Mon, April 11: Exam Review

(1) Evaluate

$$\int_0^1 \frac{1}{(x-1)(x^2+1)}\,dx$$

when $x=1$, divide by zero.

NOTES:

- P.F.D.
- Improper int.
- ~~Applications~~ ← This is not application!

Graph ⇒ asymp. at $x=1$.

<u>Initial study</u>: I need to handle both P.F.D. + Asympt. at $x=1$, so improper int.

<u>Step 1</u>: Do PFD.

<u>Step 2</u>: Handle improper int.

Step 1 will give $\dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+1}$.

So, Step 2 will consider

$$\underbrace{\int_0^1 \frac{A}{x-1}\,dx}_{\text{Here is the improper part.}} + \underbrace{\int_0^1 \frac{Bx+C}{x^2+1}\,dx}_{\text{no asymptote at } x=1 \text{ or } x=0, \text{ so I can int.}}$$

Step 1: P.F.D for $\dfrac{1}{(x-1)(x^2+1)}$ is :

$$\frac{1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$= \frac{Ax^2 + A + (Bx+C)(x-1)}{(x-1)(x^2+1)}$$

So, $1 = Ax^2 + A + Bx^2 + Cx - Bx - C$

says

$$\begin{aligned} A + B &= 0 \\ -B + C &= 0 \\ A \quad - C &= 1 \end{aligned}$$

so, $B = C$ gives

$A + C = 0$

$A - C = 1$

Now, $A = -C$ gives (in 2nd eqn)

$-C - C = 1$, so $C = \frac{-1}{2}$.

Then $A = \frac{1}{2}$, $B = -1/2$.

$$\frac{1}{(x-1)(x^2+1)} = \frac{1/2}{x-1} + \frac{-1/2 \cdot x + -1/2}{x^2+1}$$

End step 1.

<u>Step 2</u>: Consider $\displaystyle\int_0^1 \frac{1/2}{x-1}\,dx - \int_0^1 \frac{1/2 \cdot x + 1/2}{x^2+1}\,dx$

<u>Comment</u>: Right-hand term is finite, i.e. it is a real number.

First, compute improper int. + see if it diverges or converges.

$$\frac{1}{2}\int_0^1 \frac{1}{x-1}\,dx = \frac{1}{2}\int_{-1}^0 \frac{1}{u}\,du$$ ← diverges! by thm in book.

$u = x-1$
$du = dx$

$$= -\frac{1}{2}\int_0^1 \frac{1}{u}\,du$$

by symmetry w/ flip of $y = \frac{1}{u}$.

So, $\int_0^1 \frac{1}{(x-1)(x^2+1)}\,dx$ diverges.

Ex: Does $\displaystyle\int_2^\infty \frac{x}{x^3+1}\,dx$ converge?

Ideas: 1) Comparison test.

2) lim as $R \to \infty$ of $\displaystyle\int_2^R \frac{x}{x^3+1}\,dx$.

Most efficient, because $\dfrac{x}{x^3+1}$ is a difficult integrand.

Note: $\dfrac{x}{x^3+1} \le \dfrac{x}{x^3} = \dfrac{1}{x^2}$.

$$\int_2^\infty \frac{x}{x^3+1}\,dx \le \int_2^\infty \frac{1}{x^2}\,dx \leftarrow \underline{\text{converges!}}$$

So, by comp. test, original integral converges.

Ex: Find Cartesian Coords for the following polar curve:

$$r = \frac{2}{\sin\theta}.$$

- Mult. by r
- $r = \sqrt{x^2+y^2}$.
- $y = r\sin\theta$
- Plot w/ calculator, see it's a line.

Standard conversion:

$$(r,\theta) \longrightarrow (r\cos\theta,\ r\sin\theta).$$

Here, we have

$$\left(\frac{2}{\sin\theta}, \theta\right)_{pol} \longrightarrow \left(2\cdot\frac{\cos\theta}{\sin\theta},\ 2\cdot\frac{\sin\theta}{\sin\theta}\right)_{rect.}$$

$$= (2\cdot\cot\theta,\ 2)_{rect.}$$

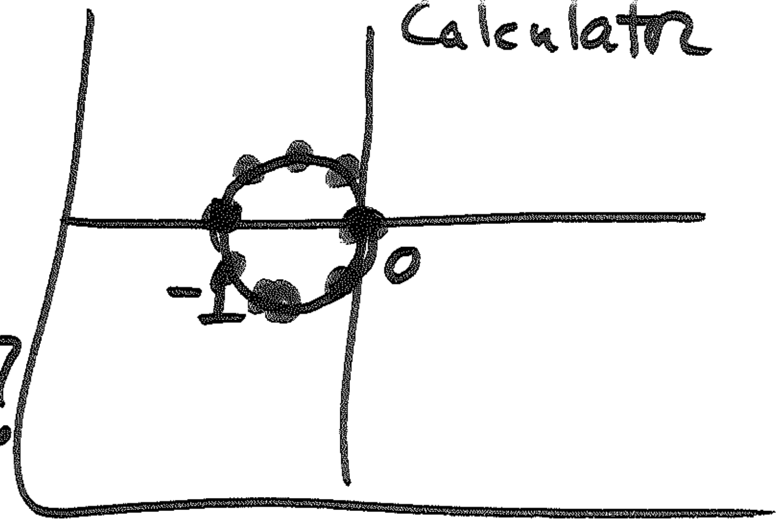
$y=2$

Answer: $y=2$.

NOTE: $y = r\sin\theta = 2$ gives $y=2$.

Ex: Sketch $r = -\cos\theta$.

Q: Why is this a circle of radius $1/2$ centered at $(-\frac{1}{2}, 0)$?



There is an example in book: $r = 2a\cos\theta$ is a radius a circle centered at $(a, 0)$. Setting $a = -\frac{1}{2}$ gives $r = -\cos\theta$, our example.

If no calc:

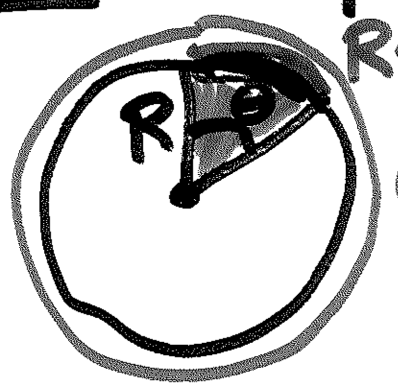
Check at $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$, etc.

Wed, April 13

§11.4: Area + Arc length in polar coords.

Q: How do we find the area of a sector of a circle of radius R?

A: Proportional Reasoning.



$R\cdot\theta$ is arc length if θ is in radians.

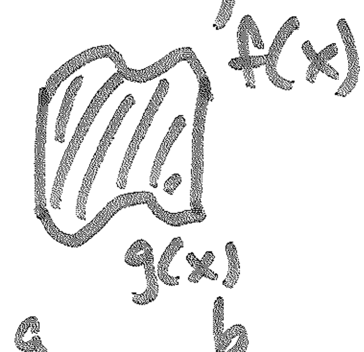
$C = 2\pi R$.

$$\frac{R\cdot\theta}{\underbrace{2\pi R}_{\text{circumference}}} = \frac{\text{Area of sector}}{\underbrace{\pi R^2}_{\text{area of disk}}}$$

Solving gives Area of Sector of radius R circle w/ angle θ is $A = \frac{R^2\theta}{2}$. ← θ in radians.

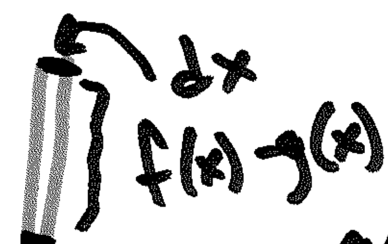
Q: what is area of sector of a polar curve?

NOTE: so far, calc. has given us areas of



This Q is about area of this shape.

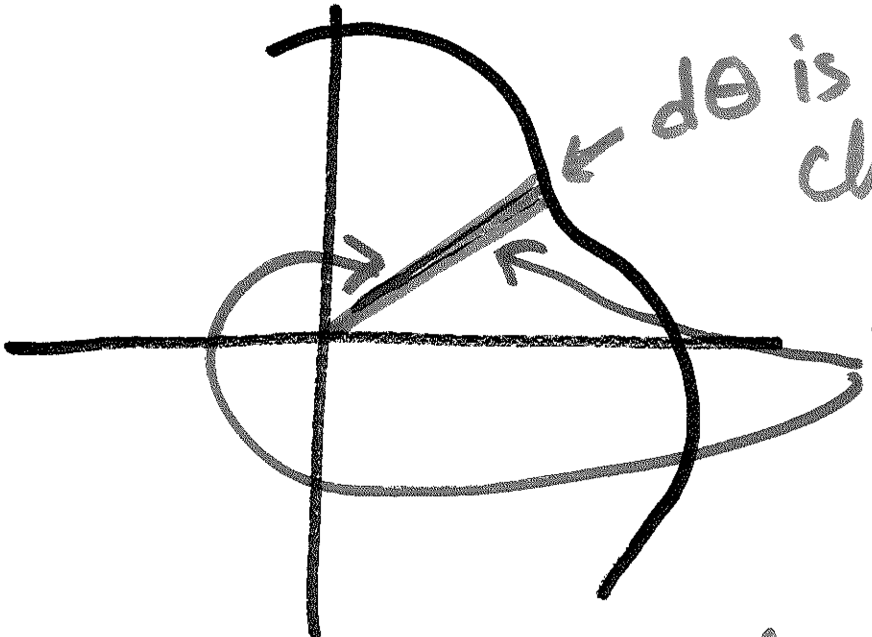
in this case, we use $\int \underbrace{[f(x)-g(x)]}_{\text{height}} \underbrace{dx}_{\text{infinitely thin width.}}$



area is product!

Same idea w/ polar curve: Say $r = f(\theta)$.

Then curve $(r, \theta) = (f(\theta), \theta)$.



$\leftarrow d\theta$ is an infinitely small change in angle

then both of these radial lines are of length $f(\theta)$.

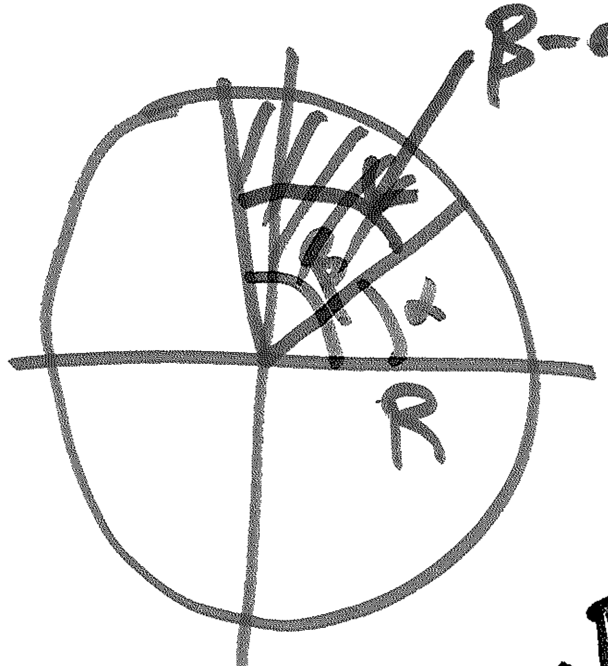
Area of green "infinitely thin" sector is $\frac{1}{2} \cdot f(\theta)^2 \, d\theta$ from our previous $A = \frac{R^2\theta}{2}$.

Thm: If $f(\theta)$ is cts, then area bounded by $r = f(\theta)$ between angles $\theta = \alpha$ and $\theta = \beta$ with $\alpha < \beta$ is

$$\int_{\alpha}^{\beta} \frac{1}{2} f(\theta)^2 \, d\theta$$

Note: If you do not like "infinitely small" as an idea, see §11.4 for an explanation of this using Riemann Sums.

Ex: Use the formula to find area of sector of circle of radius R btwn angles $\alpha < \beta$.



Step 1: What is $f(\theta)$ for a circle of radius R?

$f(\theta) = R$.

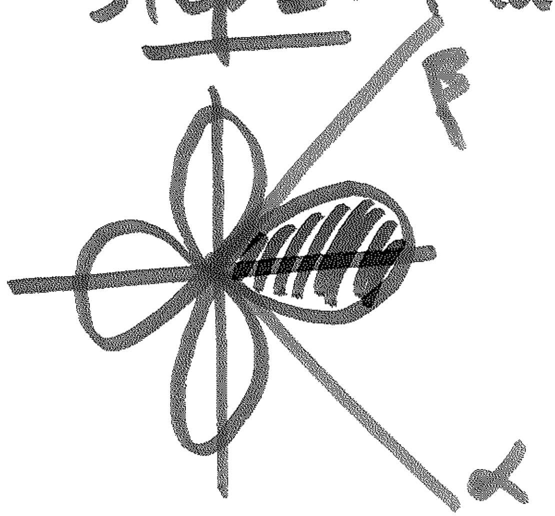
Step 2: So, plug this into our formula.

$$\int_\alpha^\beta \underbrace{\frac{1}{2}R^2}_{\text{constant w.r.t. } \theta} d\theta = \frac{1}{2}R^2 \int_\alpha^\beta d\theta = \frac{1}{2}R^2\theta \Big|_\alpha^\beta$$

$$= \frac{1}{2}R^2(\beta-\alpha).$$

Ex: Find area of $r = \cos 2\theta$ between $\alpha = -\frac{\pi}{4}$, $\beta = \frac{\pi}{4}$.

Step 1: Make sense of problem!



Area: Plug into formula!

$$\frac{1}{2}\int_{-\pi/4}^{\pi/4} r^2\, d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2(2\theta)\, d\theta$$

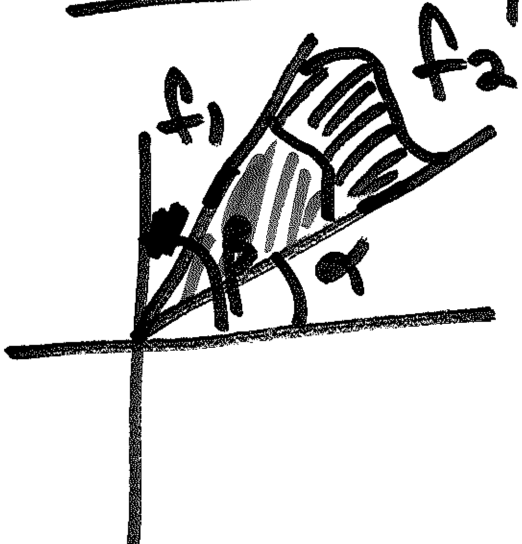
$= (*)$

$\cos^2\theta = \frac{1+\cos 2\theta}{2} \Rightarrow \cos^2(2\theta) = \frac{1+\cos(2\cdot 2\theta)}{2}$.

$$\circledast = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \frac{1+\cos 4\theta}{2}\, d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{\sin 4\theta}{4}\right]_{-\pi/4}^{\pi/4} = \pi/8 .$$

<u>Thm</u>: Say $f_2(\theta) \geq f_1(\theta)$ on $\alpha \leq \theta \leq \beta$.



The area between these two polar curves is

$$\underbrace{\int_\alpha^\beta \frac{1}{2} f_2(\theta)^2 d\theta}_{\text{entire area}} - \int_\alpha^\beta \underbrace{\frac{1}{2} f_1(\theta)^2 d\theta}_{\text{purple area}} .$$

Talk w/ your neighbors:

Start w/ a number n.

If n is even, divide by 2.

If n is odd, Multiply by 3 and add 1.

Then repeat. **Q**: What happens?

Ex: $n=3$

$3, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, \ldots$

what happens for other values of n?

$2, 5, 7, 1, 75, \ldots$

- 11, 34, 17, 52, 26, 13, 40, 20, 10,
  5, 16, 8, 4, 2, $\underbrace{1, 4, 2}_{\text{repeats}}$, 1, 4, 2, 1, ...
- 362, 181, 544, 272, 136, 68,
  34, 17,

Q: For every n, do you eventually end up repeating 1, 4, 2, 1, 4, 2, ... ?

Noone knows the answer.

It is called $3x+1$ problem and
"Collatz" conjecture.
Unsolved since 1937.
This is a <u>discrete</u> dynamical
system. Attempts to solve this
use "discrete derivatives".

Friday, April 15

1. Please fill out (short) Canvas Survey.

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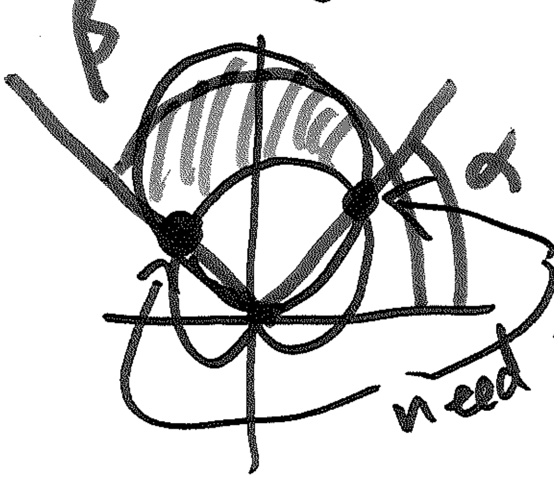
§11.4 cont.

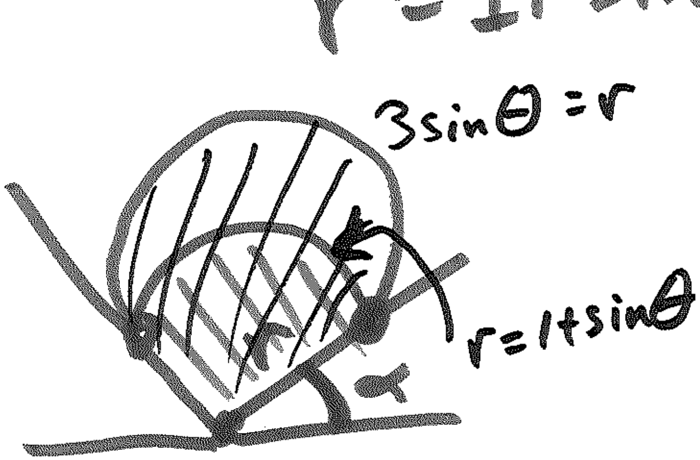
Ex: Find area ~~[illegible]~~ of region that lies inside $r = 3\sin\theta$ and outside $r = 1 + \sin\theta$.



need to find these pts of intersection.

Q: What are the angles α & β for the polar coords of pts of intersection?

we need pts where for the same θ-value, both $r = 3\sin\theta$ & $r = 1 + \sin\theta$ are equal.



So, set $3\sin\theta = 1 + \sin\theta$ and solve.

$\Rightarrow \sin\theta = \frac{1}{2}$.

So, $\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ are the two solutions between 0 & π.

So, $\alpha = \frac{\pi}{6}$, $\beta = \frac{5\pi}{6}$.

Now, we can do an area integral.

So, Area is

$$\frac{1}{2}\int_{\pi/6}^{5\pi/6} (3\sin\theta)^2\, d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1+\sin\theta)^2\, d\theta$$

$$\overset{\text{algebra}}{=} \frac{1}{2}\int_{\pi/6}^{5\pi/6} (8\underbrace{\sin^2\theta}_{\text{problem.}} - 1 - 2\sin\theta)\, d\theta$$

use: $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (4 - 4\cos 2\theta - 1 - 2\sin\theta)\, d\theta$$

$$= \pi.$$ ← Same area as unit circle.

Arc Length:

Thm: For a polar curve $r=f(\theta)$ w/ angles $\alpha \leq \theta \leq \beta$, the arc length of the curve is

$$\int_{\alpha}^{\beta} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta .$$

Ex: $r = f(\theta) = 5\cos\theta$, $0 \le \theta \le \frac{3\pi}{4}$, find length of polar arc.

$$\int_0^{3\pi/4} \sqrt{25\cos^2\theta + 25\sin^2\theta}\, d\theta = \int_0^{3\pi/4} \sqrt{25}\, d\theta = \frac{15\pi}{4}.$$

Ex: $r = 2^\theta$, $0 \le \theta \le 2\pi$.

General technique: If you have a^b, convert to base e. Do this by

$$a^b = e^{\ln(a^b)} = e^{b \cdot \underbrace{\ln(a)}}$$

If a is constant, $\ln(a)$ is constant.

Here, $2^{\theta} = e^{\ln(2)\theta} = e^{\theta \cdot \ln(2)}$.

If $f(\theta) = 2^{\theta} = e^{\ln(2)\theta}$,

$f'(\theta) = \ln(2) \cdot e^{\ln(2)\theta} = \ln(2) \cdot 2^{\theta}$.

Arc length is

$$\int_0^{2\pi} \sqrt{2^{2\theta} + (\ln(2))^2 \cdot 2^{2\theta}}\, d\theta =$$

$$\int_0^{2\pi} \sqrt{2^{2\theta}\left(1 + \ln(2)^2\right)}\, d\theta =$$

$$\sqrt{1+\ln(2)^2}\int_0^{2\pi} 2^\theta \, d\theta =$$

$$\sqrt{1+\ln(2)^2}\left[\frac{2^\theta}{\ln(2)}\right]\Bigg|_0^{2\pi} = \sqrt{1+\ln(2)^2}\cdot\left(\frac{4^\pi-1}{\ln(2)}\right).$$

<u>Recall</u>: $2^\theta = e^{\ln(2^\theta)} = e^{\theta\cdot\ln(2)}$.

$$\left(\square = e^{\ln(\square)}\right)$$

$$\ln(2^\theta)$$
$$= \theta\cdot\ln(2)$$

Q: Where does formula $\sqrt{f^2 + f'^2}$ come from?

A: Use arc length formula for parametrized curves, where our θ is our t.

If $(f(\theta), \theta)$ is a polar curve, then rect. coords are $\underbrace{\left(f(\theta)\cos\theta,\ f(\theta)\sin\theta\right)}_{\text{This is a par. curve.}}$.

Say $\alpha \leq \theta \leq \beta$.

We know arc length is

$$\int_\alpha^\beta \sqrt{x'(\theta)^2 + y'(\theta)^2}\, d\theta \underset{\uparrow}{=} \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta.$$

messy computations + trig $\leftarrow$ in book.

Monday, April 18, 2016

- Course Evals, aka TCE's, are open! Please fill it out!
- Today we start §9.1.

Q: what function $f(x)$ satisfies

$$f'(x) = k \cdot f(x) \quad \text{with } f(0) = 1 ?$$

(for a constant k)

$$f(x) = e^{kx} \Rightarrow f'(x) = k \cdot e^{kx}$$ by chain rule.

Model population growth:

Population will $\underbrace{\text{increase}}_{\text{more}}$ if you have more bacteria in a petri dish.

Suppose you have a room w/ temperature S + a cup of coffee w/ coffee temp $f(t)$ at time t. Newton's Law of Cooling says

some constant.

$$f'(t) = \frac{df}{dt} = k\big(\underbrace{f(t) - S}_{\substack{\text{diff. btwn} \\ \text{coffee + air temp.}}}\big)$$

__Ex__: In "control + feedback" system theory, an important __differential equation__ is

$$\frac{df}{dx} = \frac{f(x) \cdot f(\sqrt{x})}{2x}.$$

__Note__: If $f(x) = \frac{\text{\# of prime \#'s} \leq x}{x}$,

= % of #'s between 1 & x that are prime,

$f(x)$ __almost__ satisfies the above eqn.

__Goal__: Consider a __differential eqn__, i.e. an equation involving a function $y = f(x)$, its derivatives, and other functions of x, and try to determine what y is.

__Ex__: $y'' = (\cos x)\,y' + 1$.

__NOTE__: The __order__ of a D.E. is the highest der. that appears. So this example is order 2.

A DE is <u>linear</u> if it can be written

$$a_n(x)\,y^{(n)} + a_{n-1}(x)\,y^{(n-1)} + \cdots + a_1(x)\,y^{(1)} + a_0(x)\,y = b(x).$$

<u>Ex</u>:

$$y''' = e^y + x^2 y' \Rightarrow 1\cdot y''' - x^2 y' = e^y$$

not linear!

$$y'' + x^2 y' + e^x \cos x\, y = 2 + 7x^3 - y'$$

$$\Rightarrow 1\cdot y'' + (x^3+1)\,y' + (e^x \cos x)\,y = 2 + 7x^3$$

functions involving only x, not y's.

__General Rule__: We don't know how to solve DE's. We know how to approximate solns.

__Ex__: Look up Navier-Stokes eqns for fluid flow. ~~The~~ Determining if these eqns have a soln or not has a \$1 million bounty.

We can solve __separable__ eqns.

A D.E. is separable if ⟷ separate x + y variables!

$$y' = \frac{dy}{dx} = f(x) \cdot g(y).$$

General soln technique: Treat dy + dx are separate quantities + rewrite

$$\frac{dy}{g(y)} = dx \cdot f(x)$$

Then, integrate both sides

$$\int \frac{1}{g(y)}\, dy = \int f(x)\, dx.$$

Finally, try to solve for y.

Ex: $xy\frac{dy}{dx} - 3 = 0$. Solve for y.

$\Rightarrow \quad \frac{dy}{dx} = \frac{3}{x} \cdot \frac{1}{y}$ (separate variables)

$\Rightarrow \quad y\,dy = \frac{3}{x}\,dx$ (set up)

$\Rightarrow \quad \int y\,dy = \int \frac{3}{x}\,dx$ (integrate

$\Rightarrow \quad \frac{y^2}{2} = 3\ln|x| + C$

$\Rightarrow \quad y = \pm\sqrt{6\ln|x| + C}$

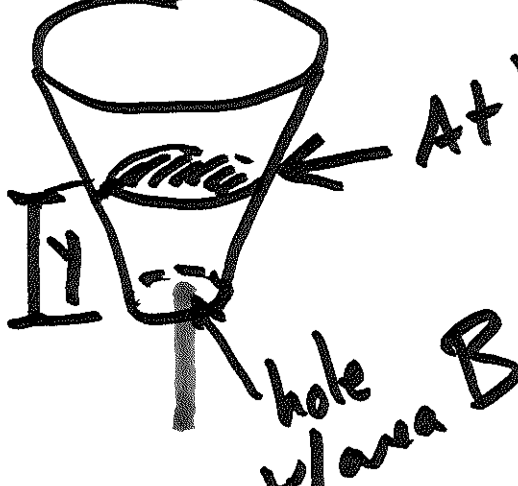
multiplied C by 2, but its still a constant so I didnt change name.

NOTE: You should check that this y satisfies $xy\frac{dy}{dx} - 3 = 0$!

Read your text, §9.1, for more examples <u>and</u> general vs particular solutions.

<u>Point</u>: DE's are used for modeling!

<u>Ex</u>: Water in a tank draining out a small hole at bottom.



At height y, cross sectional area is $A(y)$.

Height $y(t)$ of water in tank is a function of time.

Let $v(y)$ = velocity of water flowing through hole when H_2O at height $y(t)$.

2 ways to measure change in volume over instantaneous time change dt:

$$B \cdot v(y(t)) \cdot dt \quad \leftarrow \text{first way}$$

or $\quad A(y) \cdot dy$

<u>So</u>: $\quad A(y)\,dy = B \cdot v(y)\,dt$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{B \cdot v(y)}{A(y)}.$$

$v(y)$ is given by Torricelli's Law:

$$v(y) = -\sqrt{2gy} = -\sqrt{2 \cdot 9.8\, y}.$$

This leads to separable D.E. See Example 3 in §9.1.

Wednesday, April 20 §9.2 Models with $y' = k(y-b)$

Recall a quantity grows or decays exponentially if

$$Q(t) = Ce^{kt}$$

Then $\frac{dQ}{dt} = k \cdot \underbrace{Ce^{kt}} = kQ(t)$

Using our terminology, we say $Q(t)$ satisfies the differential equation $y' = k \cdot y$

Consider $y' = k(y-b)$ $\quad$ k, b constants

$k \neq 0$

Find general solution using separation of variables.

$$\frac{dy}{dt} = k(y-b)$$

$$\int \frac{dy}{y-b} = \int k\,dt$$

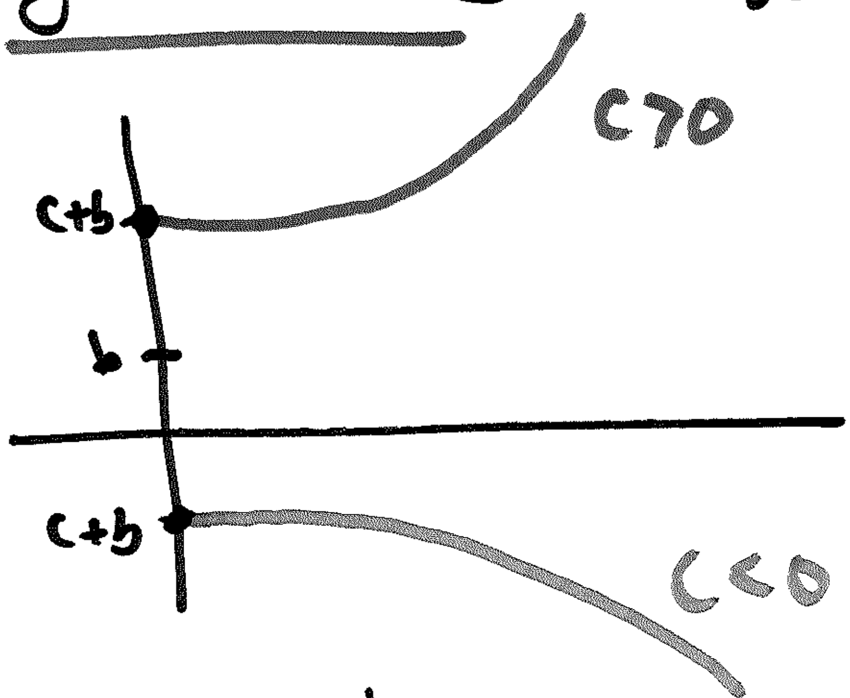
$$e^{\ln|y-b|} = e^{kt+C}$$

$$|y-b| = e^{kt} \cdot \underbrace{e^{C}}$$

$$y-b = Ce^{kt}$$

$$\boxed{y = Ce^{kt} + b}$$

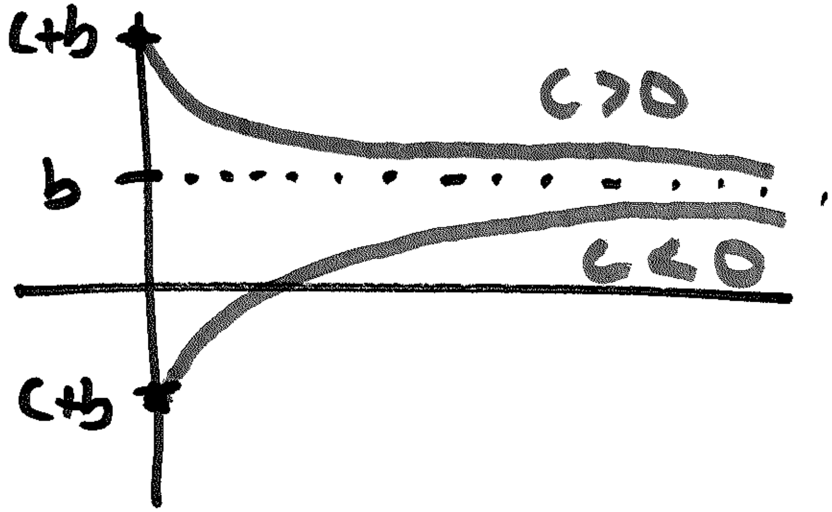
$y = Ce^{kt} + b$ is solution to $y' = k(y-b)$



Note: ending behavior is $+\infty$ or $-\infty$

models annuity
(interest on investment)

$y = Ce^{-kt} + b$ is solution to $y' = -k(y-b)$



Note:

$$\lim_{t\to\infty} Ce^{-kt} + b = b$$

- Newton's Law of Cooling
- free-fall

Application Newton's Law of Cooling

Let $y(t)$ be temp. of a hot object cooling

T_0 ambient temp.

Newton says that $y(t)$ satisfies

$$y' = -k(y - T_0)$$

k is cooling constant

units of $k = (\cancel{\text{min}}\ \text{time})^{-1}$

By what we just showed, general solution is

$$y(t) = Ce^{-kt} + T_0$$

Observe $\lim_{t \to \infty} y(t) = T_0$

Problem A hot metal bar with cooling constant $k = 3.1\ \text{min}^{-1}$ is submerged in water of temp. $10°C$. What is bar's temp. after 30 sec. if the initial temp. was $200°C$?

Solution Temp. of bar satisfies

$$y' = -k(y - T_0) = -3.1(y - 10)$$

General solution

$$y(t) = Ce^{-3.1t} + 10.$$

Find C using $y(0) = 200$. ($\leftarrow$ initial temp.)

$$200 = Ce^{-3.1(0)} + 10 = C + 10$$

$$190 = C$$

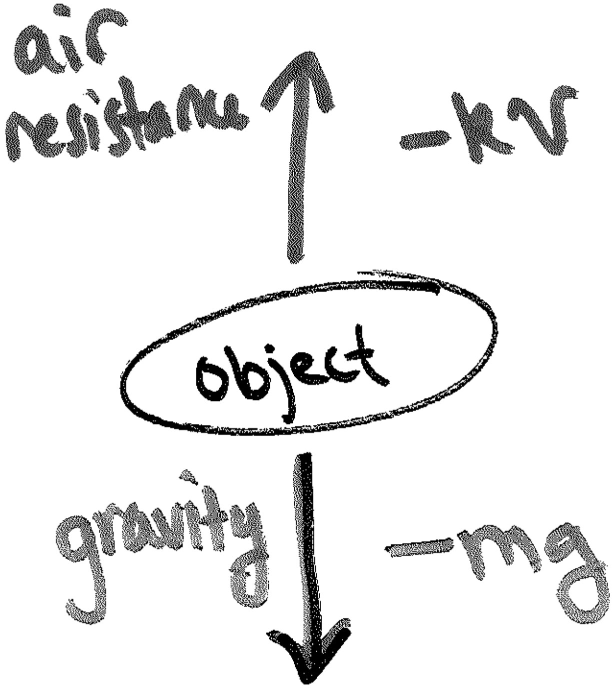
$$y(t) = 190e^{-3.1t} + 10$$

$$y(.5) = 190e^{-3.1(.5)} + 10$$

30 sec = .5 min

$\approx \boxed{50.33^{\circ}C}$ after 30 sec.

Application: Model free-fall with air resistance



m = mars

$g = 9.8\ m/s^2$

v = velocity $(v<0)$

$k>0$ constant

Total Force $= -kv - mg$

By Newton's Law, $F = m \cdot \underset{\substack{\uparrow \\ \text{accel.}}}{a} = m \cdot v'$

$$m \cdot v' = -kv - mg$$

$$v' = -\frac{kv}{m} - g = \frac{-k}{m}\left(v + \frac{g \cdot m}{k}\right)$$

Notice we have diff. eq of the form

$$v' = -k(v-b) \quad \text{where} \quad k = \frac{k}{m}$$

$$b = -\frac{mg}{k}$$

So general solution is

$$\boxed{v(t) = Ce^{-\left(\frac{k}{m}\right)t} - \frac{mg}{k}}$$

Observe

$$\lim_{t\to\infty} \underbrace{Ce^{-\left(\frac{k}{m}\right)t}}_{\to 0} - \frac{mg}{k} = -\frac{mg}{k}$$

terminal velocity.

Problem: A 70-kg skydiver jumps off a plane. How long does it take her to reach half of her terminal velocity if initial velocity is 0. Assume $k = 7$ kg/sec.

Solution Terminal velocity $= -\frac{mg}{k} = \frac{-70(9.8)}{7} = -98$ m/s

Want to find t such that $v(t) = \frac{1}{2}(-98) = -49$ m/s.

$$v(t) = Ce^{-\frac{k}{m}t} - \frac{mg}{k}$$

$$v(t) = Ce^{-\frac{7}{70}t} - 98$$

$$\underset{\overset{\parallel}{0}}{v(0)} = C \cdot e^0 - 98 \implies C = 98$$

$$v(t) = 98e^{-\frac{1}{10}t} - 98$$

$$-49 = 98e^{-\frac{1}{10}t} - 98$$

$$49 = 98e^{-\frac{1}{10}t}$$

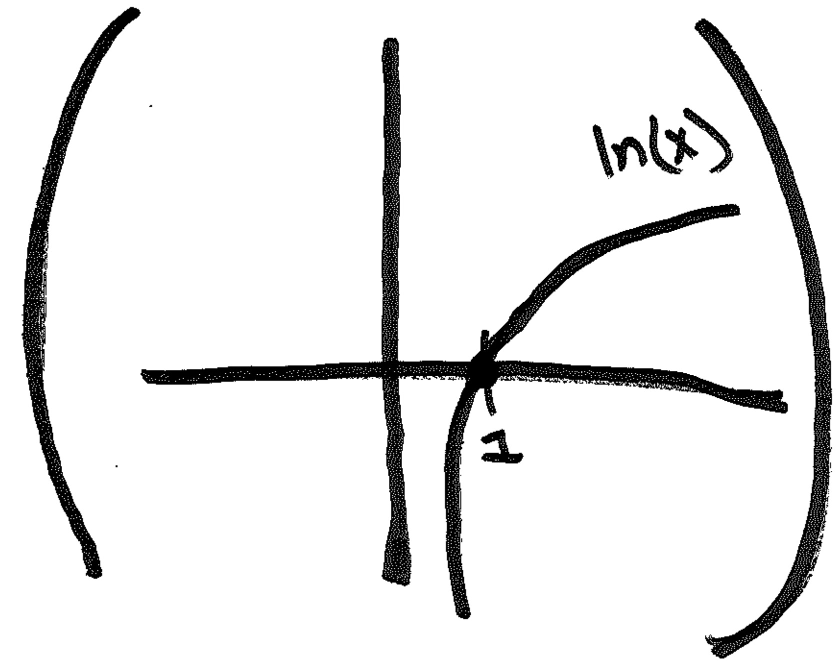
$$\frac{1}{2} = e^{-\frac{1}{10}t}$$

$$\ln\left(\frac{1}{2}\right) = \ln\left(e^{-\frac{1}{10}t}\right)$$

$$\ln\left(\frac{1}{2}\right) = -\frac{1}{10}t$$

$$t = -10\ln\left(\frac{1}{2}\right)$$

$$\boxed{t \approx 6.93 \text{ sec.}}$$



Friday, April 22 §9.3 Graphical Method

Find general solution to $\frac{dy}{dt} = -ty$.

$(\dot{y} = -ty)$

$$\int \frac{dy}{y} = \int -t\,dt$$

$$e^{\ln|y|} = e^{-\frac{1}{2}t^2 + C}$$

$$|y| = e^{-\frac{1}{2}t^2 + C}$$

$$y = \pm e^C e^{-\frac{1}{2}t^2}$$

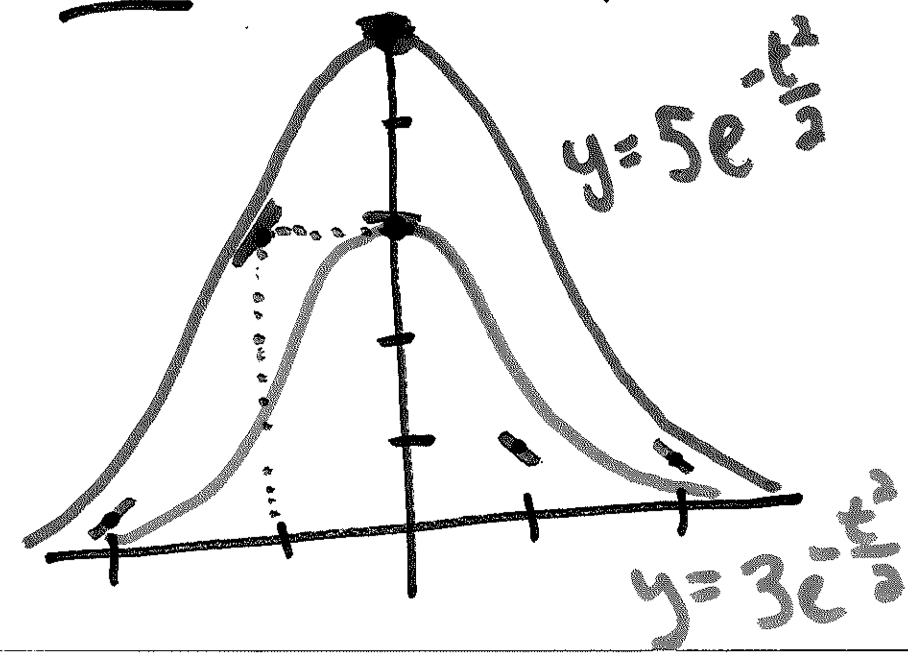
$$\boxed{y = Ce^{-\frac{1}{2}t^2}}$$

Is there a way to visualize all solutions at a glance? YES!

<u>Defn:</u> A <u>slope field</u> is an array of small segments of slope $\dot{y} = F(t,y)$ at points (t,y).

(Newton's notation $\frac{dy}{dt} = \dot{y} = F(t,y)$ t is an independ. variable)

<u>Ex:</u> Sketch slope field of $\dot{y} = -ty$.



pt.	slope
$(0,5)$	$-(0)\cdot 5 = 0$
$(-1,3)$	$-(-1)\cdot 3 = 3$
$(2,.5)$	$-(2)\cdot .5 = -1$
$(0,3)$	$-0\cdot 3 = 0$
$(1,1)$	$-1\cdot 1 = -1$
$(-2,.25)$	$-(-2)(.25) = \frac{1}{2}$

Fact: Most differential equations cannot be solved explicitly.

Good news: We can always draw a slope field and follow the slope segments to get a solution. (Like a road map.)

Tool: To sketch a slope field, find **curves** where all points on that curve have the same slope. (called **isoclines**)

Ex: $\dot{y} = y + t$ (**not** separable)

$\dot{y} = y + t$

- pts. where slope is 0.

$\dot{y} = 0$

$y + t = 0$

$y = -t$

- pts. where slope is 1

$\dot{y} = 1$

$y + t = 1$

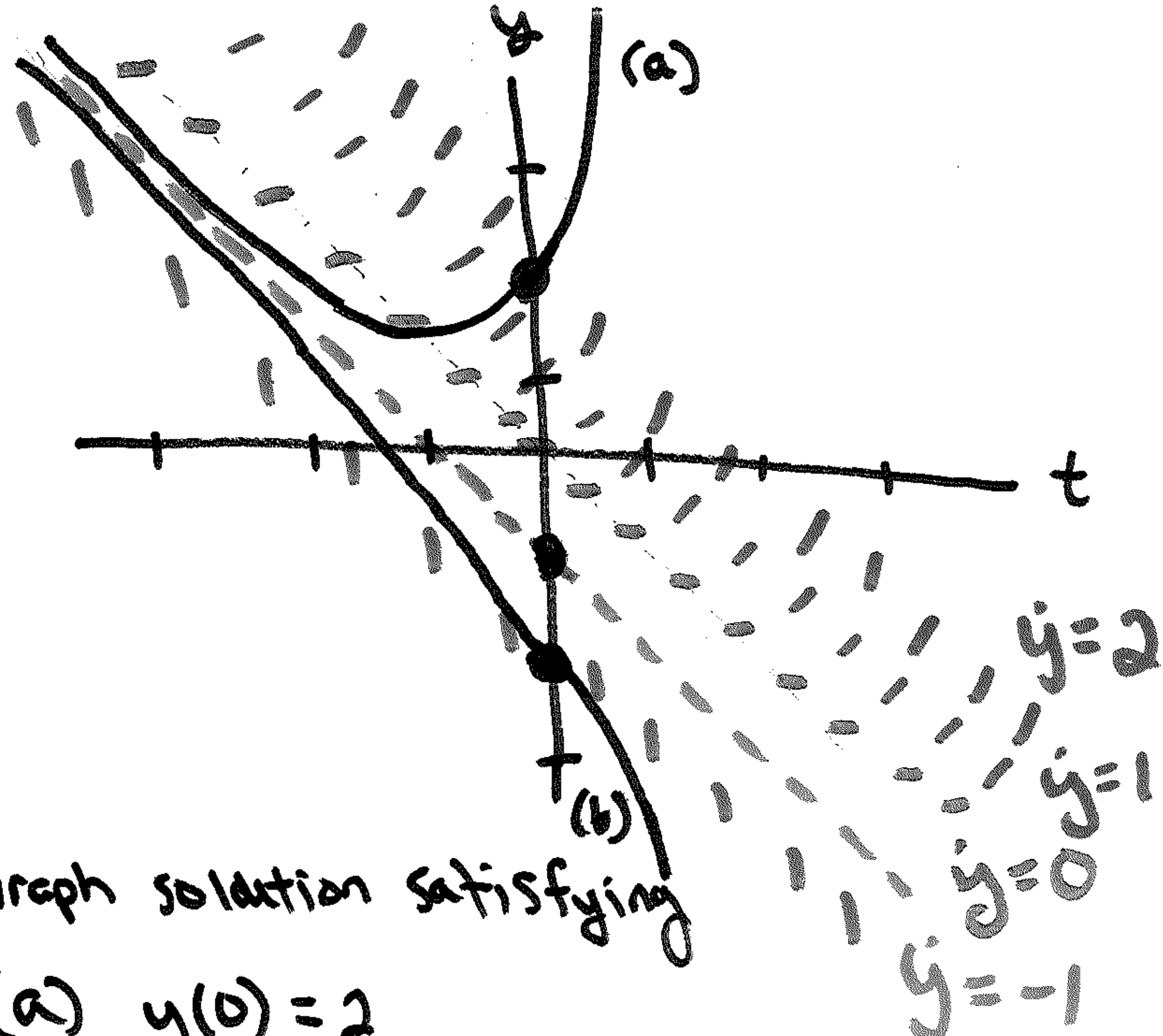
$y = 1 - t$

- pts. where slope is d.

$\dot{y} = d$

$y + t = d$

$y = d - t$



Graph solution satisfying

(a) $y(0) = 2$

(b) $y(0) = -2$

(c) $y(0) = -1 \xrightarrow[\text{solution}]{} y = -1 - t$

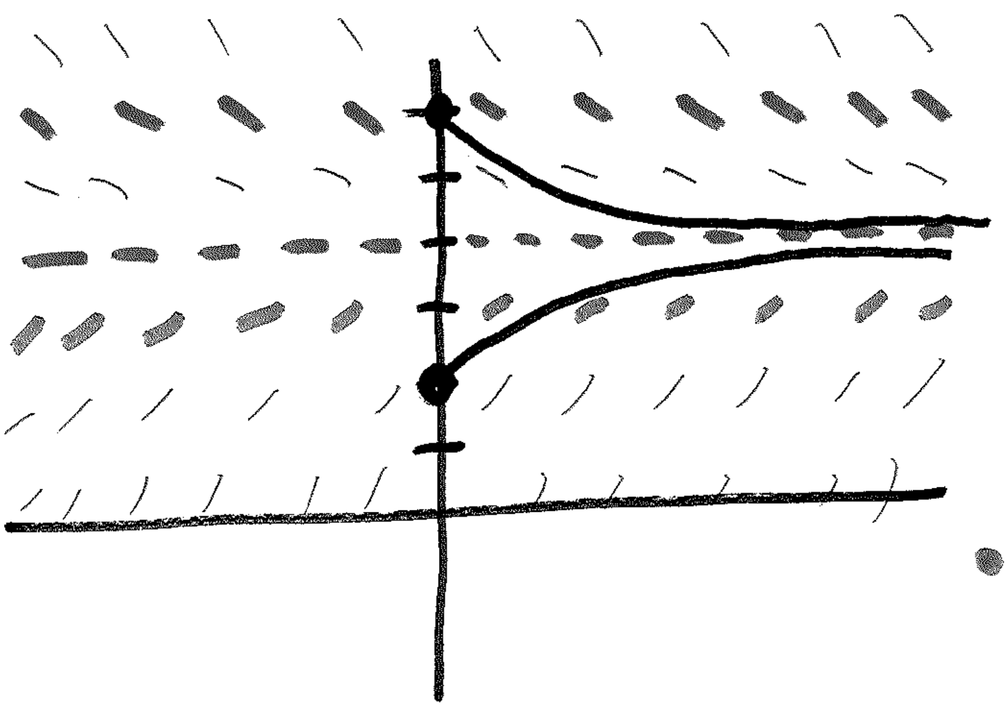
why? $\dot{y} = -1 \stackrel{?}{=} (-1-t) + t = -1$ ✓

Slope fields reveal behavior of solutions.

<u>Ex</u>: Say the temp. of an object placed in fridge satisfies

$$\dot{y} = -.5(y-4).$$

Draw S.F. and describe behavior.



<u>Note</u> $\dot{y}$ is independent of t.

$$\dot{y} = 0$$
$$-.5(y-4) = 0$$
$$y = 4$$

- When $y = 3$, $\dot{y} = -.5(3-4)$
  $\dot{y} = .5$
- When $y = 6$, $\dot{y} = -.5(6-4)$
  $\dot{y} = -1$

- If initial temp is > 4, object cools to 4 eventually.

  $y(t) \longrightarrow 4$

- If initial temp is < 4, $y(t)$ increases to 4.

- $\dot{y} = F(t,y) > 0$ whenever $y < 4$.
- $\dot{y} = F(t,y) < 0$ whenever $y > 4$.

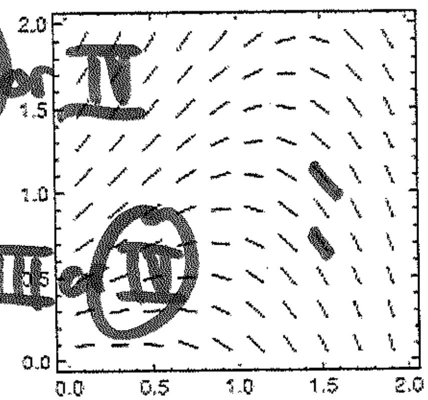
Match the diff. eq. with the correct slope field

(a) $y' = xy$ III or IV

(b) $y' = x^2 + y^2$ III or IV

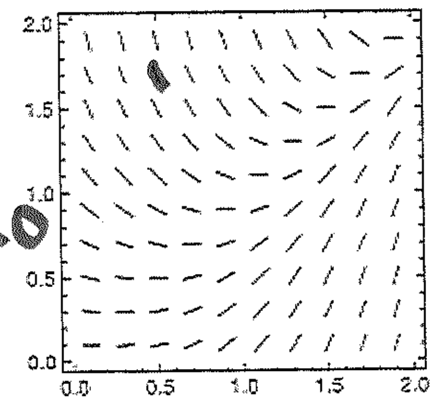
(c) $y' = x^2 - y^2$

(d) $y' = y - x^2$



$y' < 0$

(I)



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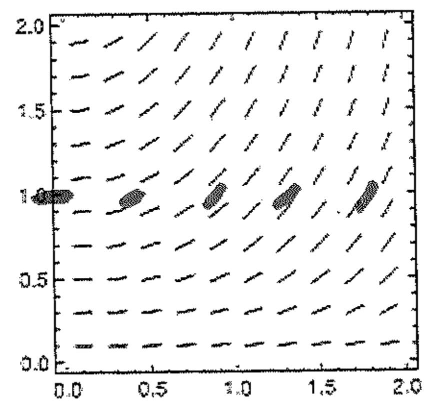

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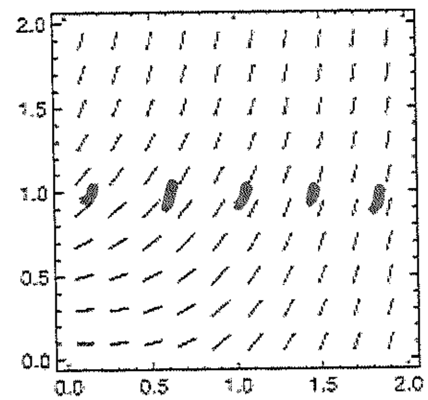

(IV)