{x^3+x+1} = \lim_{x \to \infty} \frac{\frac{1}{x^3}(x^2+2)}{\frac{1}{x^3}(x^3+x+1)} =$$

$$\lim_{x \to \infty} \frac{\frac{1}{x} + \frac{2}{x^3}}{1 + \frac{1}{x^2} + \frac{1}{x^3}} = \frac{0}{1} = 0.$$

w/ e^x, can't use this technique... So.

L'Hospital's Rule: Suppose f + g are diff' and $g'(x) = 0$ near a (a might be $\pm\infty$).

If either (A) $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$

or (B) $\lim_{x\to a} f(x) = \pm\infty$ and

$\lim_{x\to a} g(x) = \pm\infty$

then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$, if right-hand limit exists.

$\left.\begin{array}{l}\text{Condition (A)} \Rightarrow \text{type } \frac{0}{0} \text{ in limit.} \\ \text{Condition (B)} \Rightarrow \text{type } \frac{\pm\infty}{\pm\infty} \text{ in limit.}\end{array}\right\}$ L'H rule bypasses this issue.

Ex: $\lim_{x\to\infty} \dfrac{x^2+2}{x^3+x+1} \underset{\substack{\text{type } \frac{\infty}{\infty},\\ L'H}}{=} \lim_{x\to\infty} \dfrac{2x}{3x^2+1} \underset{\substack{\text{type } \frac{\infty}{\infty}\\ L'H}}{=} \lim_{x\to\infty} \dfrac{2}{6x} = 0.$

Ex: $\lim_{x\to\infty} \dfrac{x^2}{e^x} \underset{\substack{\text{type } \frac{\infty}{\infty},\\ L'H}}{=} \lim_{x\to\infty} \dfrac{2x}{e^x} \underset{\substack{\text{type } \frac{\infty}{\infty}\\ L'H}}{=} \lim_{x\to\infty} \dfrac{2}{e^x} = 0.$

Ex: $\lim_{x\to 2} \dfrac{4-x^2}{\sin(\pi x)} \underset{\substack{\text{type } \frac{0}{0}\\ L'H}}{=} \lim_{x\to 2} \dfrac{-2x}{\pi\cdot\cos(\pi x)} = \dfrac{-4}{\pi\cdot\cos(2\pi)}$

$= \dfrac{-4}{\pi}.$

Ex: $\lim_{x\to 1} \dfrac{x^2-9}{x+3} = \dfrac{1^2-9}{1+3} = \dfrac{-8}{4} = -2.$ by continuity.

NO L'H here!

__Ex:__ $\lim_{x\to 1} \frac{x-1}{\ln(x)} = \lim_{x\to 1} \frac{1}{1/x} = 1.$

type $\frac{0}{0}$
L'H

__Ex:__ $\lim_{x\to 2} \frac{x^3-8}{x^4+2x-20} = \lim_{x\to 2} \frac{3x^2}{4x^3+2} = \frac{6}{17}$

type $\frac{0}{0}$,
L'H

continuity

__L'H Rule w/ products:__ Say

$\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \pm\infty$.

What might $\lim_{x\to a} f(x)g(x)$ be?

__Fact:__ Might be any real #, or $\pm\infty$.

Rewrite $f \cdot g$ as either $\dfrac{f}{1/g}$ or $\dfrac{g}{1/f}$,

and you can apply L'H rule.

Ex: $\lim\limits_{x \to 0^+} \underbrace{x}_{0} \cdot \underbrace{\ln(x)}_{-\infty} = \lim\limits_{x \to 0^+} \dfrac{\ln(x)}{1/x} \underset{\substack{\text{type } \frac{-\infty}{\infty} \\ \text{L'H}}}{=} \lim\limits_{x \to 0^+} \dfrac{\frac{1}{x}}{-1/x^2}$

$= \lim\limits_{x \to 0^+} -x = 0.$

"x goes to zero faster than ln(x) goes to $-\infty$ as $x \to 0^+$."

Ex: Find $\lim_{x\to\infty} \frac{x^7}{e^x} \underset{\frac{\infty}{\infty}\text{ L'H}}{=} \lim_{x\to\infty} \frac{7x^6}{e^x} \underset{\frac{\infty}{\infty}\text{ L'H}}{=}$

$$\lim_{x\to\infty} \frac{7\cdot 6\cdot x^5}{e^x} = \underset{\text{L'H's applications}}{- - - -} = \lim_{x\to\infty} \frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{e^x}$$

$= 0$ since $e^x \to \infty$, numerator constant.

Two outcomes:

① Notation $n\cdot(n-1)(n-2)\cdot\ldots\cdot 3\cdot 2\cdot 1 = n!$ n factorial.

② $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$ by a similar argument.

$\Rightarrow$ exponentials grow faster than polynomials.