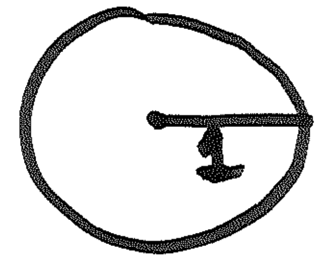
9/19/16 9AM

1. EXAM I is tomorrow night! See Canvas announcement for info.

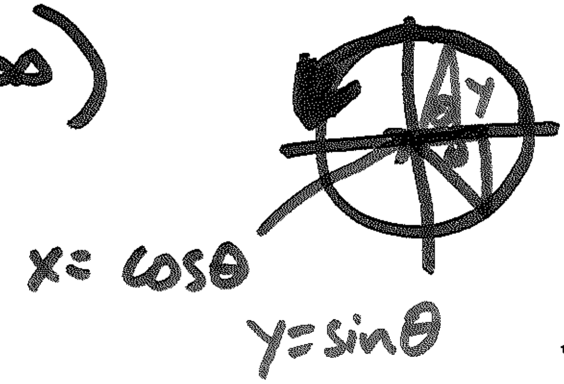
2. MA 113 review: 6-8 PM tonight, FB200

3. With neighbors; without looking it up! What are domains + ranges of sin, arcsin, cos, arccos, tan, arctan? Why?



sin: domain: $(-\infty, \infty)$

range: $[-1, 1]$



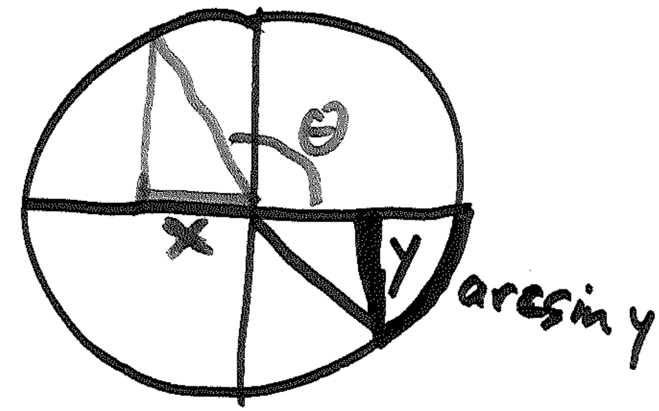
radian input is any radian measure.

cos: domain: $(-\infty, \infty)$

range: $[-1, 1]$

arcsin: domain: $[-1, 1]$

range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

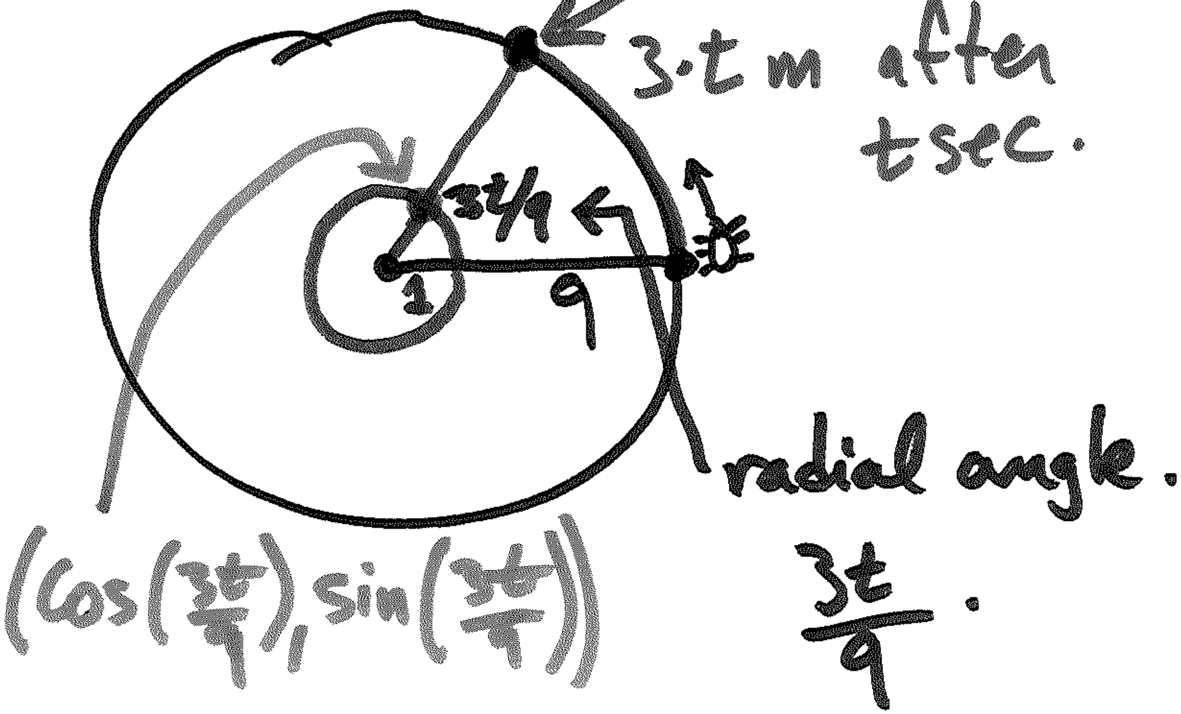


arcsin(y) = length of arc for vertical length y in unit circle.

arccos: domain: $[-1, 1]$

range: $[0, \pi]$

Ex: Bug at $(9,0)$ at $t=0$ ← seconds
9 is in meters. Crawls at 3 m/sec
counterclockwise along circle of radius 9.
What are x-y coords of bug after
t seconds?



This is pt I want!

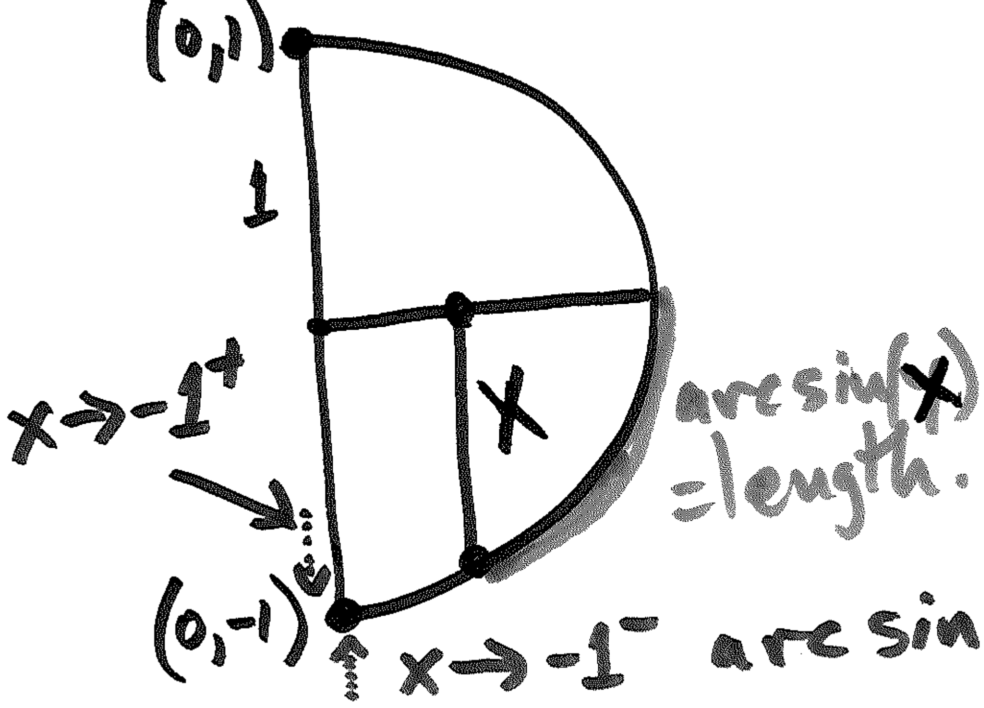
$$\Rightarrow \left(9\cos\left(\frac{3t}{9}\right), 9\sin\left(\frac{3t}{9}\right)\right)$$

This is it!

__Ex__: Find $\displaystyle\lim_{x \to -1^+} \frac{1}{\arcsin(x)}$.

__Q__: Could we also consider $\displaystyle\lim_{x \to -1^-} \frac{1}{\arcsin(x)}$? Why or why not?

NO, since $x < -1$ not in domain of arcsin.



arcsin is not defined < -1!

$$\lim_{x\to -1^+} \frac{1}{\arcsin(x)} = \frac{1}{\arcsin(-1)} = \frac{1}{-\pi/2} = \frac{-2}{\pi}.$$

Since $\arcsin(x)$ is cts on its domain, it is right-cts at $x=-1$. $[-1,1]$

Ex: Model limit law soln.

NOTE: Inst. vel. for $f(x) = \frac{1}{x^2}$ at $x=2$.

$$\lim_{h\to 0} \frac{\frac{1}{(2+h)^2} - \frac{1}{4}}{h} = \lim_{h\to 0} \frac{\frac{4-(2+h)^2}{4(2+h)^2}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{4} - \cancel{4} - 4h - h^2}{h \cdot 4 \cdot (2+h)^2} = \lim_{h\to 0} \frac{-4\cancel{h} - h^{\cancel{2}}}{\cancel{h} \cdot 4 \cdot (2+h)^2}$$

$$= \lim_{h \to 0} \frac{-4-h}{4\cdot(2+h)^2} = \circledast .$$

<u>2 options:</u> <u>$\circledast$ 1:</u> Invoke continuity of $\frac{-4-h}{4(2+h)^2}$.

This results in $\circledast = \frac{-4}{4\cdot(2)^2} = \frac{-1}{4}$.

<u>$\circledast$ 2:</u> Use limit laws.

$$\circledast = \frac{\lim_{h\to 0} -4-h}{\lim_{h\to 0} 4\cdot(2+h)^2} = \frac{-4-\lim_{h\to 0} h}{4\cdot \lim_{h\to 0} (2+h)^2} =$$

Quot. law if den. $\neq 0$

const. mult law, diff. law, const. law

$$\underset{\substack{\text{power law +}\\ \text{sum law +}\\ \text{const. law}}}{=} \frac{-4-\lim_{h\to 0} h}{4\cdot\left(2+\lim_{h\to 0} h\right)^2} \underset{\substack{\text{evaluate}\\ \text{limits}}}{=} \frac{-4}{4(2+0)^2} = \frac{-1}{4}.$$

Ex: Suppose f is cts and values for f are

t	0	1	2	3	4	5	6	7	8
f	-1	2	3	0	-2	7	2	1	4

Q: For which intervals $[n, n+1]$ does $f(t) = 2.5$ definitely have a soln?

cts + soln $\Rightarrow$ IVT.

5 intervals above definitely contain solns.

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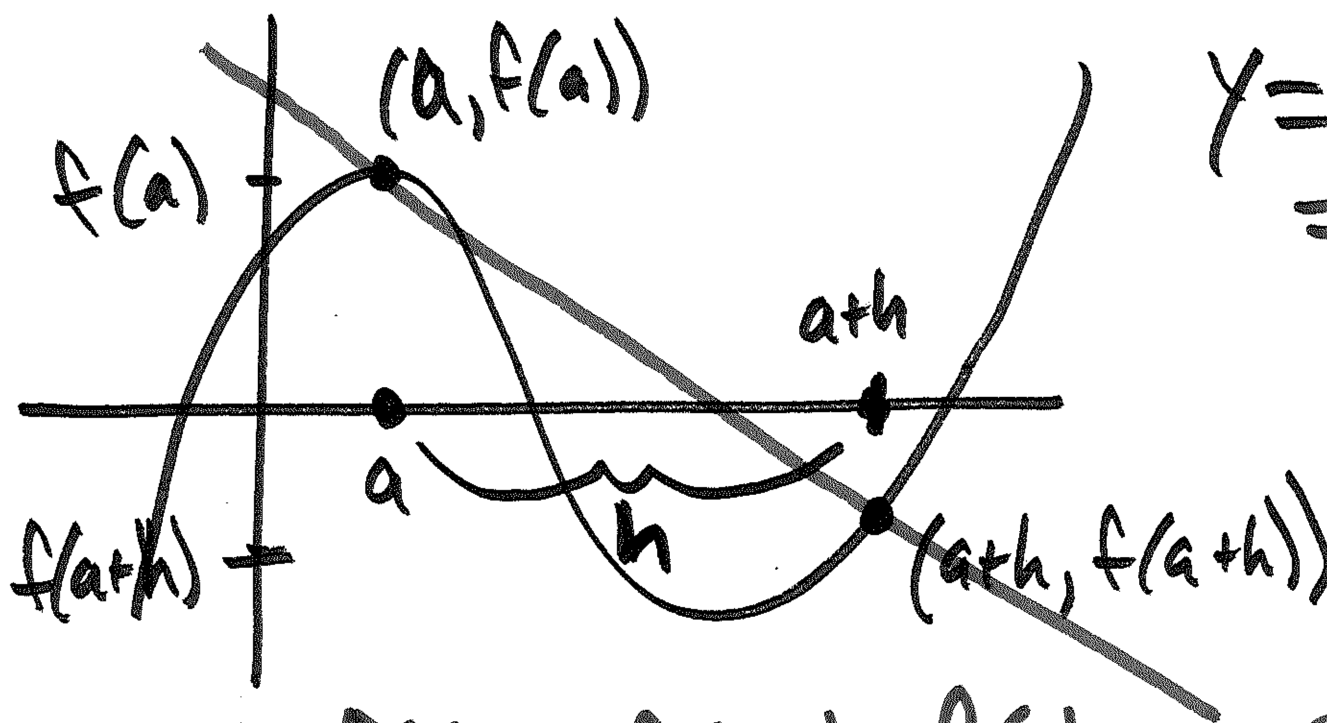
§2.7: Derivatives

Key Pt: You've already been doing this!

Recall: For a position fn $f(x)$ over interval $[a, a+h]$, we have

$$\frac{\Delta f}{\Delta x} = \frac{f(a+h)-f(a)}{h} \quad \text{and} \quad \text{inst. vel} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

Mathematically, we want to connect this to the graph of f.



$y = f(x)$
= position fn.

$$\frac{f(a+h)-f(a)}{a+h-a} = \frac{f(a+h)-f(a)}{h}$$ = slope of secant line btwn $(a, f(a))$ + $(a+h, f(a+h))$

= **average velocity** on $[a, a+h]$.

Q: What is $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$? → Desmos.

$$\lim_{h \to 0} \frac{f(a+h)-f(a)}{h} = \text{slope of line tangent to graph of } f \text{ at } (a, f(a)).$$

- For most f, this limit is tough to compute.

<u>Our goal:</u> For simple functions f that come up in applications, e.g. polynomials, trig, exponential, inverses, etc, try to systematically determine these limits at various a's.

(Next few weeks of class).

<u>NOTE</u>: Two ways to represent $\frac{\Delta f}{\Delta x}$.

(1) $[a,x] \longmapsto \dfrac{f(x)-f(a)}{x-a}$.

(2) $[a,a+h] \longmapsto \dfrac{f(a+h)-f(a)}{h}$.

If $x=a+h$, then $x-a=h$.

<u>Def$^{\underline{n}}$</u>: Given points $P=(a,f(a))$ & $Q=(x,f(x))$, the <u>slope of secant line</u> is $m_{PQ}=\dfrac{f(x)-f(a)}{x-a}$.

The <u>tangent line</u> to f at P has slope

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h},$$ if limit exists!

Dictionary:

Position at time x $\longleftrightarrow$ function $f(x)$

Average vel. on $[a,x]$ $\longleftrightarrow$ $\dfrac{f(x)-f(a)}{x-a} = m_{PQ}$

Inst. vel. on $[a,x]$ $\longleftrightarrow$ slope of tangent line.

Def$^{\underline{n}}$: The derivative of f at a, if it exists, is

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a).$$

Ex: Compute $f'(3)$ for $f(x) = x^2 + x - 3$.

Use limits! **Hint:** 70% of time (I think), h is better than $x-a$.

$$f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} =$$

$$\lim_{h \to 0} \frac{(3+h)^2 + (3+h) - 3 - (3^2 + 3 - 3)}{h} =$$

$$\lim_{h \to 0} \frac{2 \cdot 3 \cdot \cancel{h} + h^{\cancel{2}} + \cancel{h} \cdot 1}{\cancel{h}} = \lim_{h \to 0} 7 + h = 7.$$

So, $f'(3) = 7$.

$\boxed{\S 2.8}$ If we let "a" vary, then

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$ is a function of a.

Switch to convention of x for input:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Q: What does function $f'(x)$ look like?

Desmos $\longrightarrow$ link on Canvas (remind me!)

Key observation: Domain of f' does not have to be same as domain of f.

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$\boxed{1.}$ Curve is +6 on Exam 1

$\boxed{2}$ See 2 canvas announcements from this morning.

Derivatives:

- $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

- f is position of object, $f'(x)$ = inst. vel at time x.

Ex: $f(x) = \sqrt{x}$, compute $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{\left(\sqrt{x+h} - \sqrt{x}\right)}{h} \cdot \frac{\left(\sqrt{x+h} + \sqrt{x}\right)}{\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + h - \cancel{x}}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \underset{\text{lim. laws}}{=} \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Q abt Hw: $f = \frac{1}{\sqrt{x}}$ $f'(a)$ at $x = a$

slope of tang. line. $a = 7, 11, 13.$ $(a, f(a))$

$$f'(a) = \lim_{h\to 0} \frac{\frac{1}{\sqrt{a+h}} - \frac{1}{\sqrt{a}}}{h} = \lim_{h\to 0} \frac{\frac{\sqrt{a} - \sqrt{a+h}}{\sqrt{a}\cdot\sqrt{a+h}}}{h}$$

h→0 inden.

$$= \lim_{h\to 0} \frac{(\sqrt{a} - \sqrt{a+h})}{h\cdot\sqrt{a}\cdot\sqrt{a+h}} \cdot \frac{(\sqrt{a} + \sqrt{a+h})}{(\sqrt{a} + \sqrt{a+h})}$$

$$= \lim_{h\to 0} \frac{\cancel{a} - (\cancel{a}+h)}{h\cdot\sqrt{a}\sqrt{a+h}(\sqrt{a} + \sqrt{a+h})}$$

$$= \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}\cdot\sqrt{a}\sqrt{a+h}(\sqrt{a} + \sqrt{a+h})}$$ = lim laws will finish this.

$$= \frac{-1}{2a\sqrt{a}} = \frac{-1}{2a^{3/2}}.$$

NOTE: Domain of $\sqrt{x}$ is $[0, \infty)$.

But $(\sqrt{x})' = \dfrac{1}{2\sqrt{x}}$ w/ domain $(0, \infty)$.

So, domain of f' does not have to equal domain of f.

NOTATION: If $y = f(x)$, we write any of the following for $f'(x)$:

$$f'(x),\ \frac{df}{dx},\ y',\ \frac{dy}{dx},\ \frac{d}{dx}f(x),\ Df(x),\ D_x f(x),\ \text{and } \dot{y}$$

For us, primarily use the first five of these.

<u>Def$^{\underline{n}}$</u>: A fn $f(x)$ is <u>differentiable</u> on (a,b) if $f'(x)$ exists for all x in (a,b).

<u>Ex</u>: $f(x) = |x^3 - 1|$ is not diff at $x=1$.

why? $$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = 3.$$

but $$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = -3.$$

These are not equal, so $\lim\limits_{h\to 0} \frac{f(1+h)-f(1)}{h}$ does not exist.

__NOTE__: One failure of differentiability at a pt is having a "corner" of the graph at that pt.

If f is not cts at a, then f is not diff. at a. Why?

__Thm__: If f is diff, then f is cts.

__Idea of proof:__ $\lim_{x \to a} f(x) = f(a)$ is continuity.

Consider instead $\lim_{x \to a} \frac{[f(x)-f(a)] \cdot (x-a)}{(x-a)} =$ Prod. law of limits

$= \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \cdot \lim_{x \to a} x-a = \underbrace{f'(a)}_{\text{use } f'(a) \text{ exists.}} \cdot 0 = 0.$ ∎

Higher derivatives:

$f \longrightarrow f'$.

Take derivative of f', get $(f')' = f''$

f "double prime"

$$\frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d^2 f}{dx^2}$$

second derivative.

In general, for the n^{th} derivative, write either $\frac{d^n f}{dx^n}$ or $f^{(n)}(x)$.

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1. See Canvas from Friday for reminders.

2. **Q**: Evaluate $\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$.

Q: What is $f(x)$ if this is $f'(x)$?

$f(x) = x^2$, then this limit is $f'(x)$. So, this is a very simple polynomial.

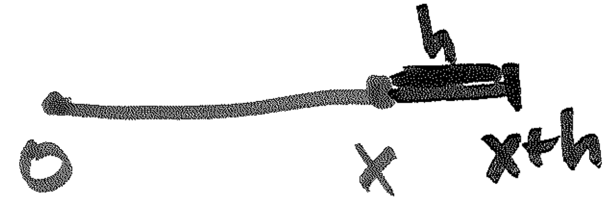
Let's take a step back...

Today:
- der. of polynomials
- der. of e^x
- Properties of $\frac{d}{dx}$.

$\underline{f(x) = x}$: $f'(x) = \lim_{h \to 0} \frac{(\cancel{x}+h) - \cancel{x}}{h} = \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}}$

$= 1.$

<u>Picture</u>:



$= (x+h) - x.$

$f(x) = x$ = length of line.

As length changes, what is the inst. change in length?

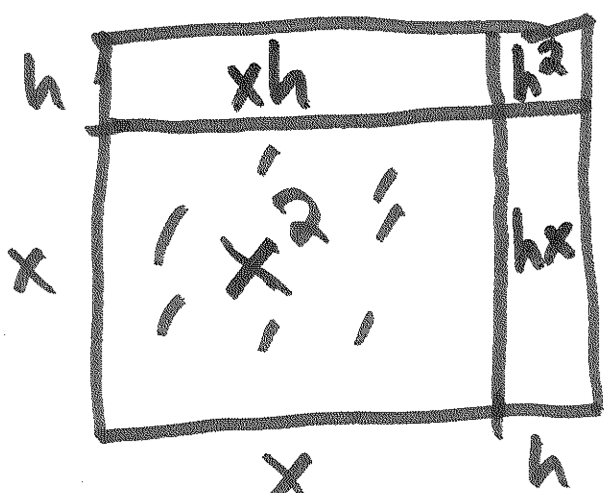
$$\frac{f(x+h) - f(x)}{h} = \frac{\text{small length change}}{\text{small time change}}$$

$$\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}.$$

Set $f(x)=x^2$ now...

x^2 = area of square of side length x.

So, $(x+h)^2 - x^2$ is area of top, right, + upper right regions left area.



This should be $xh + hx + h^2$
$= 2xh + h^2$.

This matches w/ our algebra!

$(x+h)^2$ = area of large square above.

x^2 = area of lower left square above.

<u>**Guess**</u>: $f'(x) = x + x$
$= 2x$.

$$\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h} =$$

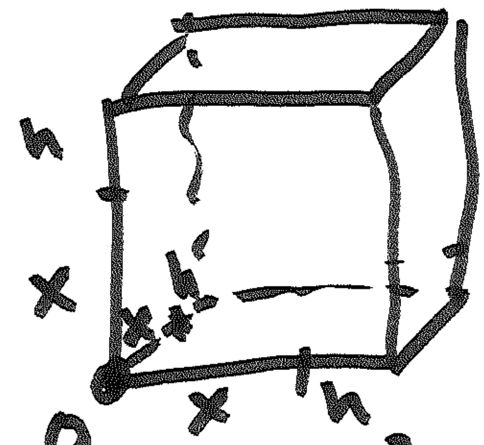
$$\lim_{h\to 0} 2x+h = 2x.$$

Q: $f(x)=x^3$. What is $f'(x)$?

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)(x+h)(x+h) - x^3}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 \cancel{-x^3}}{h}$$



$(x+h)^3$ = volume of this cube.

$$= \lim_{h \to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2 = 3x^2.$$

Thm: For any real number n,

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}.$$

Rmk: If n is a positive integer, simple proof using binomial theorem, look it up and read the text.

Pascal's triangle shows up.

Q: what about $f(x) = b^x$?

$$f'(x) = \lim_{h\to 0} \frac{b^{x+h} - b^x}{h} = \lim_{h\to 0} \frac{b^x \cdot b^h - b^x}{h}$$

$$= b^x \cdot \underbrace{\lim_{h\to 0} \frac{b^h - 1}{h}}_{= \lim_{h\to 0} \frac{b^{0+h} - b^0}{h} = f'(0).} = b^x \cdot f'(0) = f(x) \cdot f'(0).$$

factor out b^x

NOTE: An exponential fn is almost its own derivative! If $f'(0) = 1$, then it is!

Define e to be the number satisfying

$$1 = \lim_{h\to 0} \frac{e^h - 1}{h}.$$

Thm: $\frac{d}{dx}(e^x) = e^x$.

Thm: Let c be a constant, f and g be diff.

Then $\frac{d}{dx}(c \cdot f \pm g) = c\frac{d}{dx}(f) \pm \frac{d}{dx}(g)$.

NOTE: In textbook, this is listed as 3 separate rules.

Application to functions:

move across sums + differences

Ex: $\frac{d}{dx}(3e^x + x^4 - 7x^2 + x) =$

$\frac{d}{dx}(3e^x) + \frac{d}{dx}(x^4) - \frac{d}{dx}(7x^2) + \frac{d}{dx}(x) =$

pull out constants, use rules.

$3 \cdot e^x + 4x^3 - 7 \cdot 2x + 1$.

Ex: $\frac{d}{dx}\left(x^{\pi} - 4e^{x} + 17x^{16}\right) =$ ← sums + differences

$\frac{d}{dx}\left(x^{\pi}\right) - \frac{d}{dx}\left(4e^{x}\right) + \frac{d}{dx}\left(17x^{16}\right) =$ ← pull out constants, use laws

$\pi x^{\pi-1} - 4e^{x} + 17 \cdot 16x^{15}.$