Monday, April 25, 2016:

- Final Exam next week → Different room + time!
- Tell me about exam conflicts via email **NOW!**
- Final Exam is cumulative!

Q: How many people can live on the Earth?

(Flawed) estimate of upper bd. Surface area of Earth $\approx 510{,}072{,}000\ km^2$ (land + H_2O)

$= 510{,}072 \times 10^9\ m^2$. ← If each person got 1 sq meter of room.

§9.4: Logistic Eqn

Population growth last classes: $y(t)$ = pop. at time t.

$$\frac{dy}{dt} = k \cdot y(t)$$

constant.

$$\Rightarrow y = \underbrace{Ce^{kt}}.$$

unbounded, i.e. population skyrockets w/ time.

A better model:

Let $A>0$ be the "carrying capacity" for a system & $y(t)$ the pop. at time t. i.e. A is max possible population. The **logistic** equation assigns a growth constant $k>0$ to the system & models change in population by

$$\frac{dy}{dt} = \underbrace{k \cdot y}_{\text{leads to usual exponential growth}} \cdot \underbrace{\left(1 - \frac{y}{A}\right)}.$$

As $y \to A$, $1-\frac{y}{A} \to 0$. This factor inhibits pop. growth.

Thm: If $\frac{dy}{dt} = ky\left(1-\frac{y}{A}\right)$ as above,

then
$$y(t) = y = \frac{A}{1 - e^{-kt}\left(\frac{y(0)-A}{y(0)}\right)}$$
where $y(0)$ = initial population.

Q: As $t \to \infty$, what happens to $y(t)$?

the only occurrence of t in $y(t)$.

Since $k>0$, $e^{-kt} \to 0$ as $t \to \infty$. So,

$$y(t) \to \frac{A}{1} \text{ as } t \to \infty.$$

Note: • If $y=0$, we get an **unstable** equilibrium. i.e. fn $y(t)$ is constant, but if $y(0)$ is close to 0 as time increases, it moves away from 0 either to $-\infty$ or A.

• If $y=A$, we get a **stable** equil, because $y(t)=A$ is an equilibrium soln & if $y(0)$ is close to A, it gets closer as time increases.

• If $y(0)>0$, then $y(t) \to A$ as $t \to \infty$.

<u>Proof of Thm:</u> Separate Variables.

$$\frac{dy}{y(1-\frac{y}{A})} = k\,dt$$

Goal is to integrate + solve.

<u>RHS:</u> $\int k\,dt = kt + C$ ← const.

<u>LHS:</u> $\displaystyle\int \frac{1}{y(1-\frac{y}{A})}\,dy = \int \left(\frac{1}{y} - \frac{1}{y-A}\right)dy$

$= \ln|y| - \ln|y-A| + \text{const.}$

So, using log laws & combining constants

$$\ln\left|\frac{Y}{Y-A}\right| = kt + C \quad \text{New constant.}$$

$$\Rightarrow \left|\frac{Y}{Y-A}\right| = e^{kt+C} = e^{C}\cdot e^{kt}$$

$$\Rightarrow \frac{Y}{Y-A} = \underbrace{\left(\pm e^{C}\right)}_{\text{New constant, call it } C \text{ again.}} \cdot e^{kt}$$

$$\Rightarrow \frac{Y}{Y-A} = C\cdot e^{kt}.$$

Set $t=0$, + we get

$$\frac{y(0)}{y(0)-A} = C \cdot e^{k \cdot 0} = C.$$

So, $$\frac{y}{y-A} = \frac{y(0)}{y(0)-A} \cdot e^{kt}.$$ Solve for y + get

$$y(t) = \frac{A}{1 - \frac{1}{\frac{y(0)}{y(0)-A} \cdot e^{kt}}} = \frac{A}{1 - e^{-kt}\left(\frac{y(0)-A}{y(0)}\right)}.$$

$\blacksquare$

Thought exercise:

Q: How might you solve

$$\frac{dy}{dt} = k \cdot y(y-2)(y-3) \; ?$$

Q: Can you actually solve for y here?
(I don't know....)

Ex: Suppose $A = 2000$, $k = 0.6$.

Q1: what is $y(t)$ if $y(0) = 500$?

Q2: If $y(0) = 500$, how long until $y(t) = 1000$?

A1: $$y(t) = \frac{A}{1 - e^{-kt}\left(\frac{y(0) - A}{y(0)}\right)} = \frac{2000}{1 + e^{-.6t} \cdot 3} = \frac{2000}{1 + 3e^{-.6t}}$$

A2: $1000 = \dfrac{2000}{1+3e^{-.6t}}$, solve for t,

get $t = \frac{-10}{6} \ln\left(\frac{1}{3}\right) \approx 1.83$ years. assuming units.