9/9/16 9 AM

1. Turn in written assignment.
2. Next week: A5, A6, A7, Quiz 3
3. With neighbors: Explain the behavior of $\sin\left(\frac{\pi}{x}\right)$ as $x \to 0$.

   → Follow-up Q: Does $\sin\left(\frac{\pi}{x}\right)$ have any infinite limits? why or why not?

Q: What happens for $\sin(t)$ when

$$t = 10,\ 100,\ 1000,\ 10{,}000,\ \text{etc?}$$

A:
- Oscillating about x-axis
- Bound by $y=1$, $y=-1$.
- Quiet solitude ---

Observation: As $t \to \infty$, i.e. t gets large, $\sin(t)$ oscillates due to circular motion, and stays between 1 + -1.

$y = \sin\left(\frac{\pi}{x}\right)$. As $x \to 0$, x gets small.

<u>Q:</u> What happens to $\sin\left(\frac{\pi}{x}\right)$ when

$x = \cancel{10, 100, 1000}$

$\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10{,}000}$, etc.

<u>A:</u>

$$\left.\begin{aligned} \sin\left(\frac{\pi}{1/10}\right) &= 0 \\ \sin\left(\frac{\pi}{1/100}\right) &= 0 \\ \sin\left(\pi/1/1000\right) &= 0 \end{aligned}\right\}$$

Think:

$\sin\left(\pi / 1/10^a\right) =$

$\sin\left(10^a \pi\right)$

multiple of 2π.

So, 0.

If you try

$$\sin\left(\frac{\pi}{1/998{,}211}\right),$$

~~probably not zero...~~ ← Bad guess... But okay.

similarly for other small unit fractions w/ odd den.

Q: Is this true?

$$\sin\left(\frac{\pi}{1/998\,211}\right) = \underbrace{\sin(998{,}211\,\pi)}_{\text{what is this?}}$$

998,211 half-turns.

You end here!

so, value is y-coord, = 0.

Conclusion: As $x \to 0$, $\frac{\pi}{x}$ is very large + or −, so $\sin\left(\frac{\pi}{x}\right)$ rapidly oscillates from -1 to 1. So, $\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$ does not exist.

Ex: Last time,

$$f(t) = \frac{\sqrt{4+t}-2}{t}$$

gave us trouble as $t \to 0$.

Clever idea: $(a-b)(a+b) = a^2 - b^2$.

Aside: ~~FOIL~~ Distributive Law.

$$(a-b)(a+b) = (a-b)a + (a-b)b$$
$$= a^2 - ba + ab - b^2$$

$(a+b+c)(d+e+f)$ ← dist. law works here!

<u>use clever idea:</u>

$$f(t) = \frac{\sqrt{4+t}-2}{t}\,\frac{(\sqrt{4+t}+2)}{(\sqrt{4+t}+2)}$$

$$= \frac{\cancel{4}+\cancel{t}-\cancel{4}}{\cancel{t}(\sqrt{4+t}+2)} = \frac{1}{\sqrt{4+t}+2}.$$

$$\lim_{t\to 0} \frac{1}{\sqrt{4+t}+2}.$$

<u>Dream</u>: Just plug in $t=0$, get $\frac{1}{4}$, go home.

<u>Problem</u>: This doesn't always work.

Limit Laws allow us to be careful and accurate w/ this situation.

<u>Limit Laws</u>: See § 2.3 for a long list.

<u>Ex</u>: $\lim\limits_{t\to 0} \dfrac{\sqrt{4+t}-2}{t} = \lim\limits_{t\to 0} \dfrac{1}{\sqrt{4+t}+2} \underset{\substack{\uparrow \\ \text{if den.} \\ \text{limit ends} \\ \text{up} \neq 0}}{=}$

$\dfrac{1}{\lim\limits_{t\to 0}\left(\sqrt{4+t}+2\right)} \underset{\substack{\text{sum} \\ \text{and} \\ \text{constant} \\ \text{function laws}}}{=} \dfrac{1}{\left(\lim\limits_{t\to 0}\sqrt{4+t}\right)+2} \underset{\sqrt{\cdot}\ \text{law}}{=} \dfrac{1}{\sqrt{\lim\limits_{t\to 0} 4+t}+2}$

$\underset{\text{sum law}}{=} \dfrac{1}{\sqrt{4+0}+2} = \dfrac{1}{4}.$

(I) Direct Substitution Thm:

If f is a polynomial or a rational function, and if "a" is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a).$$

why? You apply Quotient, sum, const. mult, product laws repeatedly.

(II) SQUEEZE THM:

If $f(x) \le g(x) \le h(x)$ near a, $x \ne a$, and if $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$ then $\lim_{x \to a} g(x) = L$ as well.

Ex: Show that $\lim_{x \to 0} x^4 \cdot \sin\left(\frac{\pi}{x}\right) = 0.$

Step 1: ID a good bound for one or more fns.

$$-1 \leq \sin\left(\frac{\pi}{x}\right) \leq 1.$$

Step 2: Multiply your inequalities by an appropriate fn.

$$\underbrace{-x^4}_{f(x)} \leq \underbrace{x^4 \sin\left(\frac{\pi}{x}\right)}_{g(x)} \leq \underbrace{x^4}_{h(x).} \quad \text{since } x^4 \geq 0$$

Since $\lim_{x \to 0} -x^4 = 0 = \lim_{x \to 0} x^4$, we know $\lim_{x \to 0} x^4 \sin\left(\frac{\pi}{x}\right) = 0.$

9/12/16 9 AM

1. Sign attendance
2. 3 WebWorks this week, 1 Quiz Thurs.
3. Exam info will be sent this week for all students

→ google "weierstrass nowhere differentiable"

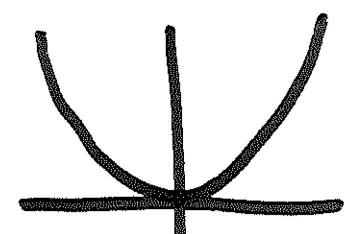
→ <u>Def^{n}</u>: A function f is <u>continuous</u> (cts) <u>at a number a</u> if

$$\lim_{x \to a} f(x) = f(a).$$

i.e. direct sub. can be used for limit at a.

<u>Note</u>: Most "standard" fns are cts.

<u>Ex</u>: $f(x) = x^2$



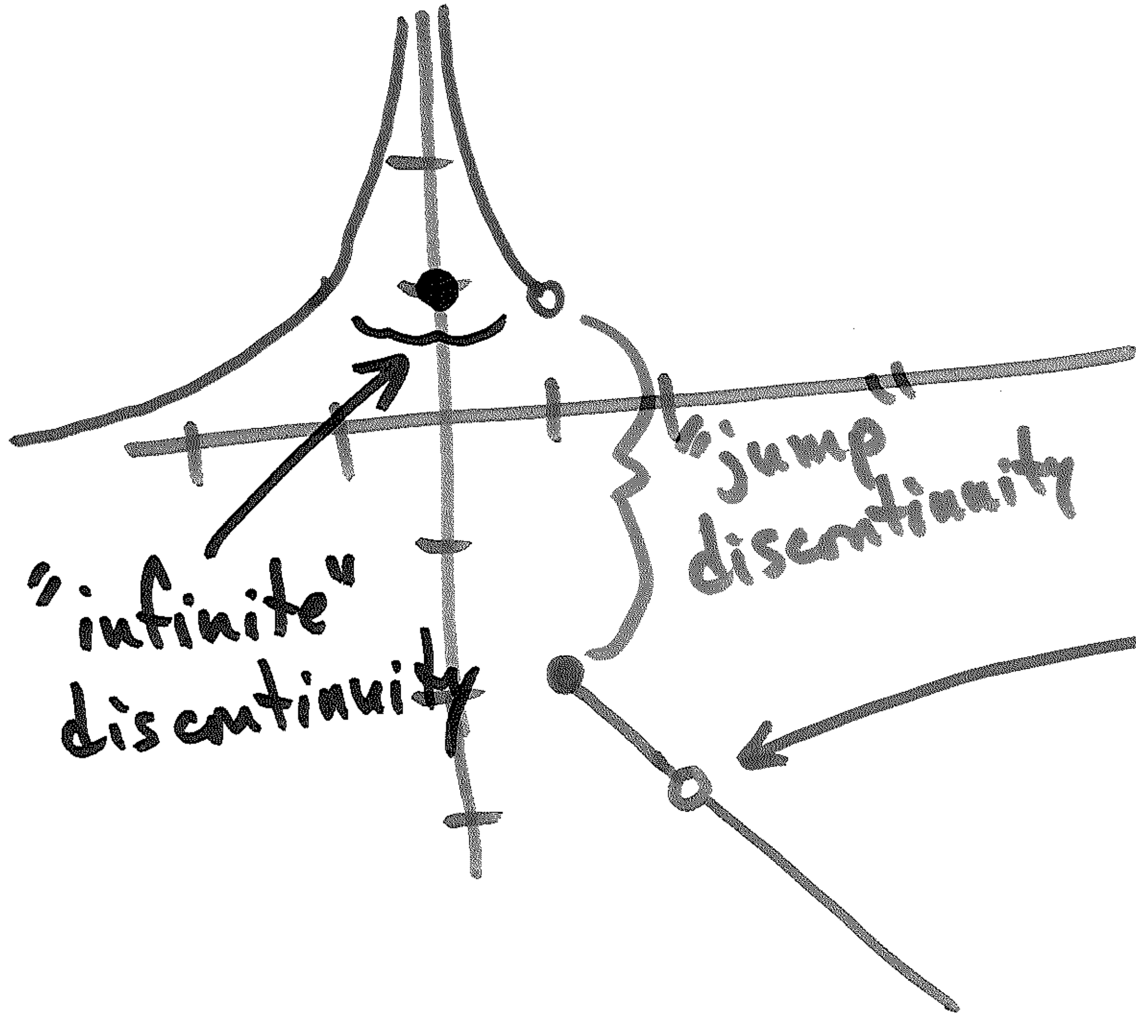
Need 3 things for continuity:

1. $f(a)$ is defined
2. $\lim\limits_{x \to a} f(x)$ exists
3. $\lim\limits_{x \to a} f(x) = f(a)$.

$\Big\downarrow$ check in this order.

<u>Ex:</u> $f(x) = \begin{cases} \dfrac{-x^2+x+2}{x-2} & x \geq 1 \\ 1/x^2 & x<0 \text{ or } 0<x<1 \\ 1 & x=0. \end{cases}$

w/ neighbors: draw this graph.



$x=0,\ f(x)=1$

$1/x^2 \quad x<0$ or $0<x<1$

$$\frac{-x^2+x+2}{x-2} \qquad x \geq 1$$

$= -x-1$

except at $x=2$.

removable discontinuity.

b/c $f(2)$ not defined, but $\lim_{x \to 2} f(x)$ exists.

To fix this, define $f(2)$ to be -3, & get cts at 2.

Thm: Continuity at a is preserved by $+, -, \times, \div$, & mult. by a constant.

as long as you divide by $\neq 0$ values.

Thm: These fns are cts at all pts in their domain:

- polynomials & rational fns
- trig & inverse trig fns
- exponential & log fns.

Ex: $\lim\limits_{x \to 2\pi} \underbrace{\dfrac{\cos(x) + x^2}{3 - \sin x}}$.

Q: Is this a cts fn?

A: Yes.

So, limit is $\dfrac{\cos(2\pi) + (2\pi)^2}{3 - \sin(2\pi)}$.

$= \dfrac{1 + 4\pi^2}{3}$.

Thm: If f and g are cts, g cts at a, f cts at $g(a)$, then $f \circ g$ is cts at a.

Ex: $\sin(\cos(x^4))$ is cts at all real #'s.

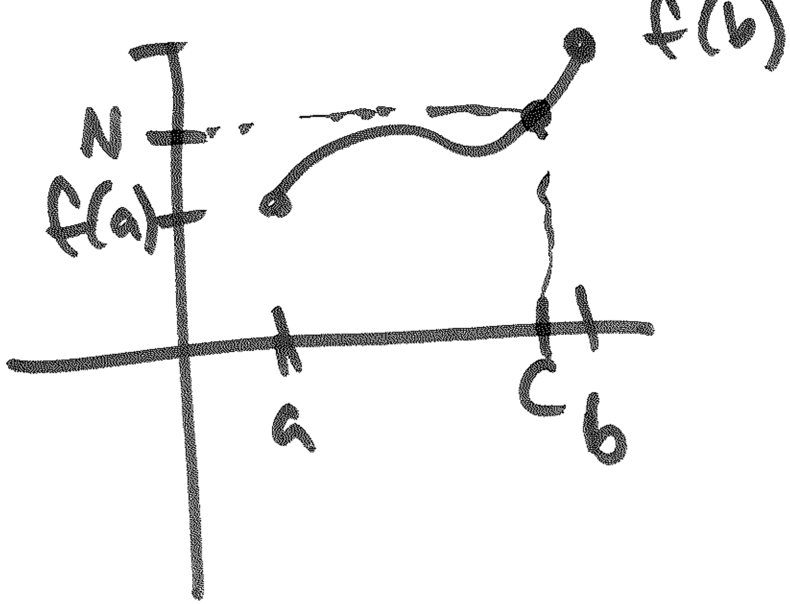
<u>Note</u>:

- Read §2.5 for def$^{\underline{n}}$ of one-sided continuity.
- $f(x)$ is cts on an interval if it is cts at every point in the interval.

  so, f is "cts" if it is cts at every point in domain of f.

Intermediate Value Thm:

Suppose f is cts on $[a,b]$, with $f(a) \neq f(b)$, and N satisfies $f(a) < N < f(b)$. Then there is a c in (a,b) such that $f(c) = N$.



Note: This matters for showing solutions exist.

Ex: If $f(x) = x^2 + 10\sin x$, show there is a soln to $f(x) = 1000$.

Step1: Is f cts?

Yes, since x^2, $\sin x$ are cts

& we use + & const. mult.

Step2: Find a w/ $f(a) < 1000$,

Find b w/ $f(b) > 1000$.

Guess & check now...

$f(0) = 0 < 1000$. Use $a = 0$.

$f(100) = 100^2 + 10\sin(100)$ use $b = 100$.

$\geq 10{,}000 - 10 > 1000$.

So, there is a soln in $(0, 100)$.

9/14/16 9AM

1. Quiz + W.A. solutions are now posted.
2. **Exam 1**: Review Session Monday, 6-8PM, FB 200.
3. **Q:** with neighbors:

Graph $\frac{x^2-1}{x^2+1}$, $\frac{x-1}{x^2+1}$, $\frac{x^2-1}{x+2}$.

Why do these 3 functions behave as they do when x gets very large?

Exam 1:

5 True/False

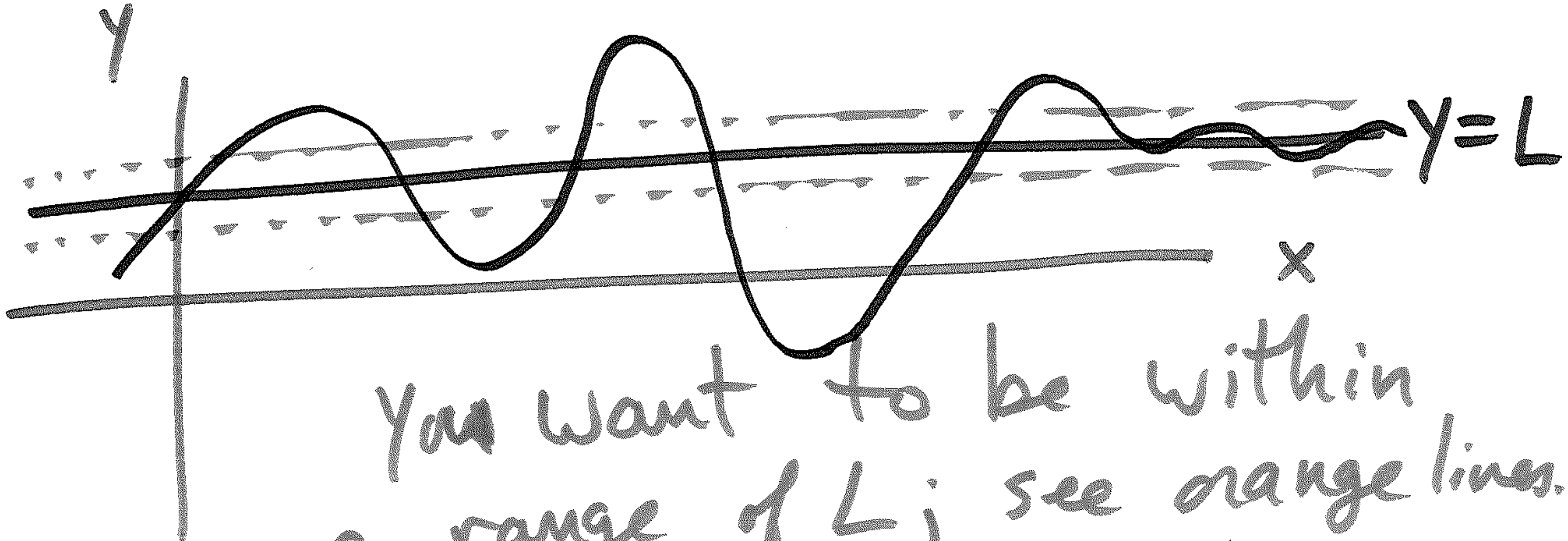
10 Mult. Choice

4 Free Response

Def$^{\underline{n}}$: Let f be defined on (a, ∞) for some a. Then we write

$$\lim_{x \to \infty} f(x) = L$$

if the value of $f(x)$ can be made as close as we want to L by requiring x to be sufficiently large.



You want to be within a range of L; see orange lines.

Graph of f must eventually stay within those bounds, no matter how tight the bound is.

<u>Note:</u>
- Similar def^n for $\lim_{x \to -\infty} f(x) = L$.
- If $\lim_{x \to \pm\infty} f(x) = L$, we call L a horizontal asymptote for f.

  ($\pm$: one or the other or both)

__Q__:

- What is $\lim \frac{1}{x^5}$ as $x \to \infty$ and $x \to -\infty$?

- What is $\lim \frac{1}{\sqrt{x}}$ as $x \to \infty$ and $x \to -\infty$?

__$\frac{1}{x^5}$__:

As $x \to \infty$, x^5 gets larger + larger.

i.e. $x^5 \to \infty$.

So, $\frac{1}{x^5} \to 0$.

As $x \to -\infty$, x^5 gets very large negative.

i.e. $x^5 \to -\infty$

So, $\frac{1}{x^5} \to 0$.

$\frac{1}{\sqrt{x}}$ behaves the same as $x \to \infty$,

i.e. $\sqrt{x} \to \infty$ as $x \to \infty$.

So, $\frac{1}{\sqrt{x}} \to 0$.

<u>NOTE</u>: $\lim_{x \to -\infty} \frac{1}{\sqrt{x}}$ does not exist because domain of $\sqrt{x}$ is $[0, \infty)$.

<u>Thm</u>:
- If $r > 0$ is a rational #, then $\lim_{x \to \infty} \frac{1}{x^r} = 0$
- If $r > 0$ is a rational # and x^r is defined for x in $(-\infty, \infty)$, then $\lim_{x \to -\infty} \frac{1}{x^r} = 0$.

Key idea: Use algebra to reveal geometric behavior of graph of f as $x \to \infty$.

Ex: $\lim\limits_{x\to\infty} \dfrac{x^2-1}{x^2+1} = \lim\limits_{x\to\infty} \dfrac{\frac{1}{x^2}}{\frac{1}{x^2}} \cdot \dfrac{x^2-1}{x^2+1}$

$$= \lim_{x\to\infty} \frac{\frac{x^2}{x^2} - \frac{1}{x^2}}{\frac{x^2}{x^2} + \frac{1}{x^2}} = \lim_{x\to\infty} \frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}} = \frac{1-0}{1+0} = 1$$

use limit laws

Ex: $$\lim_{x\to\infty} \frac{4x^7 - x^6 + 1}{\frac{1}{5}x^7 - x + 3} =$$

I see: highest power in num + den is x^7. so, divide num + den by x^7

$$\lim_{x\to\infty} \frac{4 - \frac{1}{x} + \frac{1}{x^7}}{\frac{1}{5} - \frac{1}{x^6} + \frac{3}{x^7}} \underset{\text{limit laws}}{=} (*)$$

limit laws ← on exam, show all steps here.

$$\frac{4 - 0 + 0}{\frac{1}{5} - 0 + 0} = 20.$$

Details:

$$(*) \underset{\text{if den limit is} \neq 0}{=} \frac{\lim_{x\to\infty}\left(4 - \frac{1}{x} + \frac{1}{x^7}\right)}{\lim_{x\to\infty}\left(\frac{1}{5} - \frac{1}{x^6} + \frac{3}{x^7}\right)} \underset{\text{sum law}}{=} \frac{\lim_{x\to\infty} 4 - \lim_{x\to\infty} \frac{1}{x} + \lim_{x\to\infty} \frac{1}{x^7}}{\lim_{x\to\infty} \frac{1}{5} - \lim_{x\to\infty} \frac{1}{x^6} + \lim_{x\to\infty} \frac{3}{x^7}}$$

$$= (\star)$$

$$(\star) = \frac{4-0+0}{\frac{1}{5}-0+3\cdot 0} = \frac{4}{1/5} = 20.$$

const. law, const. mult. law, Thm about $\frac{1}{x^r}$

<u>Ex</u>: $\lim\limits_{x\to\infty} \dfrac{\sqrt{3x^2-x+1}}{2x+2}$.

<u>I see</u>: $\sqrt{3x^2-x+1} \approx \sqrt{3x^2}$ for large x.

$= \sqrt{3}\,x$.

Thus, "highest term" in num + den is x. I divide num + den by x.

$$\lim_{x\to\infty} \frac{\frac{1}{x}\sqrt{3x^2-x+1}}{\frac{1}{x}(2x+2)} = \lim_{x\to\infty} \frac{\sqrt{3-\frac{1}{x}+\frac{1}{x^2}}}{2+\frac{2}{x}} = \frac{\sqrt{3}}{2}.$$

9/16/16 9AM

1. See Canvas Announcement for Exam 1 info.

2. Sign attendance

3. Discuss w/ neighbors: What topic are you most + least concerned about for Exam 1? Why?

A few more limits at infinity problems:

Ex: $$\lim_{x\to\infty} \frac{x^2-1}{x+2} = \lim_{x\to\infty} \frac{x-\frac{1}{x}}{1+\frac{2}{x}} = \frac{\lim_{x\to\infty} x-\frac{1}{x}}{1} = \infty.$$

<u>Q</u>: What would you normally do?

- divide by $x \rightarrow \dfrac{x-\frac{1}{x}}{1+\frac{2}{x}}$ (*)
- divide by $x^2 \rightarrow \left(1-\frac{1}{x^2}\right) / \left(\frac{1}{x}+\frac{2}{x^2}\right)$.
- divide by highest term in den.

<u>Claim:</u> (*) is better, because you want den. to go to a non-zero value in the limit.

Observation: $(a-b)(a+b) = a^2 - b^2$.

is super useful!

Ex: $\lim_{x\to\infty}\left[\sqrt{3x^2-x+1} - (2x+2)\right]$

Thought experiment: $\sqrt{3x^2-x+1} \approx \sqrt{3x^2} = \sqrt{3}\,x$ for large x.

So, $\sqrt{3x^2-x+1} - (2x+2) \approx \sqrt{3}\,x - 2x$ for large x.

$= \underbrace{(\sqrt{3}-2)}_{<0}x$.

Seems limit goes to $-\infty$.

NOT A SOLUTION!
INTUITION!

Q: How do we actually justify this?

use $(a-b)\frac{(a+b)}{a+b}$.

$$\left(\sqrt{3x^2-x+1}-(2x+2)\right)\frac{\left(\sqrt{3x^2-x+1}+2x+2\right)}{\sqrt{3x^2-x+1}+2x+2} =$$

$$\frac{3x^2-x+1-(2x+2)^2}{\sqrt{3x^2-x+1}+2x+2} \underset{\text{algebra}}{=} \frac{-x^2-9x-3}{\underbrace{\sqrt{3x^2-x+1}}+2x+2}.$$

For large x, $\approx \sqrt{3}\,x$.
So, divide by x.

$$\underset{\text{algebra}}{=} \frac{-x-9-\frac{3}{x}}{\sqrt{3-\frac{1}{x}+\frac{1}{x^2}}+2+\frac{2}{x}}$$

because

$$\frac{1}{x}\sqrt{3x^2-x+1} = \sqrt{\frac{3x^2-x+1}{x^2}}$$

Thus, $\lim\limits_{x\to\infty}\left[\sqrt{3x^2-x+1}-(2x+2)\right] =$ ← an algebra

$$\lim_{x\to\infty}\frac{-x-9-\frac{3}{x}}{\sqrt{3-\frac{1}{x}+\frac{1}{x^2}}+2+\frac{2}{x}} = \cdots\cdots =$$

limit laws

$$\frac{\left(\lim\limits_{x\to\infty}-x\right)-9-0}{\sqrt{3-0+0}+2+0} = \frac{\lim\limits_{x\to\infty}-x}{\sqrt{3}+2} = -\infty.$$

Q: why not use $\lim\limits_{x\to\infty} f(x)-g(x) = \lim\limits_{x\to\infty} f(x) - \lim\limits_{x\to\infty} g(x)$?

If we tried, we get

$$\underbrace{\lim_{x\to\infty}\sqrt{3x^2-x+1}}_{\infty} - \underbrace{\lim_{x\to\infty} 2x+2}_{\infty} = ?\ldots$$

these limits must be finite.

Q: What should you do to study for ~~Exam~~ 1? on a trail

S'sanswers:

In the forest, looking at a tree.

- Old Exams for practice (topics can change... watch out!)
- Practice Problems in book (odd numbered have answers)
- Review Rec. Worksheets + complete problem sets.
- Memorize theorems
- Form a study group
- Study on mathskeller for help

- Meet w/ TA or prof
- Review notes
- Study things you thought were harder....

what is MY trail map?

First thing to do: Form a study group!

2nd thing to do: Ask "What have we learned + why?"

Organizing the answer to this is hard! Balance breadth of topics w/ concise representation you can see.

→ Concept Maps are good tools for this.