## 9/28/16 9 AM

$\boxed{1}$ Sign attendance

$\boxed{2}$ Quiz Thurs, written assig. Fri, Webwork too!

"e"

$\boxed{3}$ w/ neighbor, compute $\frac{d}{dx}\left(3e^x + x^7 - 2x^e\right)$

rewrite: $\frac{d}{dx}\left(3\underline{e^x} + x^7 - 2\underline{x^e}\right)$

$= 3e^x + 7x^6 - 2ex^{e-1}.$

# §3.2: Prod + Quot. Rules

**Products:** Q: For two functions $f, g$, both diff, what is der. of $f(x) \cdot g(x)$?

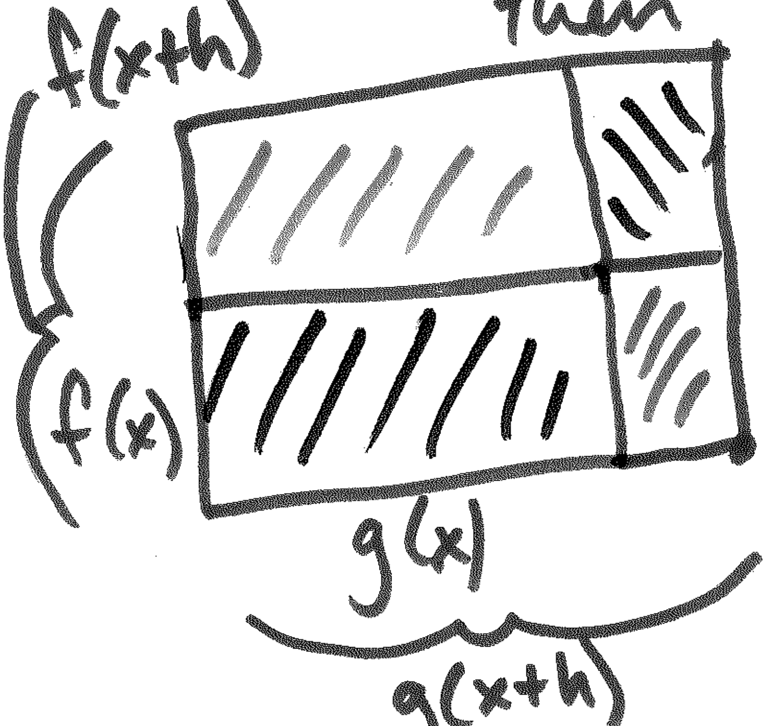
$$(f \cdot g)' = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

numerator is $\circledast$.

Suppose $F$ is diff, then

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

__Picture__: If $f$ & $g$ are both increasing, then $f(x)\cdot g(x)$ = area of rectangle:




$f(x+h)g(x+h) - f(x)g(x)$

= area in large rectangle outside shaded rectangle.

So ... $(f\cdot g)'$ is

$$\lim_{h\to 0} \frac{(f(x+h)-f(x))g(x) + (g(x+h)-g(x))f(x) + (f(x+h)-f(x))\cdot(g(x+h)-g(x))}{h} = ⊛$$

__NOTE__: Simplifying this gives our original expression.

$$\star = \lim_{h\to 0} \frac{(f(x+h)-f(x))g(x)}{h}$$

$$+ \lim_{h\to 0} \frac{(g(x+h)-g(x))}{h} f(x) +$$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\,(g(x+h)-g(x))$$

$$= g(x) \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$+ f(x)\cdot \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$+ \left(\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\right)\cdot \lim_{h\to 0} g(x+h)-g(x)$$

$$= g(x)\cdot f'(x) + f(x)\cdot g'(x) + f'(x)\cdot 0$$

__Thm (Product Rule):__ If $f, g$ diff, then $(f \cdot g)' = g \cdot f' + f \cdot g'$.

__NOTE:__ $\lim\limits_{h \to 0} \big( g(x+h) - g(x) \big) \overset{\text{diff law}}{=} \lim\limits_{h \to 0} g(x+h) - \lim\limits_{h \to 0} g(x)$

$\underset{\substack{\text{const. law} \\ \text{\& continuity of } g}}{=} g(x) - g(x) = 0.$

Ex: Find $\frac{d}{dx}(x^2)$ using product rule.

$$\frac{d}{dx}(x^2) = \frac{d}{dx}(x \cdot x) =$$

$$\frac{d}{dx}(x) \cdot x + x \cdot \frac{d}{dx}(x) = 1 \cdot x + x \cdot 1$$

$$= 2x.$$

Ex: Find $\frac{d}{dx}(x^7)$ using product rule + power rule for only $x^2$ and $x^5$.

$$\frac{d}{dx}(x^7) = \frac{d}{dx}(x^{2+5}) = \frac{d}{dx}(x^2 \cdot x^5) =$$

$$\frac{d}{dx}(x^2) \cdot x^5 + x^2 \cdot \frac{d}{dx}(x^5) = 2x \cdot x^5 + x^2 \cdot 5x^4 = 7x^6.$$

**Ex:** $\frac{d}{dx}\Big[\underbrace{(x^3+x)}_{f}\cdot\underbrace{e^x}_{g}\Big] = \underbrace{(3x^2+1)e^x}_{f'\cdot g} + \underbrace{(x^3+x)e^x}_{f\cdot g'}$

**Ex:** $\frac{d}{dx}\left(\frac{e^x}{\sqrt{x}}\right) = \frac{d}{dx}\Big(\underbrace{e^x}_{f}\cdot\underbrace{x^{-1/2}}_{g}\Big)$

$= \underbrace{e^x\cdot x^{-1/2}}_{f'\cdot g} + \underbrace{e^x\cdot -\frac{1}{2}\cdot x^{-3/2}}_{f\cdot g'}$.

**Note:** $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{d}{dx}\left(f\cdot(g)^{-1}\right) \overset{\text{Prod. rule}}{=}$

$\frac{df}{dx}\cdot(g)^{-1} + f\cdot\frac{d}{dx}\left((g)^{-1}\right)$.

der. of a composition. $\Rightarrow$ Chain rule!

Once you know chain rule, handle quotients w/ above formula.

Using prod. & chain rule leads to this formula:

<u>Quotient Rule:</u> $\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2}$.

<u>Ex</u>: $\frac{d}{dx}\left(\frac{e^x}{1+x^2}\right) = \frac{(1+x^2)e^x - e^x(2x)}{(1+x^2)^2}$

<u>Ex</u>: $\frac{d}{dx}\left(\frac{x^3+x-1}{x^2+1}\right) = \frac{(x^2+1)(3x^2+1) - (x^3+x-1)\cdot 2x}{(x^2+1)^2}$.

Ex: Find $f''(x)$ for $f(x) = \frac{e^x}{x^2}$.

In this case, use prod. rule!

$$f(x) = e^x \cdot x^{-2}$$

$$f'(x) = e^x \cdot x^{-2} + e^x \cdot -2x^{-3}$$

$$= e^x \cdot x^{-2} - 2e^x x^{-3}$$

$$f''(x) = \boxed{e^x \cdot x^{-2} - 2e^x x^{-3} - \left(2e^x x^{-3} + 2e^x \cdot -3x^{-4}\right)}$$

9/30/16 9 AM

$\boxed{1}$ Turn in written assignment.

$\boxed{2}$ W/ neighbors, what picture represents $\lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$?

**Goal:** $\frac{d}{dx}(\sin x) = ?$

Derivatives of trig fns are listed in §3.3.
You need to know all of them.

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~~Picture:~~ Look @ graph of $\sin(x)$,

desmos.

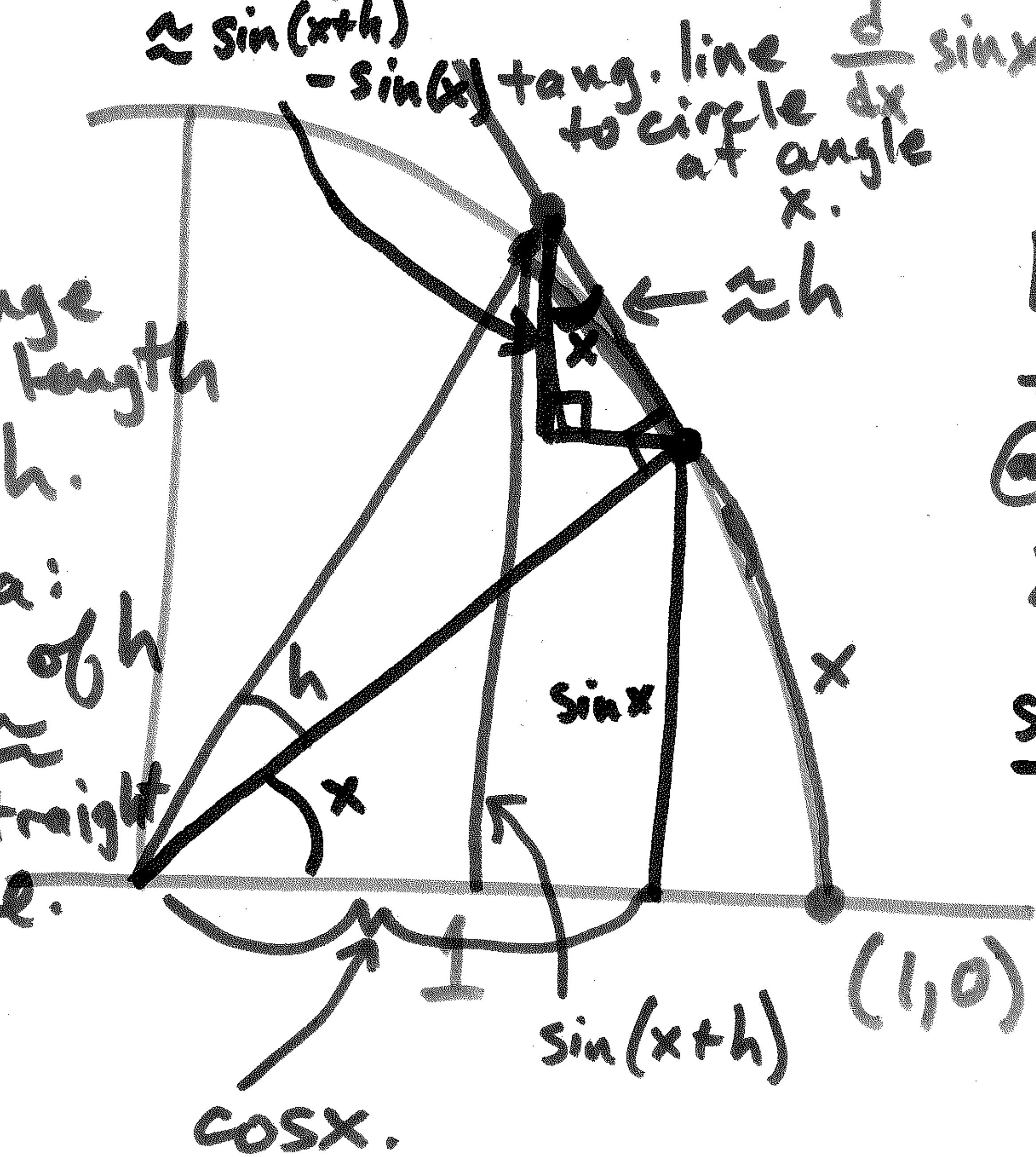
**Guess:** $\frac{d}{dx}(\sin x) \overset{?}{=} \cos x$

(1600's style discovery): look @ unit circle.

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

orange arc length is h.

Idea: arc of h is $\approx$ a straight line.




by similar triangles, look @ triangles w/ angle x,

$$\frac{\sin(x+h)-\sin(x)}{h} \approx \frac{\cos x}{1}.$$

Guess:

$$\frac{d}{dx}\sin x = \cos x.$$

Using h-form of limit,

$$\frac{d}{dx}\sin x = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} =$$

angle add. form. for sine

$$\lim_{h\to 0} \frac{\sin x\cos h + \cos x \sin h - \sin x}{h} =$$

$$\lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0} \frac{\cos x \cdot \sin h}{h} =$$

$$\sin x \cdot \lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) + \cos x \cdot \lim_{h\to 0}\frac{\sin h}{h}.$$

we want

$$= \sin x \cdot 0 + \cos x \cdot 1 = \cos x.$$

So, we need to show:

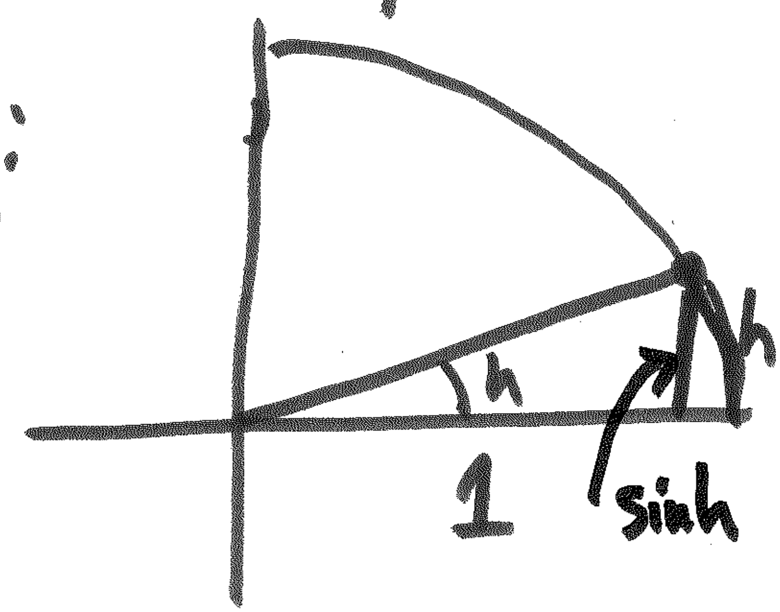
#1 • $\lim\limits_{h \to 0} \dfrac{\sin h}{h} = 1.$

and #2 • $\lim\limits_{h \to 0} \dfrac{\cos h - 1}{h} = 0.$

If this is true, then we have shown $\frac{d}{dx} \sin x = \cos x.$

<u>NOTE</u>: Use #1 to prove #2.

<u>Idea for #1</u>:




Intuitively, as $h \to 0$, arc length + vert. line length are more + more close to each other.

Proof in §3.3.

Similar analysis show:

<u>Thm:</u> $\frac{d}{dx}\sin x = \cos x$, $\frac{d}{dx}\cos x = -\sin x$.

All other trig derivatives come from Prod + chain rules (ie. quot. rule).

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<u>Ex</u>: Find $f^{(5)}(x)$ for $f(x) = -\cos x$.

$f'(x) = \sin x$.

$f''(x) = \cos x$

$f'''(x) = -\sin x$

$f^{(4)}(x) = -\cos x$

$f^{(5)}(x) = \sin x$.

<u>NOTE</u>: <u>Euler's formula:</u> $e^{ix} = \cos x + i\sin x$

**Ex**: Using $\lim_{h\to 0} \frac{\sin h}{h}$ to evaluate similar limits.

$$\lim_{x\to 0} \frac{\sin^2(3x)}{x^2} = \lim_{x\to 0} \frac{\sin(3x)}{x} \cdot \frac{\sin(3x)}{x}$$

$$= \left(\lim_{x\to 0} \frac{\sin(3x)}{x}\right)^2 = \left(\lim_{x\to 0} \frac{3 \cdot \sin(3x)}{3x}\right)^2$$

$$= \left(3 \cdot \underbrace{\lim_{x\to 0} \frac{\sin(3x)}{3x}}_{=1}\right)^2 = (3 \cdot 1)^2 = 9.$$