

\$ 10/3/16 9 AM

1] See Canvas for assignments
this week

2] Today: Chain Rule!

so far: inst. vel \Rightarrow derivatives of fns.

we know $\frac{d}{dx}$ for:

polys, exp, trig,

+, -, \times , \div of fns.

This week: compositions fns, inverses,
 \Rightarrow inv. trig + logs.

Chain Rule: If $f + g$ are diff,
then the derivative of $f \circ g$ is the
product of $f'(g(x))$ and $g'(x)$.

NOTE: This is different than $\frac{f'(x) \cdot g'(x)}{\text{not this!}}$

Form 1: $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$.

Form 2: $y = f(u)$, ~~u~~ $u = g(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{df}{du} \cdot \frac{dg}{dx}$$

Ex: $f(x) = x^3$, $g(x) = x^2$.

$(f \circ g)(x) = (x^2)^3 = x^6$.

f = "outside"
 $f' = x^2$

g = "inside"
 $g' = x^2$

$(f \circ g)'(x) = f'(g(x)) \cdot g'(x) = 3(x^2)^2 \cdot 2x = 3 \cdot 2 \cdot x^4 \cdot x = 6x^5$

Know:

$(f \circ g)' = 6x^5$

by power rule.

$y = f(u) \Rightarrow y = u^3$ $u = g(x) \Rightarrow u = x^2$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3u^2 \cdot 2x = 3(x^2)^2 \cdot 2x = 6x^5$

Ex: $F(x) = \sqrt[3]{x^4 + x}$

Ex: $F = f \circ g$
 $f = \sqrt[3]{x} = x^{1/3}$
 $g = x^4 + x$

$$F'(x) = f'(g(x)) \cdot g'(x) = \frac{1}{3}(g(x))^{-2/3} \cdot (4x^3+1) \\ = \frac{1}{3}(x^4+x)^{-2/3} \cdot (4x^3+1).$$

Ex: $\frac{d}{dx} (f \circ g \circ h) =$

$\underbrace{\hspace{10em}}_{\text{outside}} \xleftarrow{\text{inside}}$

$$= f'(g \circ h)(x) \cdot (g \circ h)'(x) =$$

$$f'(g \circ h)(x) \cdot g'(h(x)) \cdot h'(x)$$

Leibniz
notation:

$$f(u) = y,$$

$$v = g(u),$$

$$u = h(x) \Rightarrow$$

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx}$$

Ex: Find $\frac{d}{dx} (e^{\cos(x^2+1)})$.

$$y = e^v \cdot v = \cos(u) \cdot u = x^2 + 1.$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} =$$

$$= e^v \cdot (-\sin(u)) \cdot (2x)$$

$$= e^{\cos(x^2+1)} \cdot (-\sin(x^2+1)) \cdot (2x).$$

Ex: Quot. rule as prod. + chain.

$$\frac{d}{dx} \left(\frac{f}{g} \right) = \frac{d}{dx} (f \cdot g^{-1}) = \left(\frac{f}{g} \right)' \cdot \frac{d}{dx} (g^{-1}) + f \cdot \frac{d}{dx} (g^{-1}) =$$

inside = g^{-1}
outside = x^{-1}

$$\frac{f'g}{g^2} + \frac{-1}{g^2} \cdot g \cdot f =$$

$$\frac{f'g}{g^2} - \frac{f \cdot g}{g^2} = \frac{f'g - fg'}{g^2}$$

Ex: Say $b > 0$. Then $\frac{d}{dx} (b^x) = \ln(b) \cdot b^x$.

Ex: $b=1$, $\Rightarrow \frac{d}{dx} (1^x) = 0 = \ln(1) \cdot 1^x$.

Ex: $b=e$, $\Rightarrow \frac{d}{dx} e^x = \ln(e) \cdot e^x = e^x$.

Why? $b^x = e^{\ln(b) \cdot x} = e^{\ln(b) \cdot x}$.

inside = $\ln(b) \cdot x$, outside = e^{\quad} .

inside derivative $\rightarrow (b^x)^{\ln(b)} \cdot \ln(b) \cdot b^x = \ln(b) \cdot b^x$.

$\Rightarrow (b^x)^{\ln(b)} = b^x \cdot \ln(b) = \ln(b) \cdot b^x$.

10/5/16 9AM

- ① f, f', f'' as
position, velocity, acceleration.
we will discuss more on
Monday.
- discuss
w/you
neighbors.
Make sure
you know this.
- ② Sign attendance sheet.

§3.5: Implicit diffⁿ

and derivative of
inverse fns.

Ex: For $x^2 + y^2 = r^2$, want slope of
tangent line at a point (a, b) . for $\frac{dy}{dx}$
We need a formula for slope, i.e. for $\frac{dy}{dx}$
involving both x - + y - coords.

Idea: x + y are related through an
eqn, so apply $\frac{d}{dx}$ of both sides
of the eqn.

eg. $r=2$. $x^2+y^2=2^2=4$.

$$\frac{d}{dx}(4)=0, \text{ and } \frac{d}{dx}(x^2+y^2)=$$

$$2x + 2y \cdot \frac{dy}{dx}$$

chain rule.

Set these equal, since they are derivatives of same quantity:

$$0 = 2x + 2y \cdot \frac{dy}{dx} \Rightarrow -\frac{x}{y} = \frac{dy}{dx}$$

~~⊗~~ slope is $\frac{y}{x}$. perp. line has slope $-\frac{x}{y}$

Ex: Find $\frac{dy}{dx}$ for $x^4 + y^4 = 16$.

$$\frac{d}{dx}(16) = 0$$

$$\frac{d}{dx}(x^4 + y^4) = 4x^3 + 4y^3 \cdot \frac{dy}{dx}$$

chain rule w/ u^4 outside
y/d inside

set equal:

$$4x^3 + 4y^3 \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x^3}{y^3}$$

Ex: Find y' if $\cos(x+y) = x \cdot \sin(y)$.

$$\frac{d}{dx} (\cos(x+y)) = (-\sin(x+y)) \frac{dx}{dx} \quad \begin{matrix} \text{ins.} \\ \uparrow \\ \text{out.} \end{matrix} = \left(\frac{dx}{dx} + 1 \right) \cdot ((k+x) \sin(y))$$

$$\frac{d}{dx} (x \cdot \sin(y)) = \frac{d}{dx} (x) \cdot \sin(y) + x \cdot \frac{d}{dx} (\sin(y)) \quad \begin{matrix} \text{Prod.} \\ \uparrow \\ \text{chain rule} \end{matrix} = \sin(y) + x \cdot \cos(y) \cdot \frac{dy}{dx}$$

set equal:

$$[-\sin(x+y)] \cdot \left(1 + \frac{dy}{dx} \right) = \sin(y) + x \cdot \cos(y) \cdot \frac{dy}{dx}$$

Solve for $\frac{dy}{dx}$ + get

$$\frac{dy}{dx} = - \frac{\sin(y) + \sin(x+y)}{\sin(x+y) + x \cos(y)}$$

Application: Inverse fns

Thm (Problem 77(a) in §3.5): Suppose f is diff
and 1-1, and f^{-1} is also diff.

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Remember:

$$f(x) = x$$

$$\underbrace{f^{-1}(y) = x}_{\text{use implicit diff here}}$$

$$\text{And } f'(x) = y' = \frac{dy}{dx}$$

$$\frac{x^p}{p} = (x)^{\frac{x^p}{p}} = (x)^{1-p} \cdot x$$

$$\frac{x^p}{p} = \frac{x^p}{1-p} \cdot (1-p) \cdot x$$

$$\frac{x^p}{p} = \frac{x^p}{1-p} = (x)^{\frac{x^p}{1-p}}$$

$$\frac{(x)^{\frac{x^p}{1-p}}}{1-p} = (x)^{\frac{x^p}{1-p}}$$

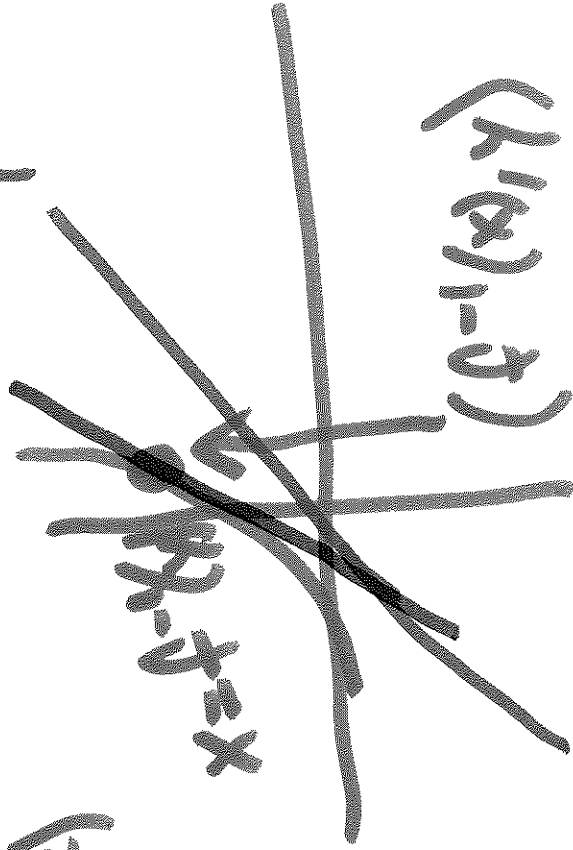
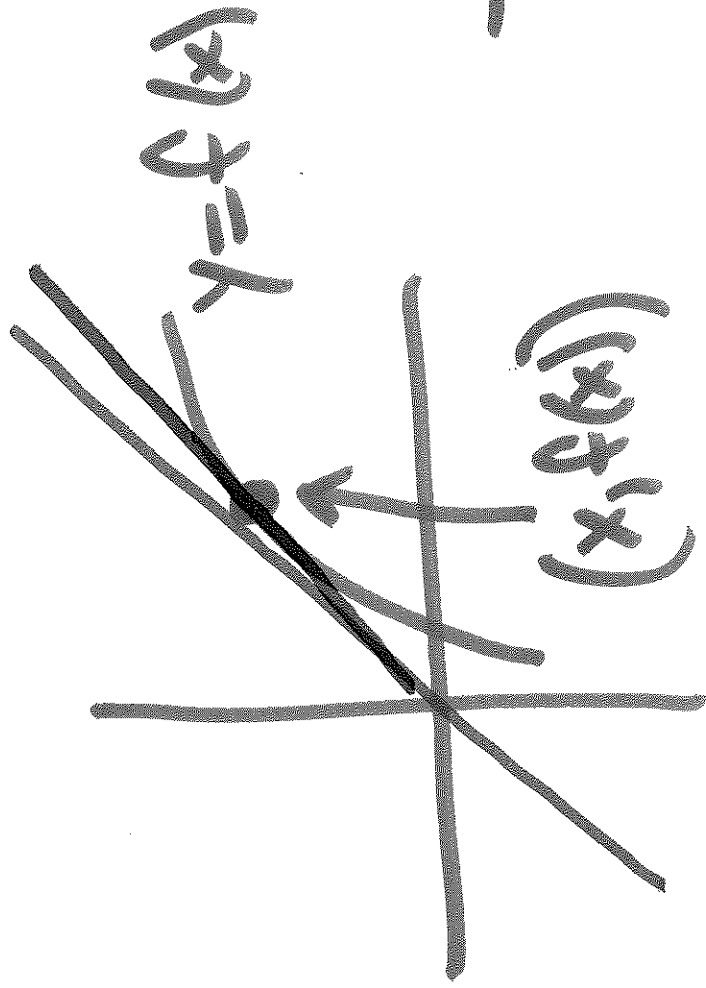
~~Arrows~~

! forget

$$f^{-1}(k, 1-t) = (k, t) = \frac{(x, t)}{t} \quad w/ \quad x = f^{-1}(k)$$

$$\Rightarrow f^{-1}(k, 1-t) = (k, t) = \frac{1}{f'(f^{-1}(k))} (k, 1-t) \quad \leftarrow \text{formula I wanted, but w/ y's instead of x's.}$$

new swap x for y.

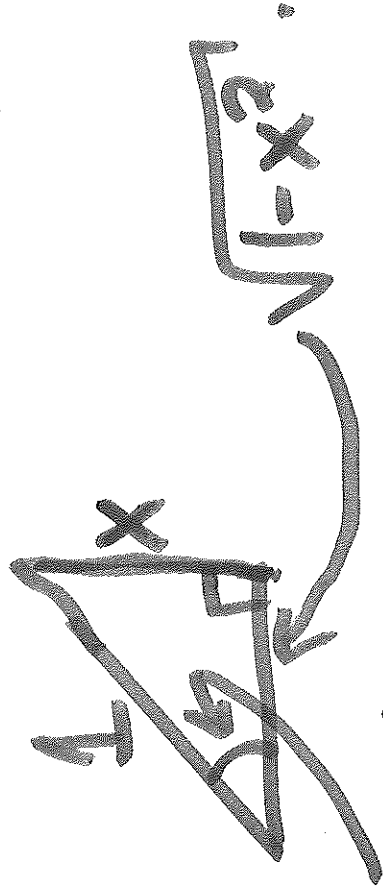


Derivatives of inverse trig fns:

See § 3.5 for table \rightarrow know them!

Ex: $f(x) = \sin(x)$ Using our formula,

$$f'(x) = \cos(x). \quad \frac{d}{dx} \arcsin(x) = \frac{1}{\cos(\arcsin(x))}$$



$\arcsin(x)$

$$\Rightarrow \frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\text{Ex: } \frac{d}{dx} \arctan(x) = \frac{1}{\sec^2(\arctan(x))}$$

$$= \frac{1}{1 + \tan^2(\arctan(x))}$$

trig id.

$$= \frac{1}{1+x^2}$$

10/7/16 9AM

1 Turn in written assign

2 What is formula for

$$(f^{-1})'(x)?$$

(Review your notes!)

$$\text{Thm: } (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Why? : Chain Rule

$$\frac{d}{dx} (f \circ f^{-1})(x) = \frac{d}{dx} (x)$$

$$\Rightarrow \underbrace{f'(f^{-1}(x)) \cdot (f^{-1})'(x)}_{\frac{d}{dx}(\text{outside})} = 1$$

$$\Rightarrow (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Recall: $\log_b(x)$ is inverse for b^x
if $b > 0, b \neq 1$.

$$f = b^x$$

$$f^{-1} = \log_b(x)$$

$$\Rightarrow \text{Apply formula: } \frac{d}{dx} \log_b(x) = \frac{1}{\ln(b) \cdot b^{\log_b(x)}}$$

$$\boxed{\text{Recall: } \frac{d}{dx} b^x = \ln(b) \cdot b^x} = \frac{1}{\ln(b) \cdot x}$$

Thm: ~~$\frac{d}{dx} (\log_b(x))$~~ $= \frac{1}{\ln(b) \cdot x}$

Recall: $\frac{d}{dx} e^x = e^x$

Q: What happens when $b = e$?

$$= \frac{1}{1 \cdot x} = \frac{1}{x}$$

$\star \rightarrow \frac{d}{dx} \ln(x) = \frac{1}{\ln(e) \cdot x}$

Neat consequence: we can actually

say what e is!

$$\text{start w/ } f(x) = \ln(x) \Rightarrow f'(x) = \frac{1}{x} \quad \text{at } x=0$$

$$\text{So, } f'(1) = 1 = \lim_{h \rightarrow 0} \frac{\ln(1+h) - \ln(1)}{h} =$$

$$\lim_{h \rightarrow 0} \frac{\ln(1+h)}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \cdot \ln(1+h) = \lim_{h \rightarrow 0} \ln((1+h)^{1/h})$$

$$\Rightarrow 1 = \lim_{h \rightarrow 0} \ln((1+h)^{1/h}) \quad \leftarrow \text{exponentiate both sides}$$

$$\Rightarrow e^1 = e^{\lim_{h \rightarrow 0} \ln((1+h)^{1/h})} = \lim_{h \rightarrow 0} e^{\ln((1+h)^{1/h})} = \lim_{h \rightarrow 0} (1+h)^{1/h}$$

Ex: $h = \frac{1}{2}$, $e \approx \left(1 + \frac{1}{2}\right)^2 = \frac{9}{4} = 2.25$

$h = \frac{1}{3}$, $e \approx \left(1 + \frac{1}{3}\right)^3 = \left(\frac{4}{3}\right)^3 = \frac{64}{27} \approx 2.37\dots$

$h = \frac{1}{7}$, $e \approx \left(1 + \frac{1}{7}\right)^7 = \left(\frac{8}{7}\right)^7$ ~~$\approx 2.37\dots$~~ $\approx 2.54\dots$

$h = \frac{1}{20}$, $e \approx \left(1 + \frac{1}{20}\right)^{20} = \left(\frac{21}{20}\right)^{20} \approx 2.65\dots$

$h = \frac{1}{101}$, $e \approx \left(1 + \frac{1}{101}\right)^{101} = \left(\frac{102}{101}\right)^{101} \approx 2.70\dots$

Note: $e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$

$$\frac{d}{dx} \ln(x) = \frac{1}{x}$$

Ex: Find $\frac{d}{dx} (\ln(2x))$.

2 ideas: (i) chain rule

$$(ii) \ln(2x) = \ln(2) + \ln(x)$$

$$(i) \frac{d}{dx} \ln(2x) = \frac{1}{2x} \cdot 2 = \frac{1}{x}$$

outside ins. der.

$$(ii) \frac{d}{dx} (\ln(2x)) = \frac{d}{dx} (\ln(2) + \ln(x)) = 0 + \frac{1}{x} = \frac{1}{x}$$

NOTE: Use log laws to simplify expressions before applying $\frac{d}{dx}$ \therefore

Ex: ~~$\frac{d}{dx} (5x^5)$~~

$$\frac{d}{dx} (\ln(x^5)) =$$

$$= \frac{d}{dx} (5 \cdot \ln(x)) = 5 \cdot \frac{1}{x} = \frac{5}{x}$$

Ex: $\frac{d}{dx} (\ln(x^n)) = \frac{d}{dx} (n \cdot \ln(x)) = \frac{n}{x}$

Ex: $\frac{d}{dx} (\ln(x^3 + x + 1)) = \frac{1}{x^3 + x + 1} \cdot (3x^2 + 1)$
chain rule

See §3.6, Example 6 for $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$.

$$\underline{\text{Ex: } \frac{d}{dx} \left[(\ln x)^{2/3} \right]} = \frac{2}{3} (\ln x)^{-1/3} \cdot \frac{1}{x}$$

- use
- power rule
 - chain rule
 - $\frac{d}{dx} \ln x = \frac{1}{x}$

$$= \frac{2}{3x^3 \sqrt[3]{\ln x}}$$

outside $u^{2/3} \rightarrow \frac{d}{du}(u^{2/3}) = \frac{2}{3} u^{\frac{2}{3}-1}$

$$\underline{\text{Ex: } \frac{d}{dx} \left[\ln \left(\frac{x^2+1}{\sqrt{x-1}} \right) \right]} = \frac{d}{dx} \left(\ln(x^2+1) - \ln(\sqrt{x-1}) \right)$$

$$\begin{aligned} &= \frac{d}{dx} \left(\ln(x^2+1) - \frac{1}{2} \ln(x-1) \right) = \frac{1}{x^2+1} \cdot 2x - \frac{1}{2} \cdot \frac{1}{x-1} \cdot 1 \\ &= \frac{2x}{x^2+1} - \frac{1}{2(x-1)} \end{aligned}$$

10/10/16 9AM

[1] See Canvas announcement for work due this week.

[2] For Exam 2, you are required to be able to state + use the limit defⁿ of a derivative, item [4] in § 2.7.

[3] Sign attendance!

§ 3.7: Rates of Change

Notation: If $f(x) = y$ is a function relating x & y , two quantities,

then if $f(x_1) = y_1$, $f(x_2) = y_2$,

we write $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$ for average change of y over $[x_1, x_2]$.

Write $\Delta y = y_2 - y_1 = \frac{f(x_2) - f(x_1)}{x_2 - x_1} +$

$$\Delta x = \frac{x_2 - x_1}{x_2 - x_1}.$$

NOTE:

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

write $\frac{\Delta y}{\Delta x}$ for average change.

we call $\frac{dy}{dx}$ the inst. rate of change, R.O.C.

Q: What is R.O.C. of area of circle with respect to radius?

→ This means write Area as a function of radius + find $\frac{dA}{dr}$

$$A(r) = \pi r^2 \quad r \text{ in meters} \quad A = m^2$$



$$\frac{dA}{dr} = 2\pi r \quad m^2/m = 2\pi r \text{ meters.}$$

↖ Circumference.

$$\text{What is } \frac{dr}{dA} ? \quad \sqrt{\frac{A}{\pi}} = r.$$

$$\Rightarrow \frac{dr}{dA} = \frac{1}{\sqrt{\pi}} \cdot \frac{1}{2\sqrt{A}}$$

Ex: (Biology):

If $f(t)$ = population of bact. at time t sec,

• average growth rate is $\frac{\Delta f}{\Delta t}$ bact./sec

• inst. growth rate is $\frac{df}{dt}$ bact./sec.

e.g. $f(t) = 203 \cdot 2^t$

After 5 sec, how quickly is pop. growing?

Approximate answer:

$$f'(5) = 203 \cdot \ln(2) \cdot 2^5 \approx 4502.68 \frac{\text{bact.}}{\text{sec.}}$$

by ~~by~~ $\frac{d}{dt} 2^t = \ln(2) \cdot 2^t$.

Technique:

$$2^t = e^{\ln(2^t)} = e^{\ln(2) \cdot t}$$

use chain rule.

Ex: (Economics): $C(x)$ = cost of producing x items.

cost fn.

$C(x_2) - C(x_1)$ = cost increase/decrease to change production from x_1 to x_2 items.

$\frac{\Delta C}{\Delta x}$ = average rate of change of cost per item

$$\begin{aligned} \text{If } x_2 = x_1 + 1, \text{ then } \frac{\Delta C}{\Delta x} &= \frac{C(x_1+1) - C(x_1)}{x_1+1 - x_1} \\ &= C(x_1+1) - C(x_1). \end{aligned}$$

Replace $\frac{\Delta C}{\Delta x}$ w/ $\frac{dC}{dx}$, called "marginal cost."

idea $C(x+1) - C(x) \approx \frac{dC}{dx}$.

eg. Say $C(x) = 2000x^2 - 12x + 100$.

Find marginal cost.

ie. Find $\frac{dC}{dx} = 6000x^2 - 12$.

To estimate extra cost of going from 5 items to 6 items, evaluate $C'(5)$.

NOTE: Read about Ohm's Law in §3.7.

$$V = I \cdot R$$

voltage current · ~~resistance~~ resistance

$$I = \frac{V}{R} \Rightarrow \frac{dI}{dR} = V \cdot \frac{d}{dR} \left(\frac{1}{R} \right)$$

$$= V \cdot \frac{-1}{R^2} = -\frac{V}{R^2}$$

e.g.

Ex: (Physics):

$f(t)$ = position in m at t sec

$f'(t)$ = velocity in m/s at t sec

$f''(t)$ = acceleration in m/s²

$f'''(t)$ = jerk

e.g. $f(t) = \frac{t^3}{3} - \frac{5}{2}t^2 + 4t$ m

$\Rightarrow f'(t) = t^2 - 5t + 4$ m/s.

when is object moving in positive direction?

Find when $f'(t) > 0$.

solve $f'(t) = 0$ + get $t = 1, 4$.

$f'(t)$	+	-	+
sign	+	-	+
	0	1	4

since $f'(t) > 0$ when

t is in $(0, 1) \cup (4, \infty)$,

that is when object moving positively.

NOTE: $f'(t) = (t-1)(t-4)$

10/12/16: 9 AM

[1] Sign attendance

[2] Reminder: Exam 2 next week!

Today: §3.9, Related Rates

Idea: Two functions, $f(x) + g(x)$, that are related by an equation.

Apply $\frac{d}{dx}$ to both sides of the equation to get a relationship involving $f(x), g(x), f'(x), + g'(x)$.

~~Use~~ use this to solve problems.

Ex: Ladder Problem's

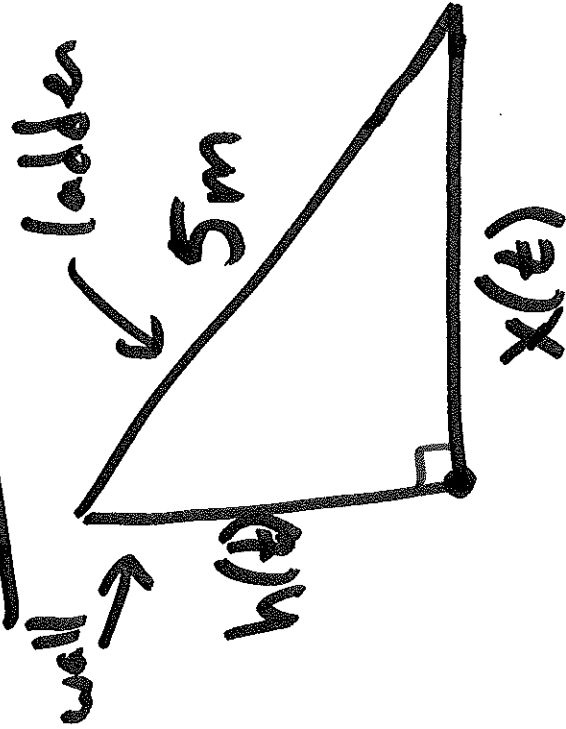
A 5-meter ladder against a wall, bottom of ladder is 1.5 m from wall at $t=0$ sec. slides away from wall at 0.8 m/s.
Q: What is velocity of top of ladder at $t=1$ sec?

Step 1: Draw + label a sketch/picture, + reframe the problem.

we know: $\frac{dx}{dt} = 0.8$.

$$x(0) = 1.5.$$

Problem: Find $\frac{dh}{dt}$ at $t=1$.



Step 2: Set up a mathematical relationship between quantities of interest.

Use Pyth. Thm $\Rightarrow x(t)^2 + h(t)^2 = 5^2$

Apply $\frac{d}{dt}$ to both sides, we get (by chain rule + implicit diff)

$$2 \cdot x(t) \cdot \frac{dx}{dt} + 2 \cdot h(t) \cdot \frac{dh}{dt} = 0$$

Solve for $\frac{dh}{dt}$ + get

$$\frac{dh}{dt} = -\frac{x(t)}{h(t)} \cdot \frac{dx}{dt}$$

Step 3: Solve the problem by evaluating.

Here, we need: ~~x~~ $x(1)$, $h(1)$, and $\frac{dx}{dt}$ at $t=1$.

This will tell us $\frac{dh}{dt}$ at $t=1$.

Here, $x(0) = 1.5$, + $\frac{dx}{dt} = 0.8 \text{ m/s}$ $\cdot \underbrace{\left(\frac{atall}{t}\right)}_{\text{constant}}$

So, $x(1) = x(0) + 0.8$
 $= 1.5 + 0.8 = 2.3 \text{ m}$.

$h(1) = \sqrt{-x(1)^2 + 5^2}$ by Pyth. Thm
 $= \sqrt{-(2.3)^2 + 25} \approx 4.44 \text{ m}$.

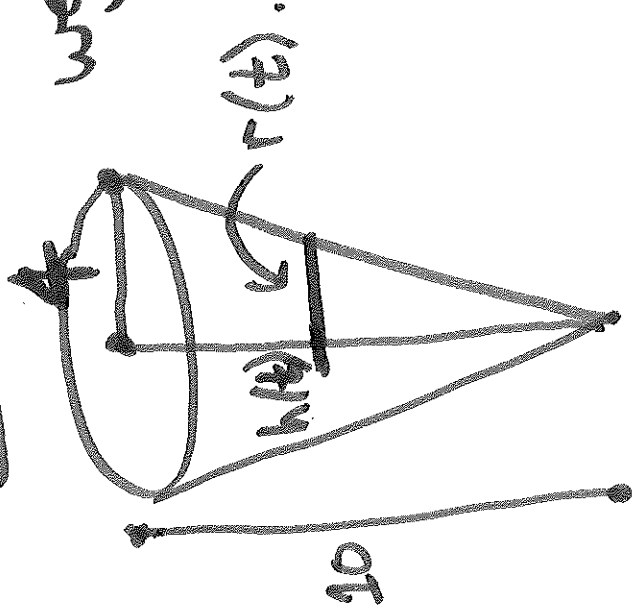
So, at $t=1$, $\frac{dh}{dt} \approx \frac{-2.3}{4.44} \cdot 0.8 \text{ m/s} = -0.42 \text{ m/s}$.

Ex: Conical Tank holding water, \downarrow change in volume.
 \uparrow H_2O is poured in at a rate of $6 \text{ m}^3/\text{min}$.

Height of tank is 10 m , radius at top is 4 m
(pointed down).

Q: How fast is the water rising at time t ?

Sketch:



In this case, $\frac{r(t)}{h(t)} = \frac{4}{10}$ by similar triangles.
 \swarrow w/ this to motivate labels.

We know ht of tank,
radius of tank,
and change in volume.

So, we need to make use of
height $h(t)$, radius $r(t)$, + volume $V(t)$.

we know $r(t) = 0.4 \cdot h(t)$. Need to involve $V(t)$.

Recall: $V = \frac{1}{3} \pi r^2 h$ for a conical tank.

3 functions! Yikes! Let's replace the r .

$$\Rightarrow V(t) = \frac{1}{3} \cdot \pi \cdot (0.4)^2 \cdot (h(t))^3$$

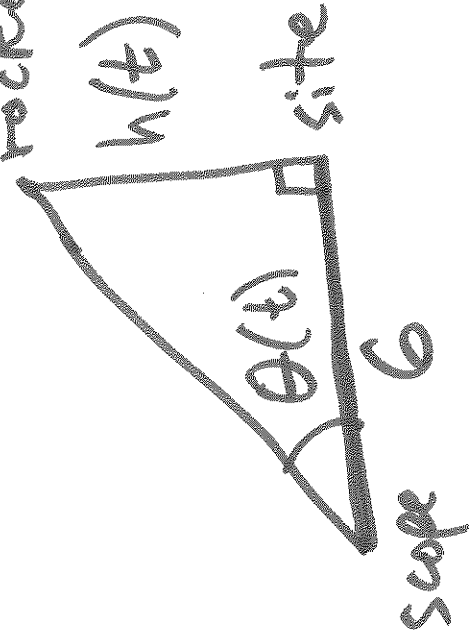
Apply $\frac{d}{dt}$ to both sides, we get

$$\frac{dV}{dt} = \frac{1}{3} \cdot \pi \cdot (0.4)^2 \cdot 3(h(t))^2 \cdot \frac{dh}{dt}$$

$$\text{use } \frac{dV}{dt} = 6 \Rightarrow \frac{dh}{dt} = \frac{18}{(0.4)^2 \cdot \pi \cdot 3 \cdot (h(t))^2} \text{ m/min}$$

Ex: Telescope tracking a rocket launch
 scope is 6km from launch site,
 rocket moves vertically. When angle
 between telescope + ground is $\frac{\pi}{3}$, ~~the~~
 the angle is changing at 0.9 rad/min.
 what is velocity of the rocket at that moment?

need to find $\frac{dh}{dt}$ at time when $\theta = \frac{\pi}{3}$



To solve, $\tan \theta = \frac{h}{6}$

$$\text{Apply } \frac{d}{dt} \Rightarrow \sec^2 \theta \cdot \frac{d\theta}{dt} = \frac{dh}{dt} \cdot \frac{1}{6}$$

$$\Rightarrow \frac{dh}{dt} = \frac{6}{\cos^2 \theta} \cdot \frac{d\theta}{dt} \quad \text{So, when } \theta = \frac{\pi}{3}, \frac{d\theta}{dt} = 0.9, \quad \text{we get } \frac{dh}{dt} = \frac{6}{(\frac{1}{2})^2} \cdot 0.9 \text{ km/min}$$