Monday, April 4, 2016

Recall: Thm said for $c(t) = (x(t), y(t))$ for suitable x, y functions of t, we have $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$.

Normally... $\frac{dy}{dx}$ is infinitesimal change in y with respect to x.

So, we want

$\frac{\text{change in } y}{\text{change in } x}$ for small changes.

So, in the limit as change $\to 0$, we get this formula.

<u>Ex</u>: Circle of radius 1, $x = \cos t$,
$y = \sin t$.

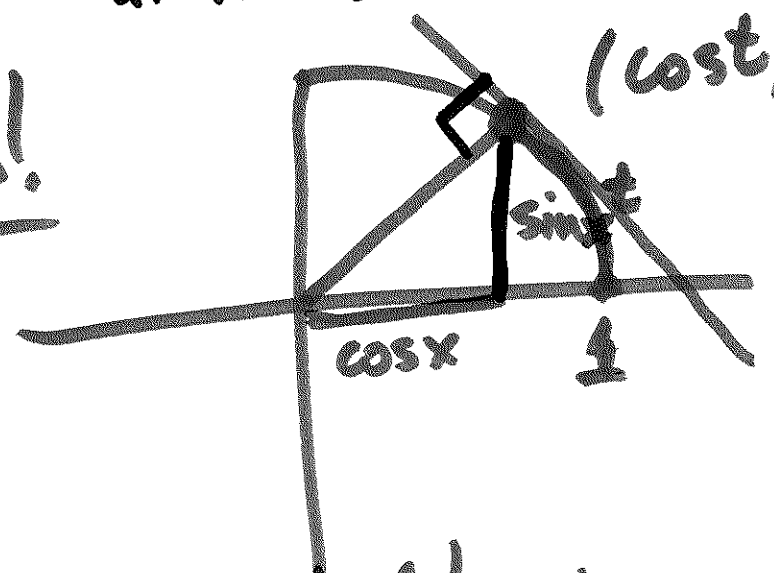
$$\left.\frac{dy}{dx}\right|_t = \frac{y'(t)}{x'(t)} = \frac{\cos t}{-\sin t} = -\frac{\cos t}{\sin t}.$$

↑at time t

This should be slope of tangent line at (cost, sint).

(cost, sint)

<u>Yes!</u>

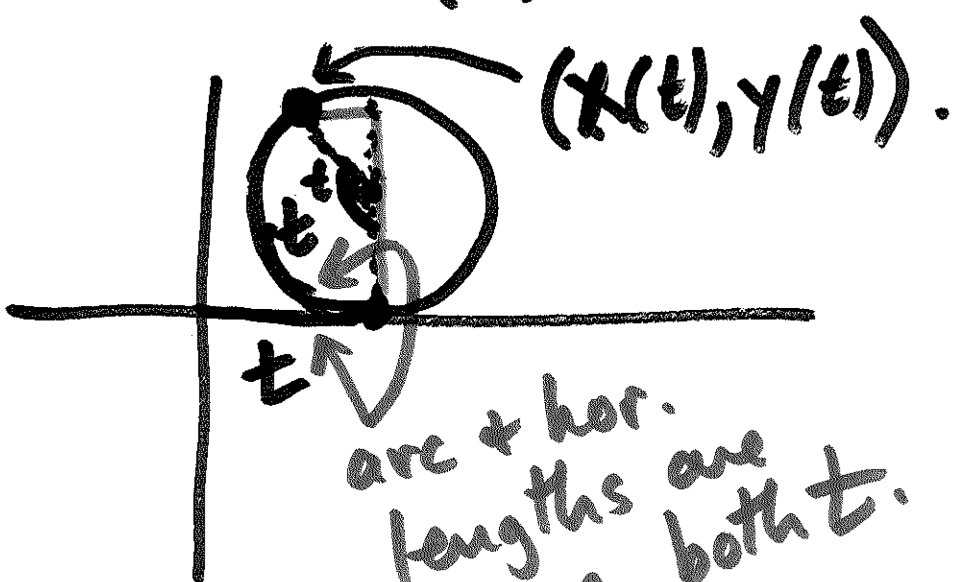


tangent line is $\perp$ to radial line, so if radial line has slope m, tangent line has slope $\frac{-1}{m}$.

Slope of radial line is $\frac{\sin t}{\cos t}$. $\Rightarrow$ slope of tangent should be $\frac{-\cos t}{\sin t}$.

<u>Ex</u>: Cycloid

$$x(t) = t - \sin t, \quad y(t) = 1 - \cos t.$$



arc & hor. lengths are same, both t.

$t=0$

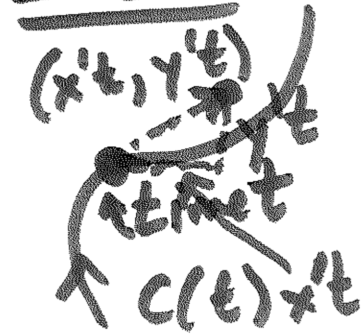
Starting here requires a shift of cos & sin to "opposite" axes from usual.

§11.2 (Arc length + Speed)

Thm: Let $c(t) = (x(t), y(t))$ where $x'(t), y'(t)$ exist and are cts. The arc length of $c(t)$ for $a \le t \le b$ is

$$s = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

Idea:



Note: $c(t) = (t, f(t))$ implies

$$s = \int_a^b \sqrt{1 + f'(t)^2}\, dt$$

our previous arc length formula!

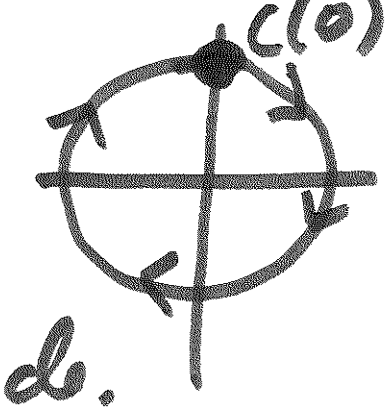
<u>Ex</u>: For $c(t) = (\sin t, \cos t)$, $t = 0$ to $t = 2\pi$, what is arc length?

$$\left.\begin{aligned} x(t) = \sin t &\Rightarrow x'(t) = \cos t \\ y(t) = \cos t &\Rightarrow y'(t) = -\sin t \end{aligned}\right\} \text{use arc length formula}$$

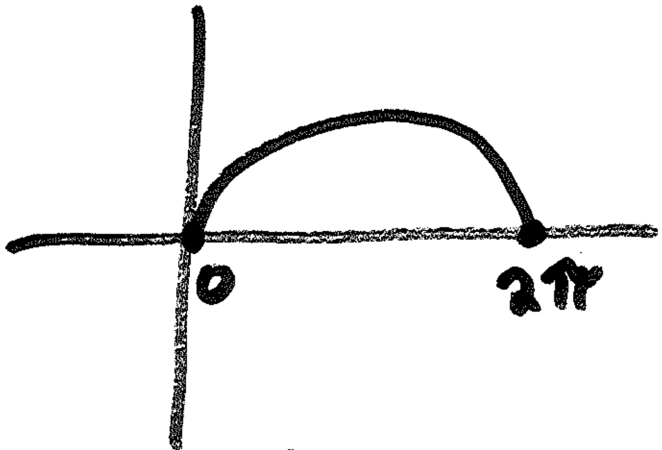
$$s = \int_0^{2\pi} \sqrt{\cos^2 t + \sin^2 t}\, dt = \int_0^{2\pi} dt = t\Big|_0^{2\pi} = 2\pi.$$

$c(t)$ = unit circle.

s = circumference of unit circle.



<u>Ex</u>: Length of cycloid, $t=0$ to $t=2\pi$.



$$x(t) = t - \sin t \Rightarrow x' = 1 - \cos t$$
$$y(t) = 1 - \cos t \Rightarrow y' = \sin t$$

$$S = \int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t}\, dt = \int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\, dt$$

$$= \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt \underset{\star}{=} \int_0^{2\pi} 2\sin\left(\frac{t}{2}\right) dt \underset{\substack{\text{standard}\\ \text{technique}}}{=} 8.$$

$\star$: $1-\cos t = 2\sin^2\left(\frac{t}{2}\right)$ is what you use.

How? $\cos^2 t = \dfrac{1+\cos 2t}{2} \Rightarrow$

$$1-\sin^2 t = \frac{1+\cos 2t}{2} \Rightarrow$$

$$\sin^2 t = \frac{1-\cos 2t}{2}.$$

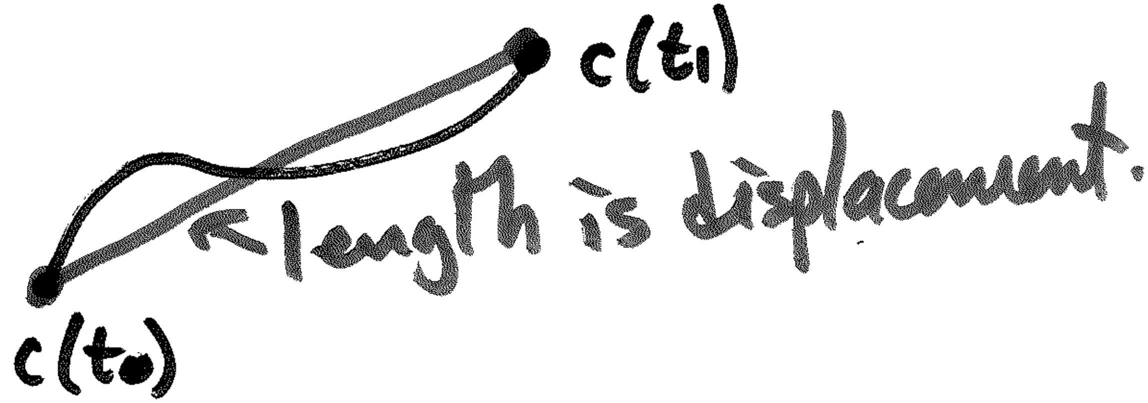
set $t \mapsto \frac{t}{2}$.

__Def__$^{\underline{n}}$: For a particle moving along $c(t) = (x(t), y(t))$, the __speed__ of the particle at time t is

$$\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2}.$$

__Def__$^{\underline{n}}$: Between time t_0 & t_1, the __displacement__ of c is the distance between $c(t_1)$ & $c(t_0)$.

__Ex__: $c(t)$



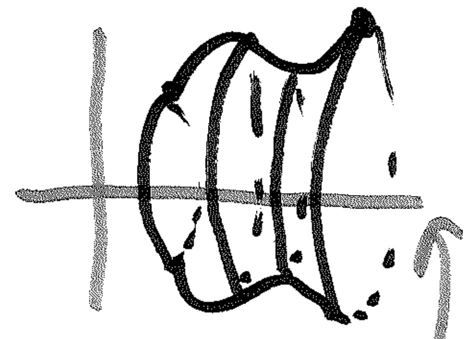
Thm: Let $c(t) = (x(t), y(t))$, where $y(t) \geq 0$ and $x(t)$ is increasing. Also, $x'(t), y'(t)$ exist and are cts. The surface obtained by rotating $c(t)$ about x-axis for $a \leq t \leq b$ has surface area

$$S = \int_a^b 2\pi\, y(t) \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

Note: For a graph of a function $(t, f(t))$, we recover our previous formula.

Wed, April 6, 2016

Recall: For $c(t) = (x(t), y(t))$ with



allows integration.

i) $y(t) \geq 0$

ii) $x(t)$ increasing ← move from left to right as t increases

iii) $x'(t), y'(t)$ cts

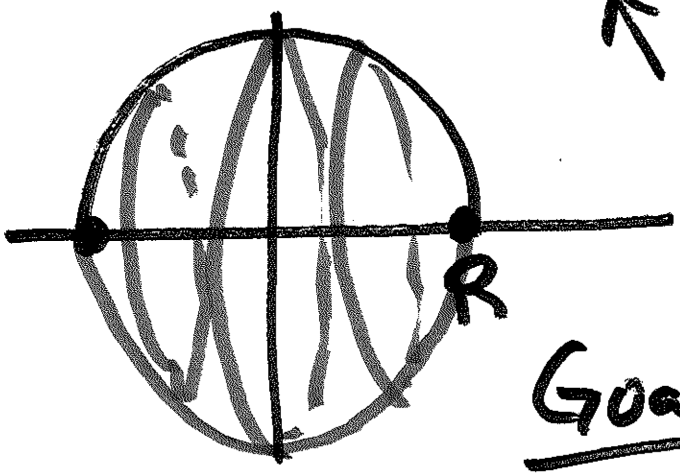
Pretend centered

Rotating $c(t)$ about x-axis on $a \leq t \leq b$ has surface area

$$S = \int_a^b 2\pi\, y(t) \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

Ex: $c(t) = (R\cos t, R\sin t)$, t in $[\underset{a}{0}, \underset{b}{\pi}]$.

semicircle of radius R.



Goal: Compute surface area, in this case we get a sphere of radius R.

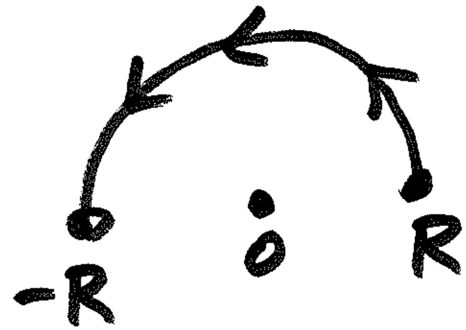
Claim: one of our assumptions fails!

Q: Which one?

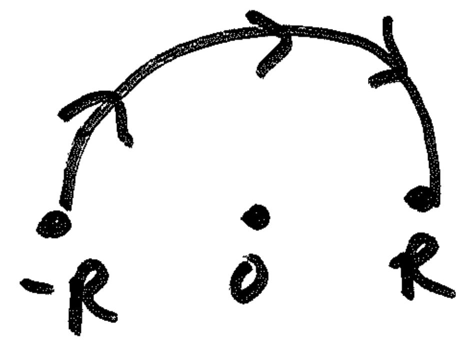
A: ii! $x(t)$ not increasing.

Q: How can I fix this?

With $c(t) = (R\cos t, R\sin t)$ we trace



Use $c(t) = (-R\cos t, R\sin t)$, I get



NOTE: Many ways to fix this!

Ex: $c(t) = \left(t, \sqrt{R^2 - t^2}\right)$

t in $[-R, R]$

Using $c(t) = (-R\cos t, R\sin t)$ satisfies all hypotheses/requirements. So,

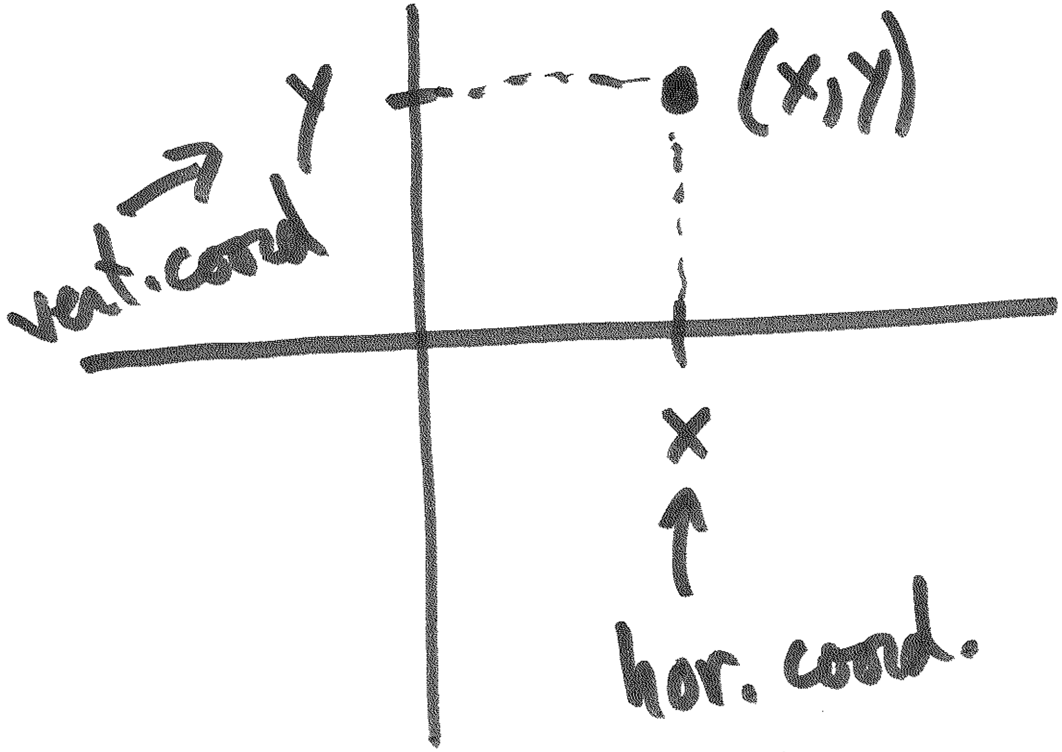
$$S = \int_0^{\pi} 2\pi R\sin t \sqrt{R^2\cos^2 t + R^2 \sin^2 t}\, dt =$$

$$= \int_0^{\pi} 2\pi R \sin t \sqrt{R^2 (\underbrace{\sin^2 t + \cos^2 t}_{=1})}\, dt$$

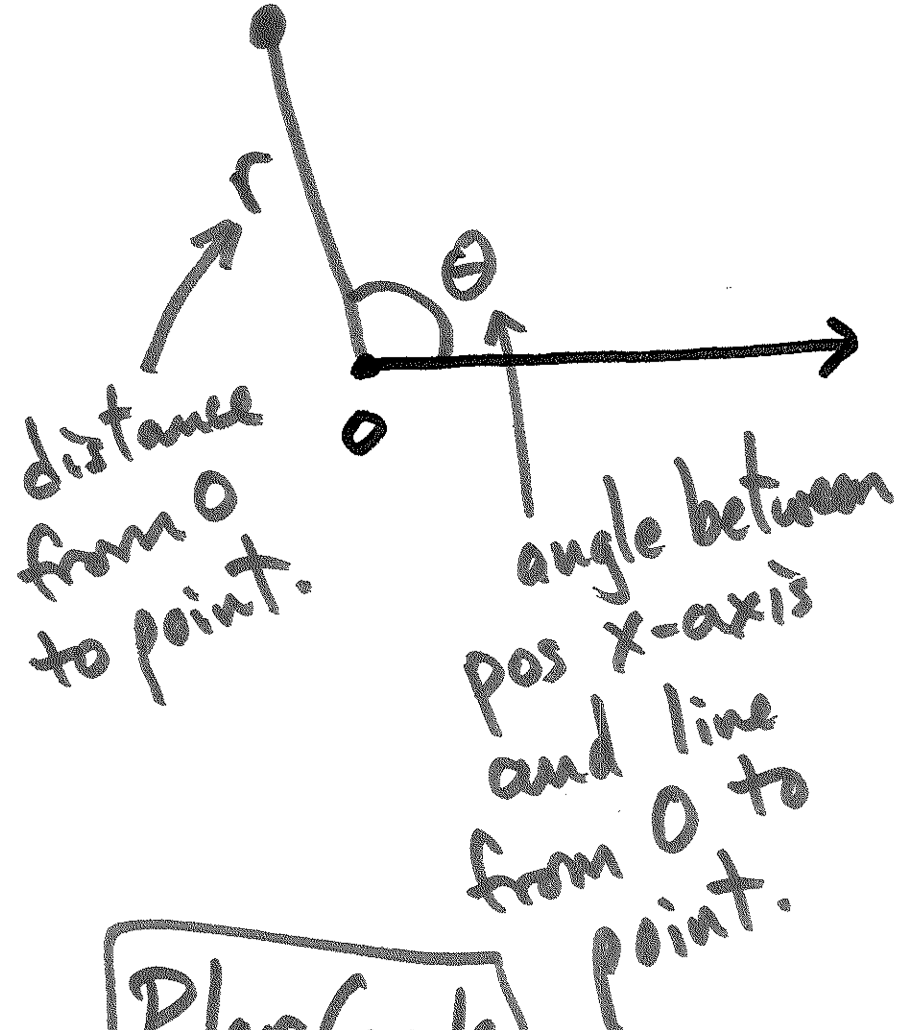
$$= \int_0^{\pi} 2\pi R \sin t \cdot R = 4\pi R^2,$$ as expected.

Takeaway: Integration isn't the main challenge! Main challenge is getting parametrization correct for formula.

§11.3: Polar Coords.



Cartesian Coords.

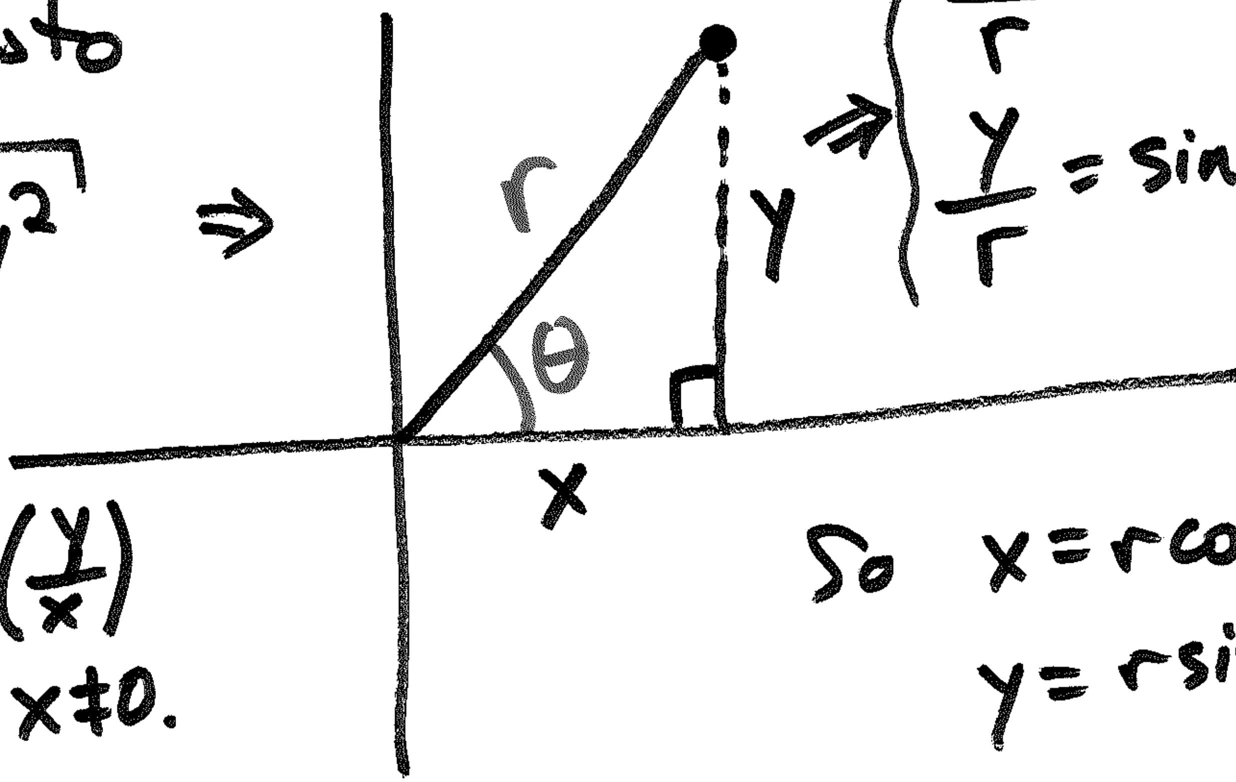


Polar Coords.

Conversion

(x,y) goes to

$r = \sqrt{x^2+y^2}$ $\Rightarrow$



$\Rightarrow$ $\left\{ \begin{array}{l} \dfrac{x}{r} = \cos\theta \\ \dfrac{y}{r} = \sin\theta \end{array} \right.$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ for $x \neq 0$.

↑ We will say more about how to specify θ.

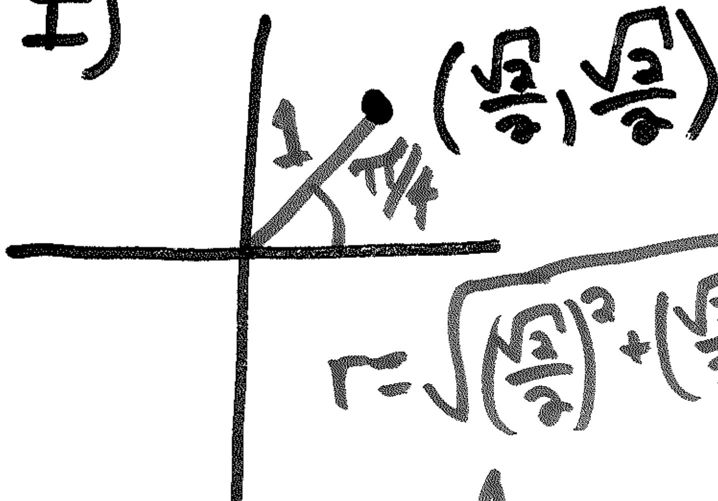
So $\left. \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array} \right\}$ If you know (r,θ)-coords, Finding rect. is this!

Find polar coords for: $\qquad r, \theta$

I) $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)_{rect} \Rightarrow \left(1, \frac{\pi}{4}\right)_{Polar}$

II) $\left(\frac{-1}{2}, -\frac{\sqrt{3}}{2}\right)_{rect} \Rightarrow \left(1, \frac{4\pi}{3}\right)_{Polar}$

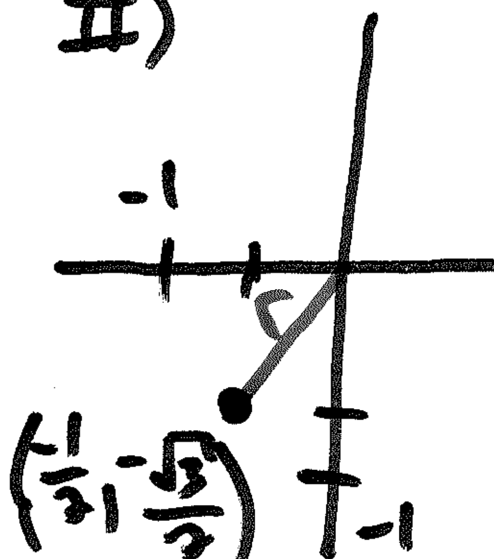
I)



$r = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2}$

$= 1$

II)



$r = 1$

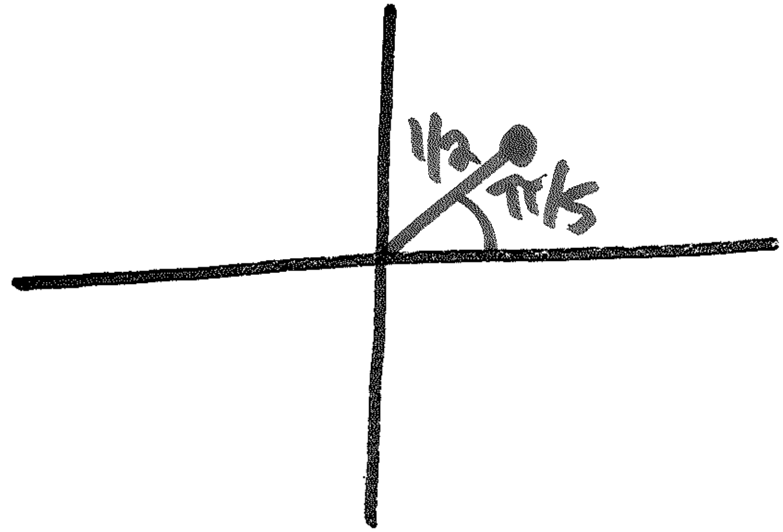
$\theta =$ the angle

w/ $\cos\theta = -1/2$,

$\sin\theta = -\frac{\sqrt{3}}{2}$.

Plot:

$\left(\frac{1}{2}, \frac{\pi}{5}\right)_{\text{Polar}}$.



(x,y)-coords are

$\left(\frac{1}{2}\cos\frac{\pi}{5}, \frac{1}{2}\sin\frac{\pi}{5}\right)$.

BIG CHALLENGE:

1) There are multiple "correct" choices for θ.

2) We want to visualize (r,θ) where r is a function of θ.

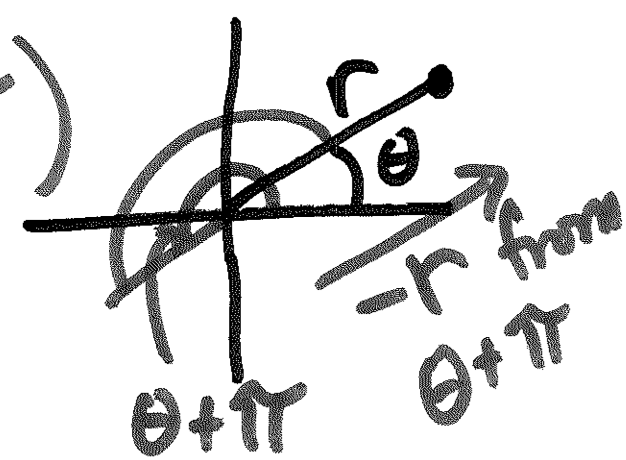
Fri, April 8

- Find <u>four</u> polar representations for $(-1,1)_{\text{rect}} = (x,y)$.

<u>Note</u>: $(r,\theta) = (r, \theta + n\cdot 2\pi)$ for any integer n.

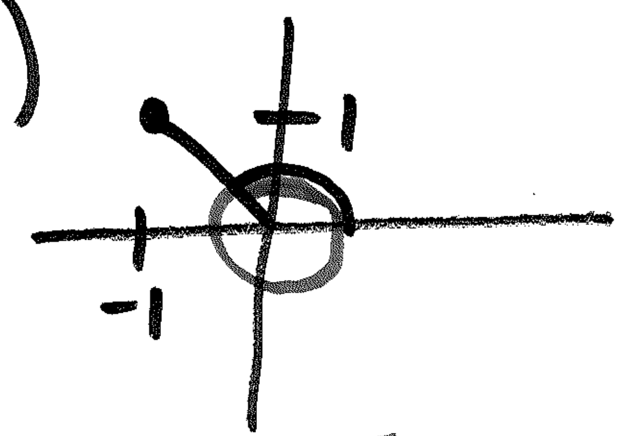
$$(r,\theta) = (-r, \theta \pm \pi)$$

If r is negative, go "backwards".



Ex: $(x,y) = (-1,1)$

has (r,θ)-coords



$r = \sqrt{2}$

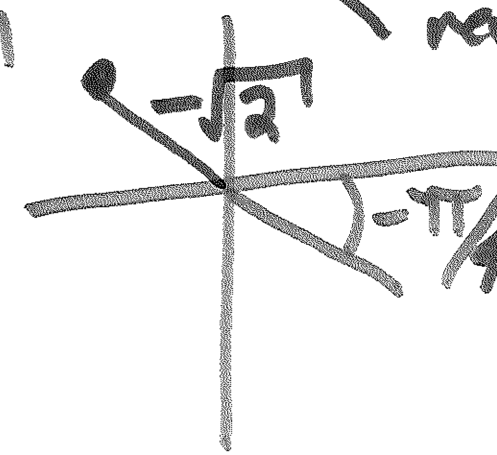
$\theta = \frac{3\pi}{4}$.

OR $r = \sqrt{2}$

$\theta = 2\pi + \frac{3\pi}{4}$

OR $r = -\sqrt{2}$

$\theta = -\frac{\pi}{4}$



"backwards" or reverse

<u>Notes</u>: 1) $(0, \theta)_{polar}$ is always the origin.

2) 2 common conventions, due to multiple descriptions of polar point.

<u>one</u>: Set $r > 0$, θ in $[0, 2\pi)$.

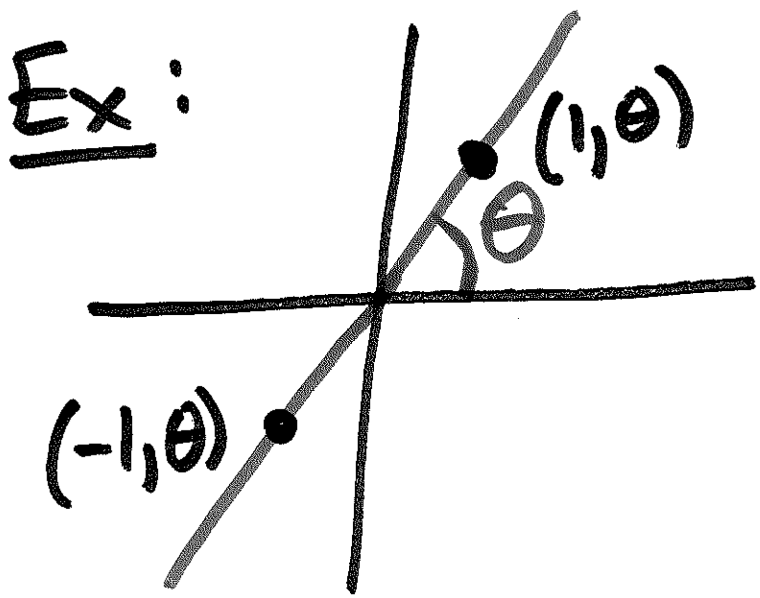
<u>two</u>: Set $r > 0$, θ in $[-\frac{\pi}{2}, \frac{3\pi}{2})$.

This is good for converting from rectangular using

$$\theta := \begin{cases} \tan^{-1}(y/x) & \text{if } x > 0 \\ \tan^{-1}(y/x) + \pi & \text{if } x < 0 \\ \frac{\pi}{2} \text{ or } -\frac{\pi}{2} & \text{if } x = 0 \text{, as appropriate.} \end{cases}$$

Polar Equations:

Ex:

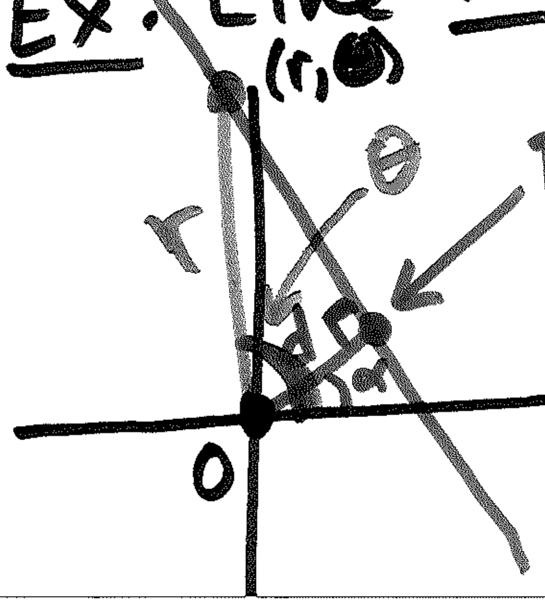


Line through origin.

(t, θ) for t in $\mathbb{R}$.

↑ real numbers.

Ex: Line not through origin, L in red.



$P_0 = (d, \alpha)$.

Need to relate (r, θ) to d and α.

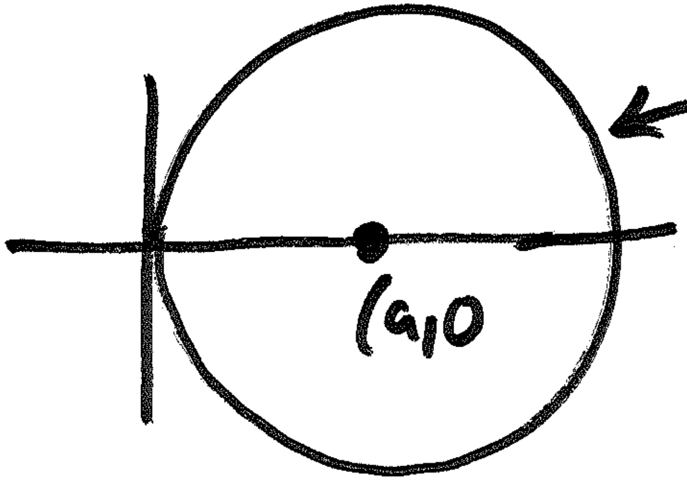
Since $\frac{d}{r} = \cos(\theta - \alpha)$, so

the line is $(r, \theta) = \left(\frac{d}{\cos(\theta - \alpha)}, \theta\right)$.

Solve for r.

__Ex:__ $(2a\cos\theta, \theta)$. $a>0$

__Claim:__ This gives circle of radius a centered at $(a,0)$.



← plot of $(2a\cos\theta, \theta)$.

__Goal:__ Match this with
$$(x-a)^2+y^2=a^2.$$

From $r=2a\cos\theta$, we know if $(x,y)=(r,\theta)$, then $x^2+y^2=r^2$, so
$$x^2+y^2 = r\cdot 2a\cos\theta. \quad (*)$$

But, $(x,y) = (2a\cos\theta \cdot \cos\theta,\ 2a\cos\theta\sin\theta)$

$= (r\cos\theta,\ r\sin\theta)$.

So, $2a \cdot r\cos\theta = 2ax$. in $\circledast$

So, $x^2+y^2 = 2ax$. We get

$x^2 - 2ax + y^2 = 0$, so complete square for x,

$\Rightarrow x^2 - 2ax + a^2 + y^2 = a^2$, hence

$(x-a)^2 + y^2 = a^2$.

Note: look at Ex. 8 in §11.3 for limaçon curve.

Ex: Leminiscate of Bernoulli.

Rect: $(x^2+y^2)^2 = x^2-y^2$.

Polar: $r^2 = \cos 2\theta$.

This is not an easy example.

Note: Arc length of this curve is

$$4\int_0^1 \frac{1}{\sqrt{1-t^4}}\,dt.$$

This is an example of an "elliptic" integral.