9/19/16 12PM

1. EXAM I is tomorrow night! See Canvas announcement for info.

2. MA113 Review: 6-8PM tonight, FB200

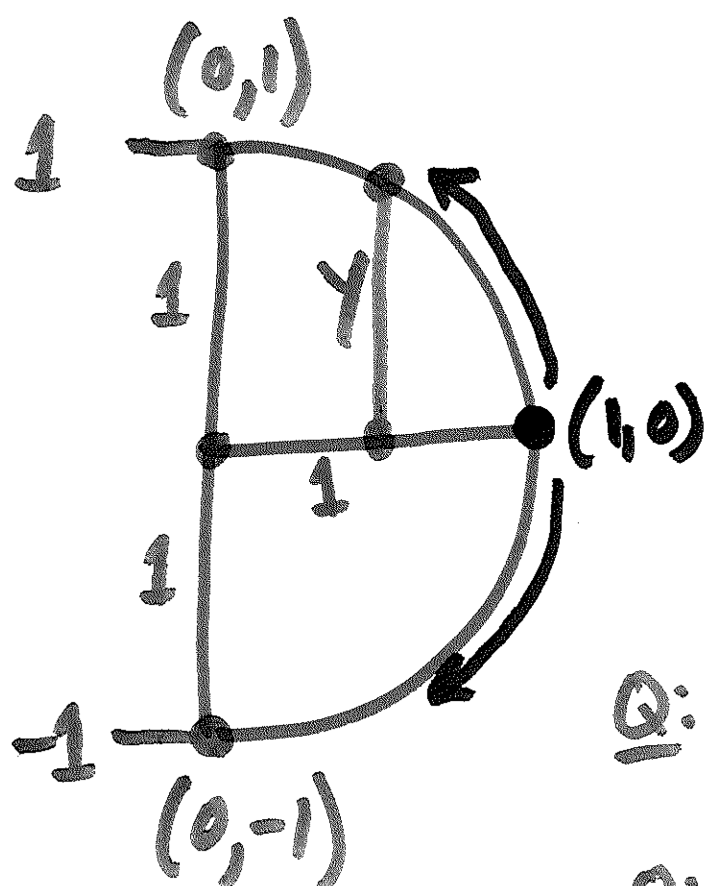
3. With neighbors; without looking it up! what are domains + ranges of sin, arcsin, cos, arccos, tan, arctan?

Why?

Ex: What is domain + range of arcsin?

2 options: 1) Draw the graph of sin

2) Draw the unit circle.



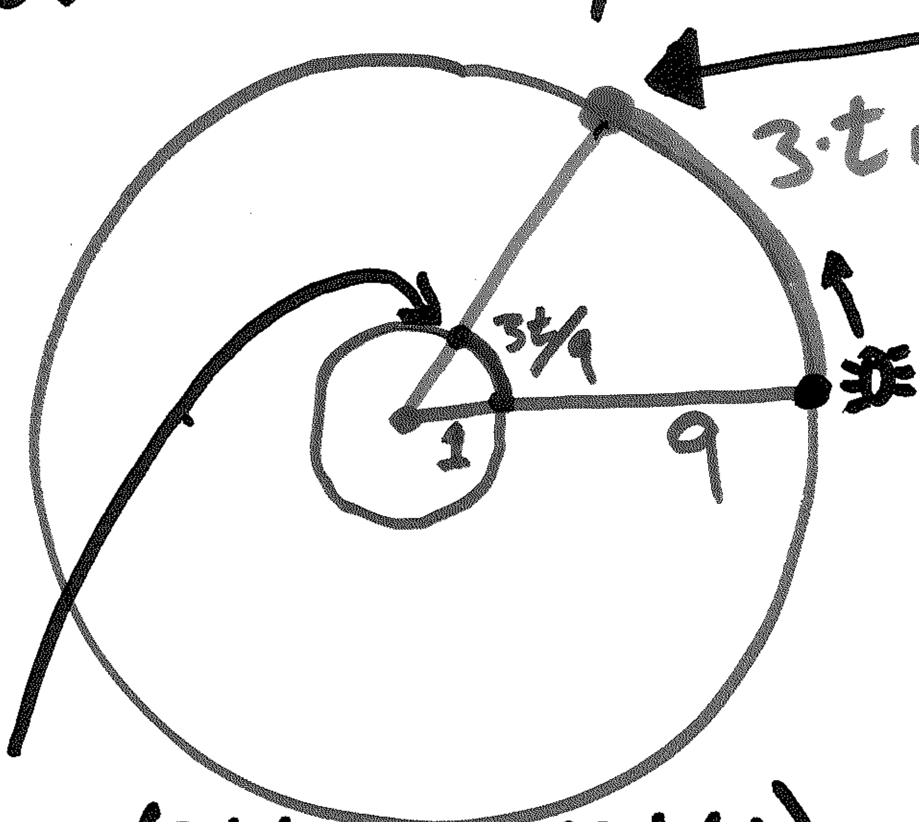
y = vertical coord of a pt

$\longmapsto$ arcsin(y) = length of corresponding arc on unit circle.

Q: what are valid y's? These form domain of arcsin. $[-1,1]$.

Q: what are possible arc lengths? These form the range. $[-\frac{\pi}{2}, \frac{\pi}{2}]$

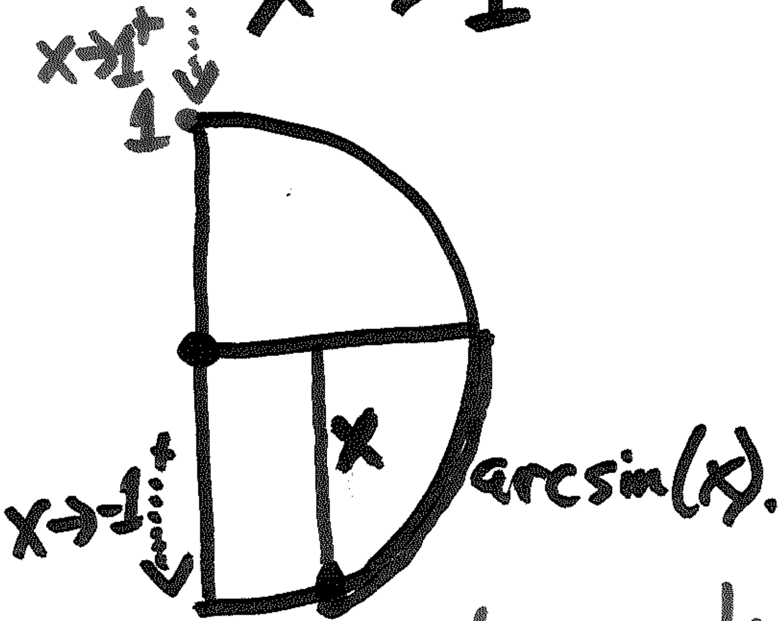
Ex: Bug at $(9,0)$, 9 meters, at $t=0$ sec. Crawls at 3 m/s counterclockwise along circle of radius 9. What are x-y coords of bug after t sec?



$\left(\cos(3t/9), \sin(3t/9)\right)$.

$$\underbrace{\left(9\cos(3t/9),\ 9\sin(3t/9)\right)}_{\text{answer}}.$$

Ex: $\lim_{x \to 1^+} \frac{1}{\arcsin(x)}$ can't do this!



Domain of arcsin(x) is $[-1, 1]$. So, $x \to 1^+$ is outside domain.

Alternatively: $\lim_{x \to -1^+} \frac{1}{\arcsin(x)} = \frac{1}{\frac{-\pi}{2}} = \frac{-2}{\pi}$.

arcsin is cts on domain, so right-cts at -1.

__Ex:__ $$\lim_{h\to 0} \frac{\frac{1}{(2+h)^2} - \frac{1}{4}}{h}$$

__NOTE:__ This is inst. vel. of $\frac{1}{x^2}$ at $x=2$.

$$= \lim_{h\to 0} \frac{\frac{4-(2+h)^2}{4\cdot(2+h)^2}}{h} = \lim_{h\to 0} \frac{\cancel{4}-\cancel{4}-4h-h^2}{h\cdot 4\cdot(2+h)^2}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(-4-h)}{\cancel{h}\cdot 4(2+h)^2} = \lim_{h\to 0} \frac{-4-h}{4\cdot(2+h)^2} \underset{\substack{\text{Quot. law} \\ \text{if den.} \neq 0.}}{=}$$

$$\frac{\lim_{h\to 0} -4-h}{\lim_{h\to 0} 4\cdot(2+h)^2} \underset{\substack{\text{diff, const.,} \\ \text{const. mult. law.}}}{=} \frac{-4-\lim_{h\to 0} h}{4\cdot\lim_{h\to 0}(2+h)^2} \underset{\substack{\text{power law, sum} \\ \text{law, evaluate}}}{=} \frac{-4}{4\cdot(2+0)^2} = \frac{-1}{4}.$$

Ex: Suppose f is $\boxed{\text{cts}}$ w/ values

t	0	1	2	3	4	5	6	7	8	9
$f(t)$	-1	2	3	0	-2	7	2	1	4	4

Q: For which intervals $[n, n+1]$ does $f(t)$ definitely $\boxed{\text{have a soln}}$ to $f(t) = 2.5$?

IVT!!!

These are your 5 intervals.

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$\boxed{\S 2.7}$ Derivatives

Q: what is $f'(x)$ for $f = x^4$.

$f'(x) = 4x^3 \leftarrow$ Fri or Mon.

Why? You evaluate either

$$\lim_{h \to 0} \frac{(x+h)^4 - x^4}{h} \text{ or } \lim_{x \to a} \frac{x^4 - a^4}{x-a}.$$

Velocity: $f(x)$ is position of object, aver. velocity over interval $[a, a+h]$ is

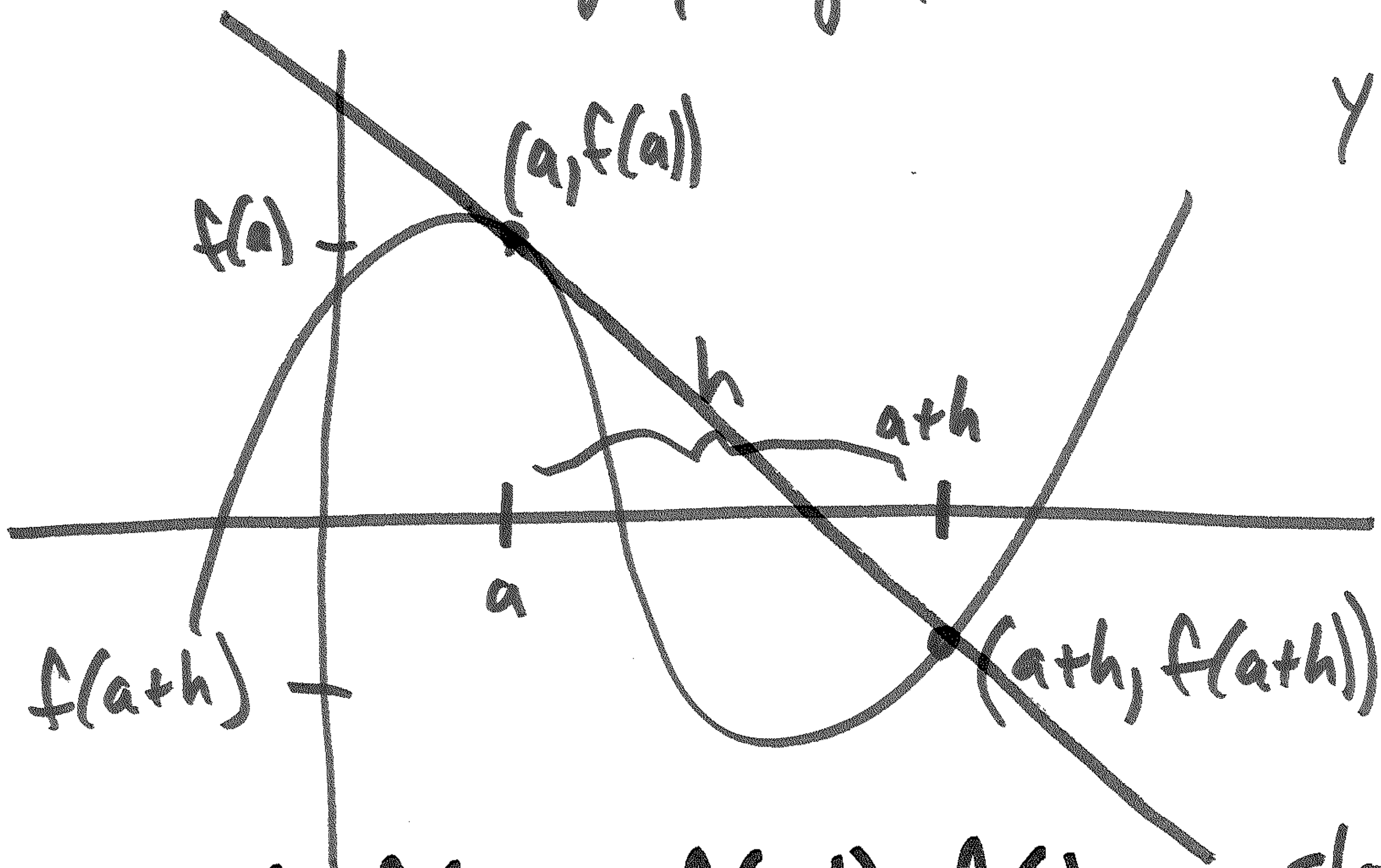
$$\frac{f(a+h)-f(a)}{h}$$

And inst. velocity is

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

Our goal: reinterpret these quantities in the context of functions f, w/out regard to physical situation. This leads to derivatives.

Connect to graph of f:



$= $ average velocity

$$\frac{f(a+h)-f(a)}{a+h-a} = \frac{f(a+h)-f(a)}{h} = $$ slope of secant line between $(a, f(a))$ + $(a+h, f(a+h))$.

Q: What is $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$?

See Desmos!

Two ways to write this:

$$[a, x] \longmapsto \frac{f(x)-f(a)}{x-a} \leftarrow \text{slope of sec. line.}$$

$$[a, a+h] \longmapsto \frac{f(a+h)-f(a)}{h}$$

<u>Here</u>: $x = a+h$, so $x-a = h$.

In general, h-form is more useful, but <u>not</u> always!

<u>Def$^{\underline{n}}$</u>: Given points $P = (a, f(a))$ & $Q = (x, f(x))$, on graph of f, the slope of <u>secant line</u> btwn P & Q is

$$m_{PQ} = \frac{f(x)-f(a)}{x-a}.$$

The tangent line to graph of f at $P=(a,f(a))$ is the line through P w/ slope

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

<u>if</u> the limit exists.

<u>Dictionary:</u>

position of object $\longleftrightarrow$ function

Ave. vel $\longleftrightarrow$ slope of secant line

Inst. vel $\longleftrightarrow$ slope of tangent line.

Def$^{\underline{n}}$: The <u>derivative</u> of f at a, if it exists, is

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

$$= f'(a). \quad \left(\begin{matrix}\text{secretly} \\ \text{inst. vel}\end{matrix}\right).$$

Ex: Compute $f'(3)$ for $f(x) = x^2 + x - 3$.

$$f'(3) = \lim_{h \to 0} \frac{f(3+h)-f(3)}{h} = \lim_{h \to 0} \frac{(3+h)^2 + (3+h) - 3 - \overset{f(3)}{9}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{9} + 6\cancel{h} + h^{\cancel{2}} + \cancel{3} + \overset{1}{\cancel{h}} - \cancel{12}}{\cancel{h}} = \lim_{h \to 0} 7 + h$$

$$= 7.$$

$\boxed{\S 2.8}$ Instead of writing $f'(a)$, we could use x as our input.

So, write $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

If we let x vary, then f' is a function.

<u>Disappointment</u> :(: For most f, this is impossible to evaluate.

Approximating w/ small h's has to suffice.

<u>Big Goal</u>: Systematically compute these limits for polynomials, trig, exp, + inv. fns.

<u>See Desmos!</u>

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$\boxed{1}$ See Canvas for 2 Announcements

$\boxed{2}$ Exam I Curve: +6 pts.

Q:

- $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

- If f is the position of an object, then $f'(x)$ is the inst. vel. of object at time x.

__Ex__: If $f(x)=\sqrt{x}$, find $f'(x)$.

$$f'(x)=\lim_{h\to 0}\frac{\left(\sqrt{x+h}-\sqrt{x}\right)}{\underbrace{h}_{\text{issue is } h\to 0 \text{ so den} \to 0.}}\cdot\frac{\left(\sqrt{x+h}+\sqrt{x}\right)}{\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$=\lim_{h\to 0}\frac{\cancel{x}+h-\cancel{x}}{h\left(\sqrt{x+h}+\sqrt{x}\right)}=\lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}\underset{\substack{\uparrow\\ \text{limit laws}}}{=}\frac{1}{\sqrt{x+0}+\sqrt{x}}=\frac{1}{2\sqrt{x}}.$$

NOTE:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- $f(x+h) = \sqrt{x+h}$
- $f(x) = \sqrt{x}$.

Ex: $f(x) = \dfrac{1}{\sqrt{x}}$. Find $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \lim_{h \to 0} \frac{\frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x} \cdot \sqrt{x+h}}}{h}$$

com. den. in num.

$$= \lim_{h \to 0} \frac{(\sqrt{x} - \sqrt{x+h})}{h \cdot \sqrt{x} \cdot \sqrt{x+h}} \cdot \frac{(\sqrt{x} + \sqrt{x+h})}{(\sqrt{x} + \sqrt{x+h})}$$

$$= \lim_{h \to 0} \frac{\cancel{x} - (\cancel{x} + h)}{h \cdot \sqrt{x} \cdot \sqrt{x+h}\left(\sqrt{x} + \sqrt{x+h}\right)}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h} \cdot \sqrt{x}\sqrt{x+h}\left(\sqrt{x} + \sqrt{x+h}\right)}$$

$$= \lim_{h \to 0} \frac{-1}{\sqrt{x}\sqrt{x+h}\left(\sqrt{x} + \sqrt{x+h}\right)} = \ldots$$ limit laws

$$= \frac{-1}{\underbrace{\sqrt{x} \cdot \sqrt{x+0}\left(\sqrt{x} + \sqrt{x+0}\right)}_{\underbrace{\sqrt{x} \cdot \sqrt{x}}_{x} \cdot \underbrace{\left(\sqrt{x} + \sqrt{x}\right)}_{2\sqrt{x}}}} = \frac{-1}{2x\sqrt{x}} = \frac{-1}{2x^{3/2}}.$$

NOTE: $f(x) = \sqrt{x}$, $f'(x) = \dfrac{1}{2\sqrt{x}}$.

$$\text{Domain}(f) = [0, \infty)$$
$$\text{Domain}(f') = (0, \infty)$$

NOT equal!

So, domain of f' might not equal domain of f.

NOTATION: The following all mean derivative of f w/ variable x:

$y = f(x)$

$$f'(x),\ \frac{df}{dx},\ y',\ \frac{dy}{dx},\ \frac{d}{dx}f(x),\ \dot{y},\ \dot{f},\ Df,\ D_x f$$

we will mainly use first five.

<u>Defⁿ</u>: We say f is <u>differentiable</u> on (a,b) if $f'(x)$ exists for all x in (a,b).

<u>Ex</u>: $f = |x^3 - 1|$ is not diff. at $x = 1$.

why?

$$\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} = 3$$

not equal!

$$\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} = -3$$

So, $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ does not exist.

NOTE: f can fail to be diff. if the graph of f has a "corner".

Thm: If f is diff, then f is cts.

Pf in book! Read this!
uses technique of mult. by $\frac{x-a}{x-a}$.

So, if f is not cts at a, then $f'(a)$ does not exist.

You can take derivatives of derivatives!

e.g. $(f')' = f''$. $\frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d^2 f}{dx^2}$.

n^{th} derivative is written $f^{(n)}(x)$ or $\frac{d^n f}{dx^n}$

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$\boxed{1}$ See Canvas from last week for reminders.

$\boxed{2}$ For what $f(x)$ is

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \;?$$

What is this limit?

<u>3 goals for today:</u>

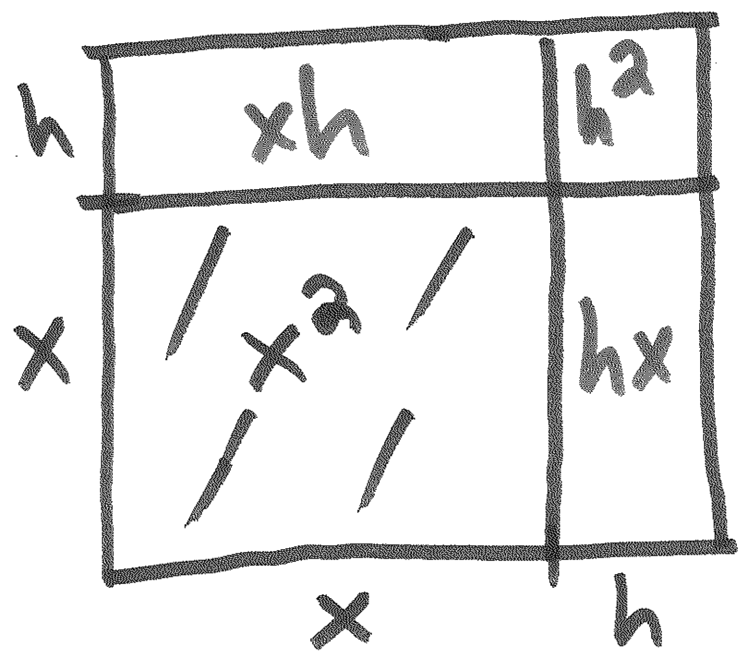
- der. of polys
- der. of e^x
- Additional rules for $\frac{d}{dx}$.

For the above limit, $f(x) = x^2$.

<u>Claim:</u> Helpful picture exists:

$(x+h)^2 - x^2$ is num. of limit.

this is area outside of x^2 within $(x+h)^2$.



$(x+h)^2 - x^2 = xh + hx + h^2$

$= 2xh + h^2$.

looking at pic, guess that $f'(x)$ should be $x + x = 2x$.

$(x+h)^2 =$ area of large square.

$x^2 =$ area of lower left square.

__Verify:__ $$\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} =$$

$$\lim_{h\to 0} \frac{\cancel{x^2} + 2\cancel{h}x + h^2 - \cancel{x^2}}{\cancel{h}} =$$

$$\lim_{h\to 0} (2x+h) = 2x.$$

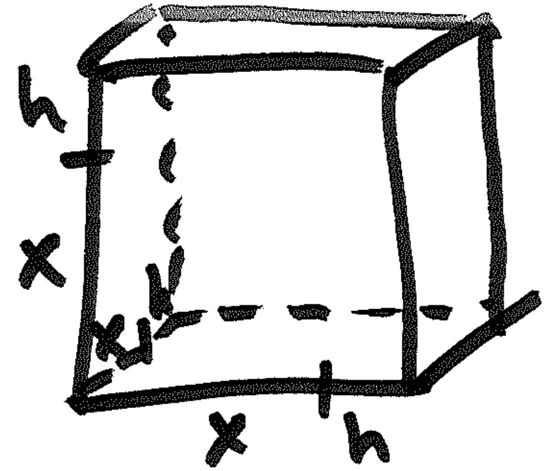
__Ex:__ $f(x) = x^3$ What is $f'(x)$?

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{(x+h)(x+h)(x+h) - x^3}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^3} + 3hx^2 + 3h^2x + h^3 - \cancel{x^3}}{h} = \lim_{h\to 0} \frac{3hx^2 + 3h^2x + h^3}{h}$$

$$= *$$

Picture:



Guess: $f'(x)$ should be $x^2+x^2+x^2=3x^2$ for 3 faces of dim$^{\underline{n}}$ 2 pushed outward.

$\circledast = \lim\limits_{h\to 0} 3x^2+3hx+h^2 = 3x^2.$

Cancel h's

Thm: For any real number n,

$$\frac{d}{dx}(x^n)=nx^{n-1}. \quad \text{(Power Rule).}$$

Aside: Look up the binomial theorem and read the proof of Power Rule in text.

__Ex__: Exponentials. $f(x)=b^x$, $b>0, b\neq 1$.

$$f'(x)=\lim_{h\to 0}\frac{b^{x+h}-b^x}{h}=\lim_{h\to 0}\frac{b^x\cdot b^h-b^x}{h}$$

$$=\lim_{h\to 0}\frac{b^x(b^h-1)}{h}=\underbrace{b^x}_{f(x).}\cdot\underbrace{\lim_{h\to 0}\frac{b^h-1}{h}}_{\text{Claim: this is } f'(0).}.$$

__Why?__ $\dfrac{b^h-1}{h}=\dfrac{b^{0+h}-b^0}{h}=\dfrac{f(0+h)-f(0)}{h}.$

__Thm:__ For $f(x)=b^x$, $f'(x)=f'(0)\cdot f(x)$.

Wouldn't it be lovely if $f'(0)$ was 1?

Def$^{\underline{n}}$: e is the number satisfies

$$1 = \lim_{h \to 0} \frac{e^h - 1}{h} \left(= f'(0) \text{ for } f(x) = e^x \right).$$

Thm: $\frac{d}{dx}(e^x) = e^x$.

NOTE: $\frac{d}{dx}(cf) = c\frac{d}{dx}(f)$ for constant c

and $\frac{d}{dx}(f \pm g) = \frac{df}{dx} \pm \frac{dg}{dx}$.

<u>Ex</u>: $\frac{d}{dx}\left(3e^x + 7x^2 + x\right) =$

$\frac{d}{dx}(3e^x) + \frac{d}{dx}(7x^2) + \frac{d}{dx}(x) =$

$3\frac{d}{dx}(e^x) + 7\frac{d}{dx}(x^2) + \frac{d}{dx}(x) =$

$3e^x + 7 \cdot 2x + 1$.

Ex: $\frac{d}{dx}\left(x^{\pi} - 17x^{16} + 1\right) =$

$$\frac{d}{dx}\left(x^{\pi}\right) - \frac{d}{dx}\left(17x^{16}\right) + \frac{d}{dx}(1) =$$

$$\pi x^{\pi-1} - 17 \cdot 16x^{15} + 0 .$$