Wed, Feb 10, 2016:

Thm 1, §10.7: If $f(x)$ is represented by a power series centered at c in an interval $|x-c|<R$ where $R>0$, then

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n .$$

$$= f(c)+\frac{f^{(1)}(c)}{1!}(x-c)+\frac{f^{(2)}(c)}{2!}(x-c)^2+\dots$$

$y=f(c)+f^{(1)}(c)(x-c)$ is eqn of tangent line to graph of $f(x)$ at $(c,f(c))$.

This is called Taylor series for $f(x)$.

Q: What is $\frac{d}{dx} e^x \Big|_{x=0} = 1$. I choose $c=0$.

If $f(x) = e^x$, $f^{(1)}(0) = 1$.

$$f^{(2)}(0) = \frac{d}{dx}\left(\frac{d}{dx} e^x\right)\Big|_{x=0} = \frac{d}{dx} e^x \Big|_{x=0} = 1.$$

$f^{(3)}(0) = 1$

In general, $f^{(n)}(0) = 1$, because $f^{(n)}(x) = e^x$.

By our theorem, if e^x has a series representation,

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Ex: $\sin(x) = f(x)$.

$f^{(1)}(x) = \cos(x)$

$f^{(2)}(x) = -\sin(x)$

$f^{(3)}(x) = -\cos(x)$

$f^{(4)}(x) = \sin(x)$

$\vdots$ repeat pattern.

Choose $c=0$:

$f^{(0)}(0) = \sin(0) = 0$

$f^{(1)}(0) = \cos(0) = 1$

$f^{(2)}(0) = -\sin(0) = 0$

$f^{(3)}(0) = -\cos(0) = -1$

$\vdots$

So, $\sin(x) = 0 + \frac{1}{1!}(x-0) + \frac{0}{2!}(x-0)^2 + \frac{-1}{3!}(x-0)^3 + \cdots$

$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$

$$\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

R of Conv. is ∞.

What is R. of Conv? Ratio Test:

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!}}{\frac{(-1)^n x^{2n+1}}{(2n+1)!}} \right| = \lim_{n\to\infty} \left| \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^2}{(2n+2)(2n+3)} \right| = |x^2| \lim_{n\to\infty} \left| \frac{1}{(2n+2)(2n+3)} \right| = 0.$$

For any x, my $\rho = 0$ ← ratio test, so converges. So, int. of Conv. is $(-\infty, \infty)$.

For $\cos(x)$, use $\frac{d}{dx}\sin(x) = \cos(x)$.

$$\cos(x) = \frac{d}{dx}\sin(x) = \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= 1 - \frac{\cancel{3}\cdot x^2}{\cancel{3}\cdot 2} + \frac{\cancel{5}\cdot x^4}{\cancel{5}\cdot 4\cdot 3\cdot 2} - \frac{\cancel{7}x^6}{\cancel{7}\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} + \cdots$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

Ex: What is $\frac{d}{dx}(\arctan(x)) = \frac{d}{dx}(\tan^{-1}(x))$?

It is $\frac{1}{1+x^2}$ for certain values of x.

Thus, $\arctan(x) = \int \frac{1}{1+x^2}\,dx$.

But, $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} \underset{\substack{\uparrow \\ \text{geom.} \\ \text{series}}}{=} 1 - x^2 + x^4 - x^6 + x^8 - \cdots$

So, $\arctan(x) = \int (1 - x^2 + x^4 - x^6 + x^8 - \cdots)\,dx$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots .$$

NOTE: $\int$ and $\frac{d}{dx}$ preserve radius of convergence.

Application:

$$\arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

$$= \frac{\pi}{4}$$

Due to Leibniz, & converges but very slowly.

Fri, Feb 12, 2016:

1. Last day of infinite series is today.
2. Reminder to read before lecture + recitation.

We have a machine: Start w/ $f(x)$, a function.

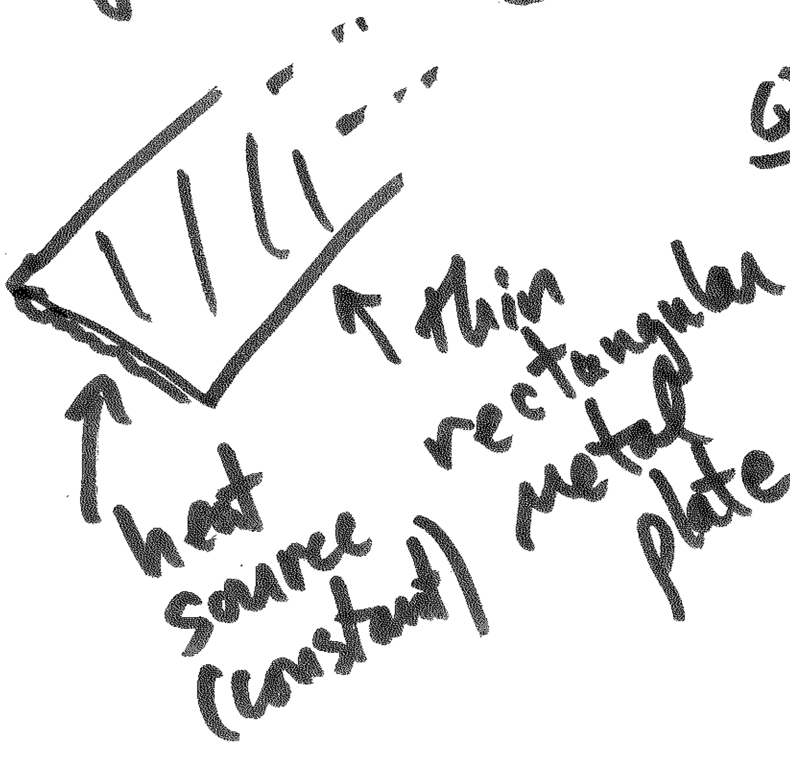
DO ONE OF:

1) Write $f(x)$ as a derivative or integral of a power series + go term-by-term.
2) Compute $\frac{f^{(n)}(c)}{n!}$ + write $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} x^n$
3) Write $f(x)$ as a substitution, e.g. e^{3x} ← $3x$ is substituted for exponent.

WARNING: Be careful w/ radius of convergence

OUTPUT: POWER Series representing $f(x)$.
(Usually.)

NOTE: This was the story 1665 – 1807.
In 1807, Fourier solved the problem of modeling heat distribution on a metal plate.



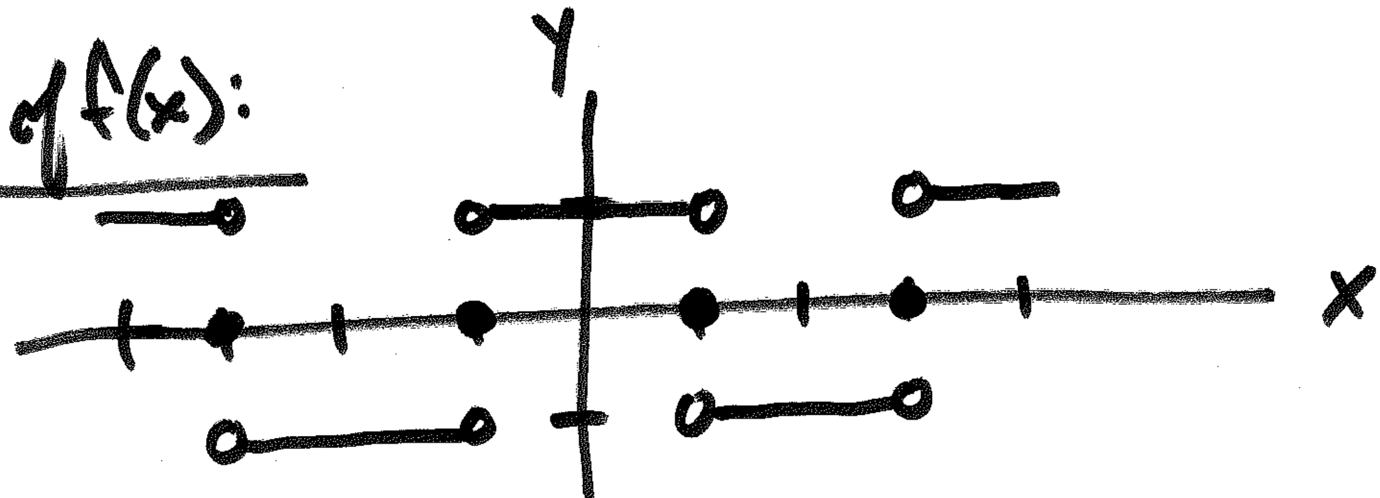
Q: How does heat distribute?

This lead to "Fourier Series"

Fourier needed:

$$f(x) = \frac{4}{\pi}\Big[\overbrace{\cos\left(\frac{\pi x}{2}\right) - \frac{1}{3}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{5}\cos\left(\frac{5\pi x}{2}\right) - \cdots}^{\text{infinite.}}\Big]$$

Graph of $f(x)$:



jump discontinuities galore!

This led Cauchy to develop a careful theory of series (see MA 471G).

<u>Ex</u>: Cauchy, 1820's

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

removable discontinuity

<u>Exercise:</u> Show $f^{(n)}(0) = 0$ for all n.

So, Taylor series for $f(x)$ is

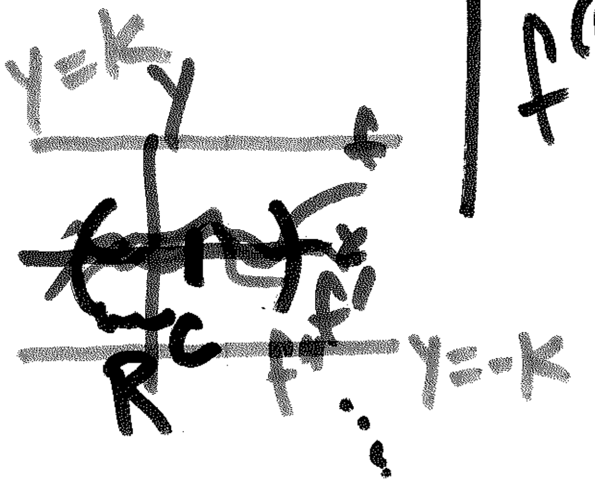
$$0 + \frac{0}{1!} \cdot x + \frac{0}{2!} \cdot x^2 + \frac{0}{3!} \cdot x^3 + \cdots = 0.$$

But, $f(x)$ isn't the zero function.

<u>TAKEAWAY:</u> Functions <u>Are</u> NOT always to their Taylor Series.

New part to machine:

Thm 2, §10.7: Let $f(x)$ be a function so that for some c, R, we have on $I = (c-R, c+R)$ a bound $K \geq 0$ such that



$$|f^{(n)}(x)| \leq K \text{ for all } x \text{ in } I \text{ and all } n = 0, 1, 2, \ldots .$$

If this happens, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \text{ for } x \text{ in } I.$$

<u>Ex</u>: Let's use this on $f(x) = \cos(x)$.

(in other words, let's show $\cos(x)$ actually equals $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ for $c=0$, $R=\infty$).

<u>Step 1</u>: Identify candidate for K.

Proposal: $K=1$.

<u>Check</u>: $f^{(n)}(x) = \frac{d^n}{dx^n}\cos(x)$ is one of:

$$\cos(x),\ -\sin(x),\ -\cos(x),\ \sin(x).$$

All of these range between $-1, 1$ in output value. So, $|f^{(n)}(x)| \le 1$, as desired.

__Step 2__: Compute Taylor Series for $f(x)$.

__NOTE__: We already did this.

__Step 3__: Conclude $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

for all x in $(-\infty, \infty)$.

__Ex__: Find Maclaurin Series (i.e. $C=0$) for

$$f(x) = \frac{1}{3x-2}.$$

__Step 1__: Decide which approach from machine you will try first.

Most efficient here is substitution...

NOTE: $\dfrac{1}{3x-2} = \dfrac{1}{2\left(\frac{3}{2}x-1\right)} = \dfrac{1}{2}\cdot\dfrac{1}{\frac{3}{2}x-1} =$

$$\frac{1}{-2}\cdot\frac{1}{1-\underbrace{\tfrac{3}{2}x}_{\text{substitution into geom. series}}} = \frac{1}{-2}\left(1+\frac{3}{2}x+\left(\frac{3}{2}x\right)^2+\left(\frac{3}{2}x\right)^3+\cdots\right)$$

Recall: If $\sum_1^\infty a_n$ conv, then $c\sum_1^\infty a_n = \sum_1^\infty ca_n$.

$$= \frac{1}{-2}\left(1+\frac{3}{2}x+\frac{3^2}{2^2}x^2+\frac{3^3}{2^3}x^3+\cdots\right)$$

$$= -\left(\frac{1}{2}+\frac{3}{2^2}x+\frac{3^2}{2^3}x^2+\frac{3^3}{2^4}x^3+\cdots\right).$$

$$= -\sum_{n=0}^{\infty}\frac{3^n}{2^{n+1}}x^n$$

R. of. Conv is $\left|\frac{3}{2}x\right|<1$.

Ex: Find Maclaurin Series for
$$f(x) = \frac{1}{(1-x)^3}.$$

Remark: Carefully read 1st 2 pages of §10.7 to motivate Taylor Series.

~~Three~~ Two starting points:

(I) Notice that $\left[(1-x)^{-3} = f(x)\right]$

$$\frac{1}{(1-x)^3} = \frac{1}{2}\frac{d}{dx}\left(\frac{1}{(1-x)^2}\right) = \frac{1}{2}\frac{d^2}{dx^2}\left(\frac{1}{1-x}\right)$$

$$= \frac{1}{2}\frac{d^2}{dx^2}\left(1+x+x^2+x^3+\cdots\right) = 1 + \frac{3\cdot 2}{2}x + \frac{4\cdot 3}{2}x^2 + \frac{5\cdot 4}{2}x^3 + \frac{6\cdot 5}{2}x^4 + \cdots$$

(II) Take derivatives of $f(x)$ & use Taylor series directly, i.e. compute $\frac{f^{(n)}(0)}{n!}$.

<u>Mon, Feb 15, 2016:</u>

Begin Applications of the Integral!

<u>1st Application</u>: Average values of functions.

<u>Average</u> value of

$$1, 3, 1, 7, 10, 5$$

is what?

$$\frac{1+3+1+7+10+5}{6} = 4.5$$

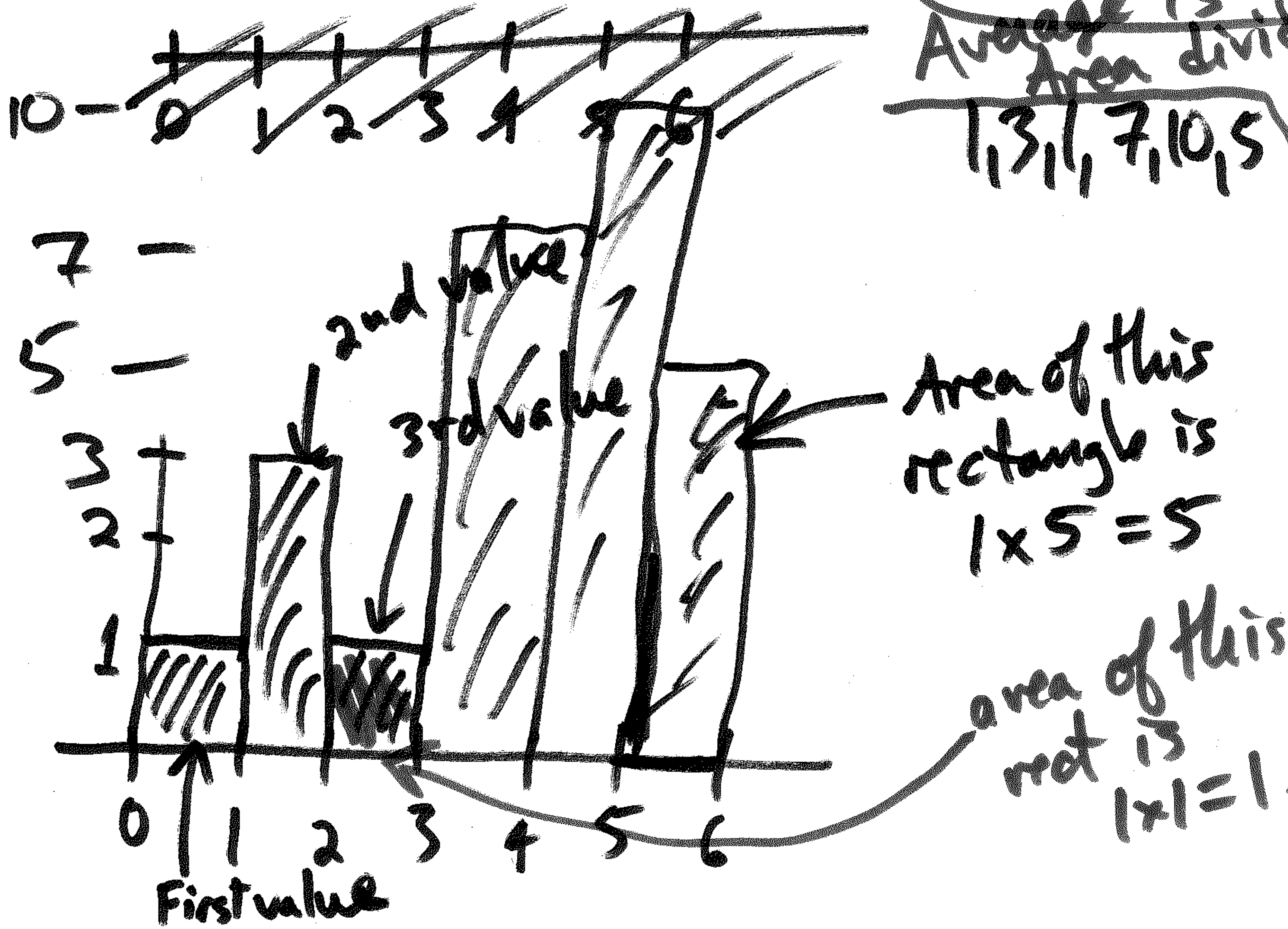
maybe... You check!

six numbers I'm adding

Picture this as an <u>area</u>.

NOTE: This is a better idea! Average is Area divided by 6.

1, 3, 1, 7, 10, 5



Problem: What is the average speed over time 1 sec to 5 seconds of a particle with position at time t given by

$$s(t) = t^3 - 6t^2 \text{ m}?$$

First Q: what is speed? Speed is absolute value of velocity.

Second Q: How do you take an average over a continuous interval?

BUT.... Speed changes continuously!....

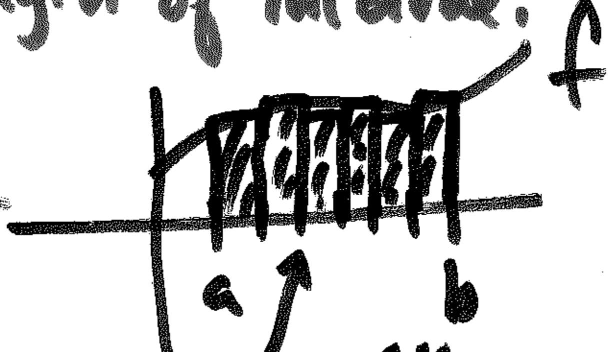
Suggestion: $\boxed{\text{Use}}$ distance function, divide by change in time.

Problem: This word is vague.

How to take average value of a cts function:
If $f(t)$ is cts on $[a,b]$, then average value of f is __defined__ to be

$$\underbrace{\frac{1}{b-a}}_{\text{divide by length of interval.}} \underbrace{\int_a^b f(t)\,dt}_{\text{Compute total value of } f \text{ on } [a,b]}.$$

Why?



x_i is i^{th} sample point.

Pick $N+1$ equally spaced sample points in $[a,b]$.
$x_0 = a, x_1, x_2, \ldots, x_N = b$.
Compute the average value of $f(x_1), \ldots, f(x_N)$

* ⟶ ALMOST.... ⟵ *

We actually compute Riemann Sum w/ right endpoints, giving that the area of the rectangles is

$$\frac{b-a}{N}\left(f(x_1)+\dots+f(x_N)\right).$$

~~Idea: ~~Average value~~ Area of f should be the limit of this as $N \to \infty$, so Area = li~~

Now, $\dfrac{f(x_1)+\dots+f(x_N)}{N}$ is average value.

So, Area of N^{th} Riem. Sum is $(b-a)\cdot$ Average value of sample pts on f.

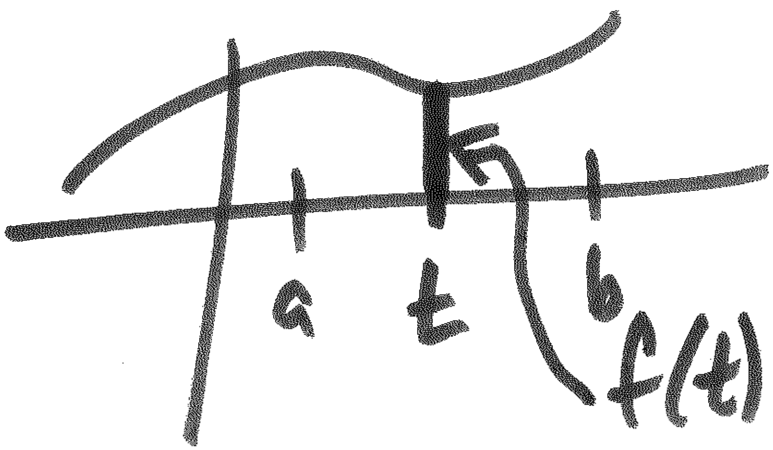
Taking limits as $N \to \infty$, we get

Area under f = $(b-a)$ · "Average value of f"

$$= \int_a^b f(t)\,dt$$

divide by $(b-a)$, get our formula.

<u>Take away:</u> "Total value" of f is $\int_a^b f(t)\,dt$, i.e.



$f(t)$ is this length, + I'm "adding up" these lengths.

Back to problem: position is $s(t) = t^3 - 6t^2$.

Speed is $|v(t)| = |\underbrace{3t^2 - 12t}_{s'(t)}| = |3 \cdot t \cdot (t-4)|$.

The average speed on $[1,5]$ is

$$\frac{1}{5-1} \int_1^5 |3 \cdot t \cdot (t-4)| \, dt \overset{\text{factor 3 out}}{=}$$

$$\frac{3}{4} \left[\int_1^4 -t(t-4)\, dt + \int_4^5 t(t-4)\, dt \right] = \frac{17}{2}.$$

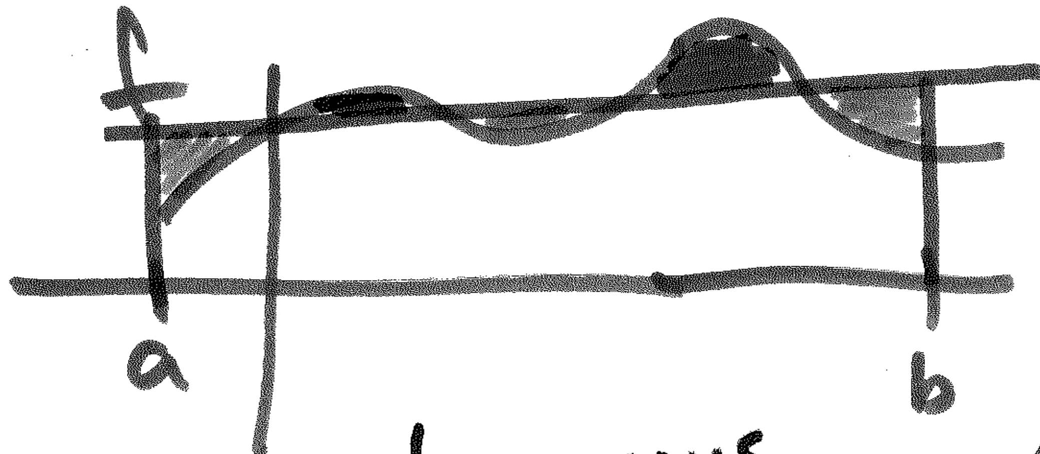
Thm (Mean Value Thm for Integrals)

If f is cts on $[a,b]$, then there is a c in $[a,b]$ such that

$$\frac{1}{b-a}\int_a^b f(t)\,dt = f(c).$$

Idea of proof:

f cts means it must cross the average value line.



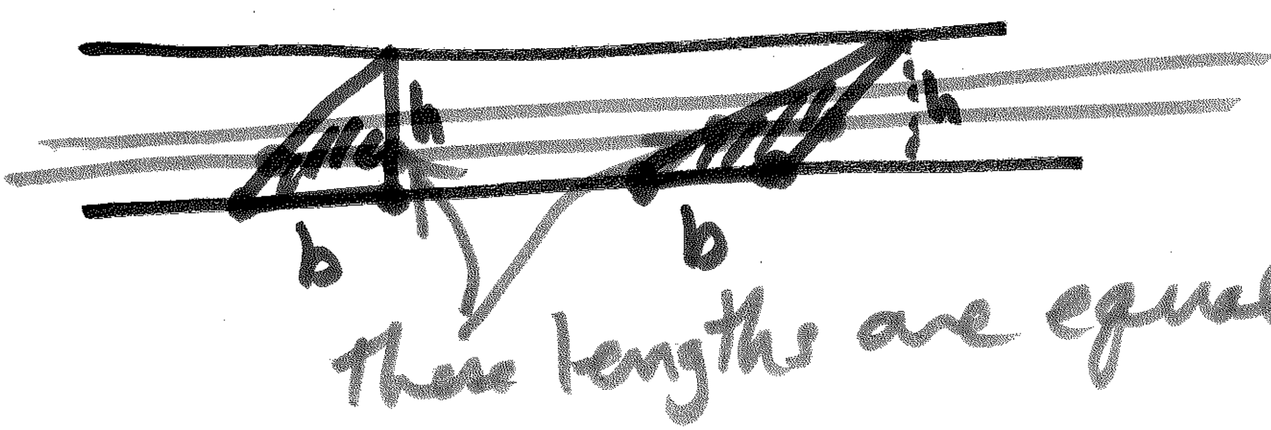
Average value means red + green are equal in area.

average value

2^{nd} Application: Area + Volume.

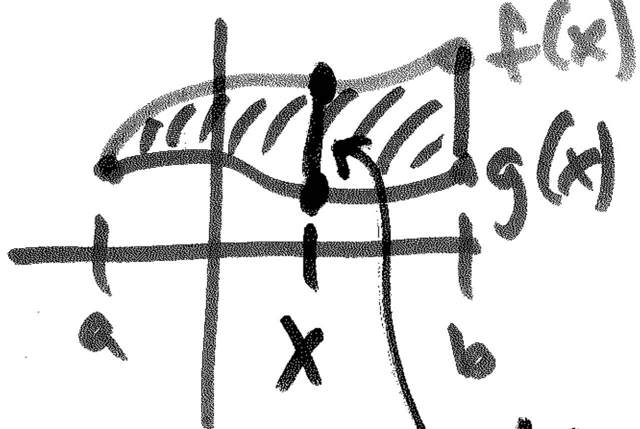
Idea: Cavalieri's Principle: A pair of shapes, either both 2-d or both 3-d, with equal cross-sectional areas have equal areas/volume.

Ex:



Recall Calc I: (§6.1 in book)

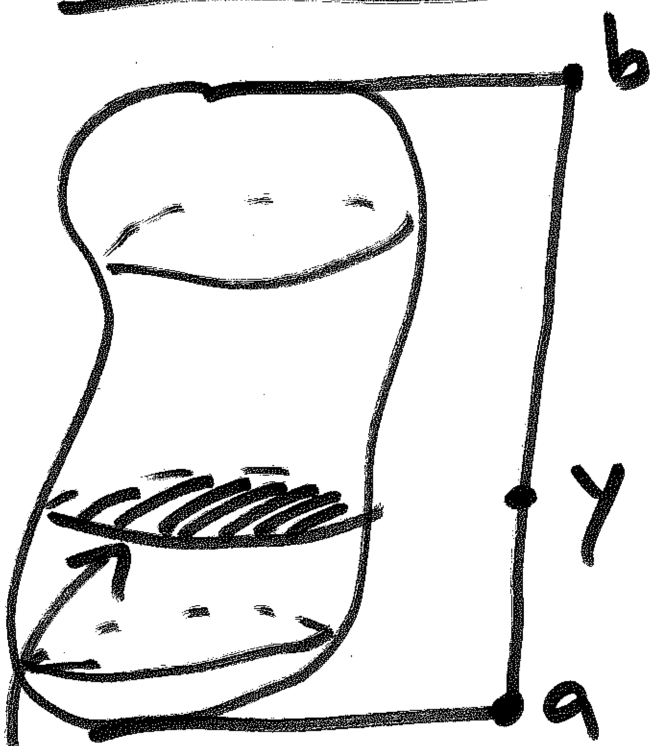
Suppose $g(x) \le f(x)$ for x in $[a,b]$, f, g cts on $[a,b]$. The area between f + g on $[a,b]$ is



$$\int_a^b (f(x)-g(x))\,dx \,.$$

this length is $f(x)-g(x)$.

__For Volume:__ Consider a Solid 3-d body.



cross-sectional area at height y is $A(y)$.

The volume is __defined__ to be

$$\int_a^b A(y)\,dy.$$