Wed, Feb 17, 2016:

Q: What is the volume of this cylinder?

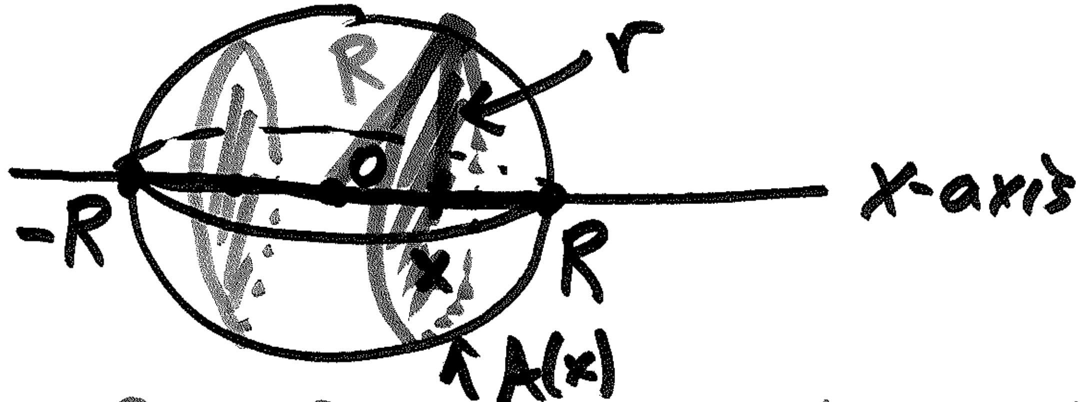
$V=\pi r^2 \cdot h$

So, volume is $\int_0^h \pi r^2\,dy = \left[\pi r^2 y\right]_0^h = \pi r^2 \cdot h$.

Ex: (Do this on your own): Verify that a rectangular box of dimensions l, w, h has volume $l \cdot w \cdot h$ using this def$^{\underline{n}}$.

<u>Ex</u>: Volume of a sphere of radius R.

$V = \frac{4}{3}\pi R^3$.



For x in $[-R, R]$, what is Area of slice of sphere at x? is. $A(x)$.

From right triangle, $x^2 + r^2 = R^2 \Rightarrow r = \sqrt{R^2 - x^2}$.

↑ radius.

So, $A(x) = \pi \cdot r^2 = \pi \cdot (R^2 - x^2)$.

So, $V = \int_{-R}^{R} A(x)\,dx = \int_{-R}^{R} \pi (R^2 - x^2)\,dx = \left(\pi R^2 x - \frac{x^3}{3}\pi\right)\Big|_{-R}^{R}$

$$= \left[\pi \cdot R^3 - \frac{R^3\pi}{3}\right] - \left[\pi(-R)^3 - \frac{(-R)^3}{3}\pi\right]$$

$$= \pi \cdot R^3 - \frac{R^3\pi}{3} + \pi R^3 - \frac{R^3\pi}{3}$$

$$= 2\pi R^3 - \frac{2}{3}R^3\pi = \frac{4}{3}\pi R^3.$$

<u>Recall</u>: If $F'(x) = f(x)$, then

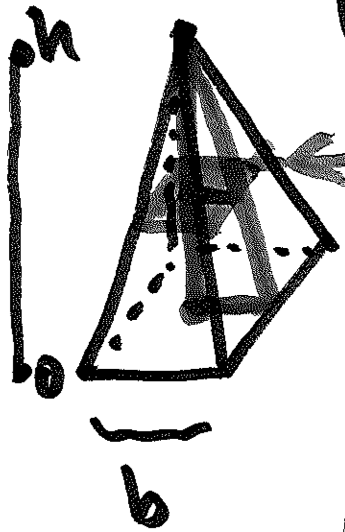
$$\int_a^b f(x)\,dx = F(b) - F(a).$$

<u>NOTE</u>: If you use $F(x) + C$ instead, you get $F(b) + c - [F(a) + c]$

cancel.

Note: Both of the examples so far were using circular cross-sections.

Ex: Consider a square pyramid w/ base $b \times b$ and height h.

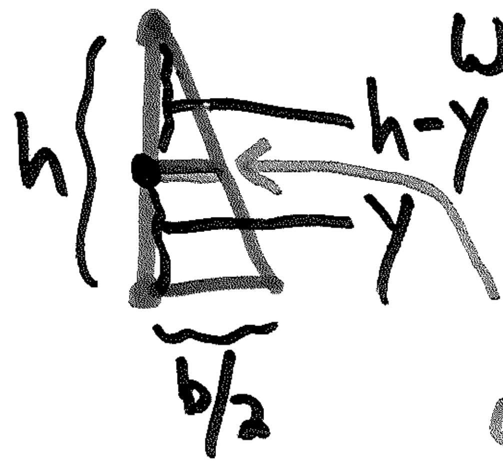


at height y, I have a square cross-section.

Q: What is $A(y)$, the area of the square?

Tool: Similar triangles.

Study the triangles at left...



Want to integrate from 0 to h.

Want this length, since it is half of the side length of the square at height y. Call this length s.

side length of green square.

Similar triangles $\Rightarrow$ $\dfrac{h-y}{s\cdot\frac{1}{2}} = \dfrac{h}{b/2}$

Solving gives $s = \dfrac{b(h-y)}{h}$.

Volume is $\displaystyle\int_0^h s^2\,dy = \int_0^h \left(b - \frac{by}{h}\right)^2 dy = \frac{b^2h}{3}$.

(§6.3)

A <u>solid of revolution</u> is obtained by rotating a region in the plane about an axis.

<u>Ex</u>: Cylinder Cone Sphere

<u>Setup</u>:

cts $f(x)$ on $[a,b]$.

length is $f(x)$.

cross-sectional area is $\pi \cdot (f(x))^2$.

<u>Idea</u>: Add up the areas of all the disks.

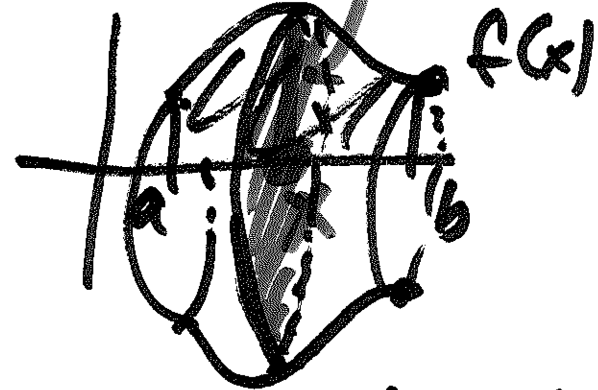
<u>Thm (Disk Method)</u>: If $f(x)$ is cts on $[a,b]$, and $f(x) \geq 0$ on $[a,b]$, then the volume of the solid of revolution about the x-axis defined by f is

$$V = \int_a^b \pi (f(x))^2 \, dx.$$

Fri, Feb 19, 2016

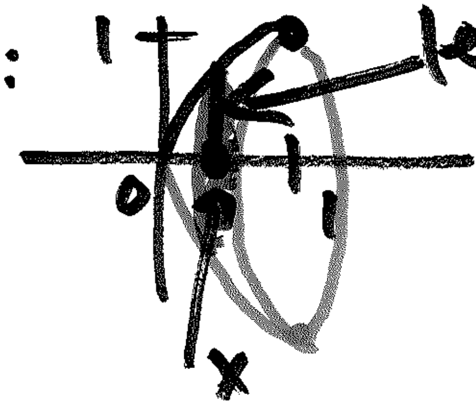
Continue "disk method":

$$V = \int_a^b \pi \cdot [f(x)]^2 \, dx$$



Ex: Find the volume of the solid obtained by rotating $y = \sqrt{x}$ about x-axis from $x = 0$ to 1.

Picture:



length is $\sqrt{x}$

Disk area is $\pi (\sqrt{x})^2$. So,

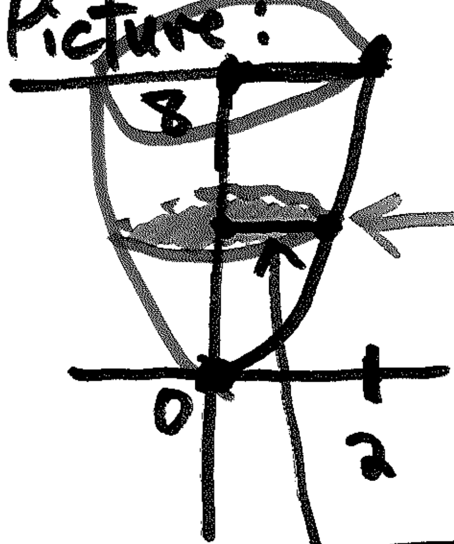
$$V = \int_0^1 \pi (\sqrt{x})^2 \, dx = \frac{\pi}{2}.$$

$$\int_0^1 \pi x\,dx = \left[\pi \frac{x^2}{2}\right]_0^1 = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

<u>Ex</u>: We can rotate about vertical axis:

Find volume when rotate $y = x^3$ about y-axis between $y = 0$ & $y = 8$.

Picture:



$y = x^3$

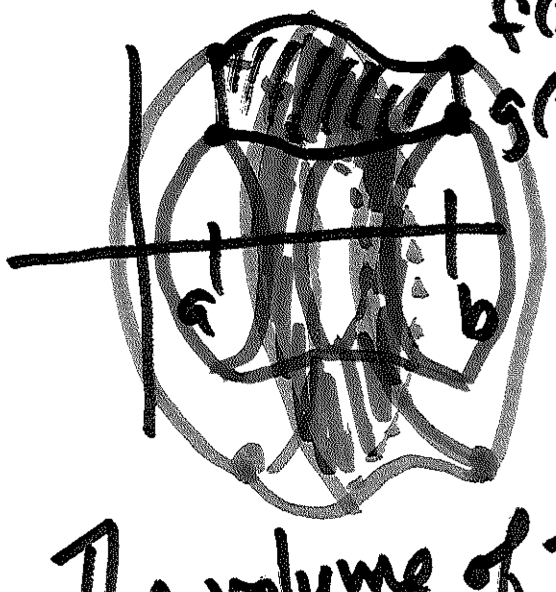
radius is a function of y..... length is: $\sqrt[3]{y}$.

$$V = \int_0^8 \pi \cdot \underbrace{(\sqrt[3]{y})^2}_{\text{Area of green disk}}\,dy$$

$$= \int_0^8 \pi \cdot y^{2/3}\,dy = \frac{96\pi}{5}.$$

Complete the steps on your own.

Thm (Washer Method)



$f(x)=y$

$g(x)=y$

Want the volume between the purple inside region + green outside region.

Suppose $0 \leq g(x) \leq f(x)$, for cts f, g on $[a,b]$.

The volume of the desired region of revolution is

$$\int_a^b \Big[\underbrace{\pi \cdot f(x)^2}_{\text{green disk}} - \underbrace{\pi\, g(x)^2}_{\text{purple disk}}\Big]\, dx.$$

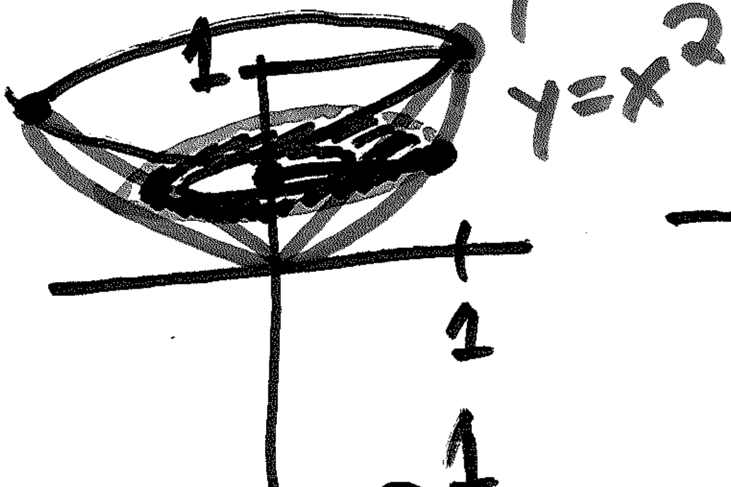
$$= \pi \int_a^b f(x)^2 - g(x)^2 \, dx.$$

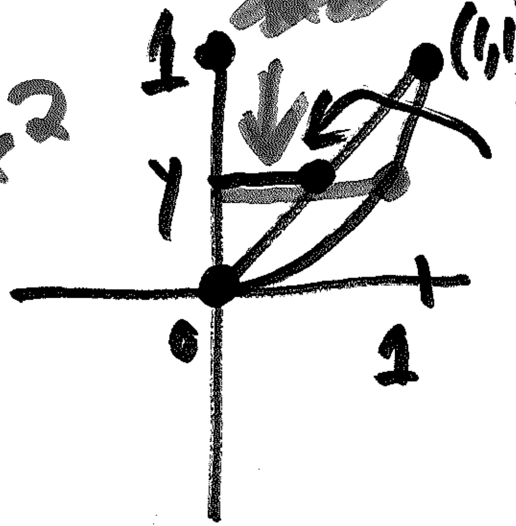
__Ex__: Consider $f(x)=x$, $g(x)=x^2$ between $x=0$, $x=1$. Rotate the region between f + g about the y-axis, + find volume.

$y=x^2 \Rightarrow \sqrt{y}=x$

As functions of y!

__Picture:__



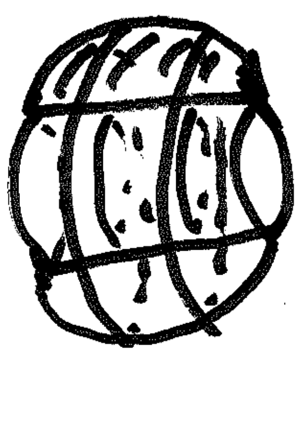


$$\int_0^1 \left[\pi\cdot(\sqrt{y})^2 - \pi y^2\right] dy = \pi\int_0^1 y-y^2\,dy$$

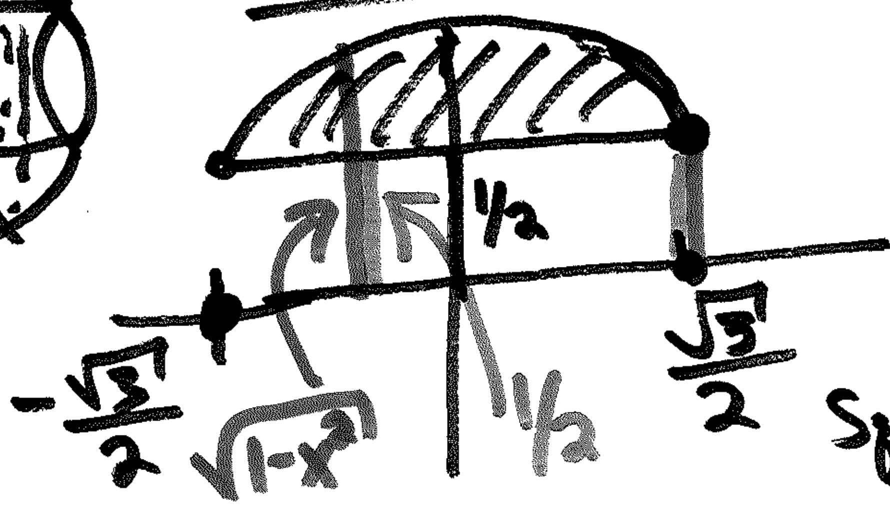
$$= \pi/6$$

<u>WARNING:</u> Sometimes bounds are hard to find.

<u>Ex</u>: Find volume of sphere of radius 1 w/ a cylinder of radius 1/2 drilled out of the center.



<u>Picture is:</u>



<u>Q</u>: what are a & b?

You need to find where

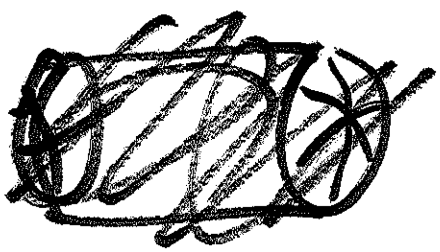
$$\frac{1}{2} = \sqrt{1-x^2}$$

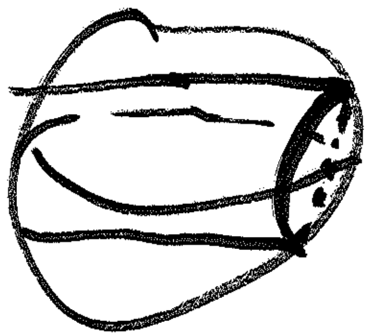
Square both sides, get

$$\frac{1}{4} = 1-x^2 \Rightarrow x = \frac{\pm\sqrt{3}}{2}.$$

$$\int_{-\frac{\sqrt{3}}{2}}^{\sqrt{3}/2} \left[\pi\left(\sqrt{1-x^2}\right)^2 - \pi\left(\frac{1}{2}\right)^2\right] dx \underset{\uparrow}{=} \pi \cdot \frac{\sqrt{3}}{2}.$$

messy details are exercise





$\left.\begin{array}{l}\text{Reading: • linear density} \\ \quad\text{• Flow rate} \\ \quad\text{• radial density}\end{array}\right\}$ You read!

Basic idea: You will need to plug into a formula & integrate.

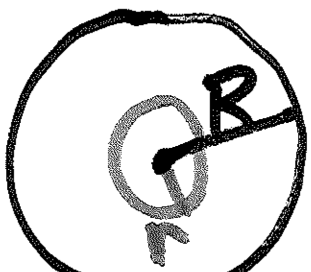
Ex: Linear density given $\rho(x)$,

$$\text{Total Mass} = \int_a^b \rho(x)\,dx.$$

Mon, Feb 22, 2016:

§6.4: Cylindrical Shells.

Recall: Area of a disk of radius R by integration:

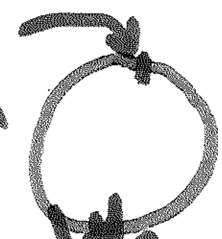


$$\text{Area} = \text{sum of lengths of fibers}$$

$$= \int_0^R \underbrace{2\pi r \, dr}_{\text{what } \underline{\underline{\text{is}}} \text{ this?}} = \left[2\pi \frac{r^2}{2}\right]_0^R = \pi R^2$$

Think of disk as being built of circular fibers.

dr is an ∞-ly small width

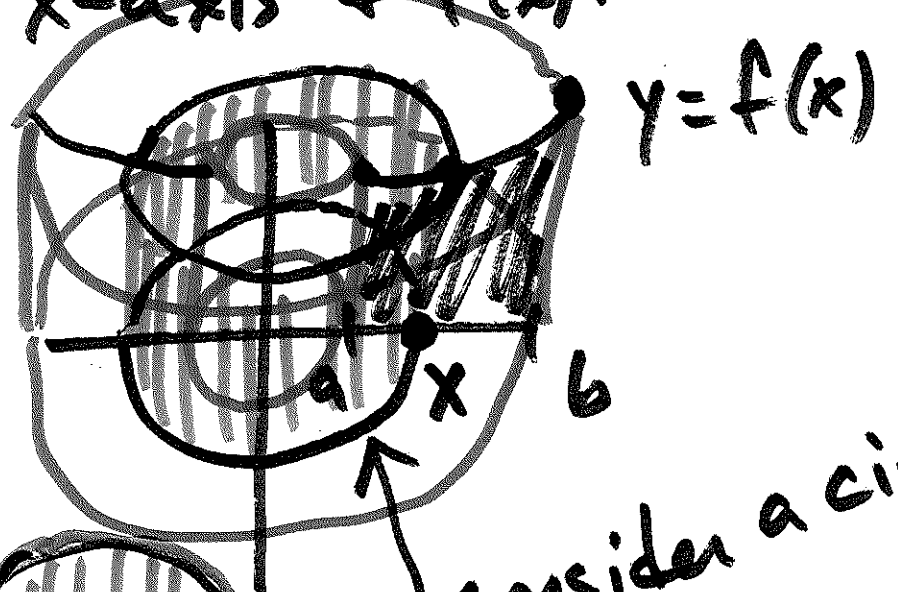


$2\pi r$ is length

$2\pi r \cdot dr$ is an "area" of an ∞-ly thin strip.

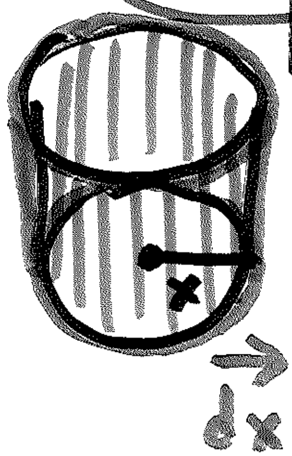
Extend this to 3-D:

Picture in general is rotate graph of $f(x)$ about y-axis, look at solid between x-axis + $f(x)$.



Idea: Add the areas of these cylinders to get volume.

consider a circle in the base, + lift a cylinder over it.



"Cylindrical shell" since hollow...

Thm (Shell Method):

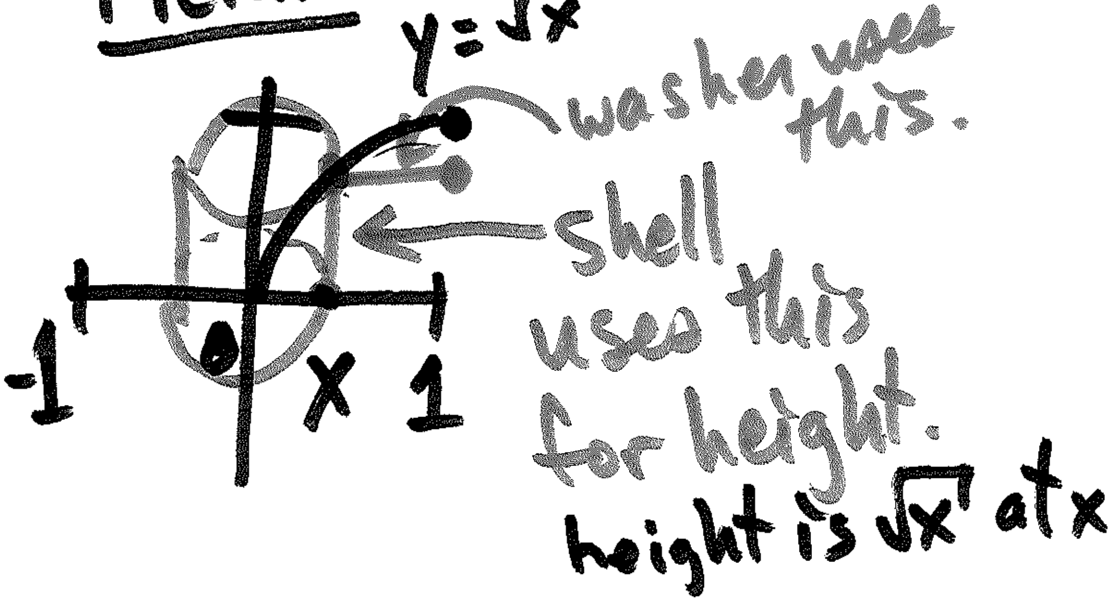
The solid obtained by rotating the region under $y = f(x)$ over the interval $[a,b]$ about the y-axis for f cts on $[a,b]$ has volume

$$V = \int_a^b \overbrace{\underbrace{2\pi x}_{\text{circumference of circle}} \; \underbrace{f(x)}_{\text{height of cylinder}}}^{\text{area of shell}} \; \underbrace{dx}_{\infty\text{-ly thin width}}$$

Ex: Find the volume of solid of revolution from rotating $y = \sqrt{x}$ on $[0,1]$ about y-axis using shell method.

NOTE: You can use "washer" method, but shell is easier.

Picture:



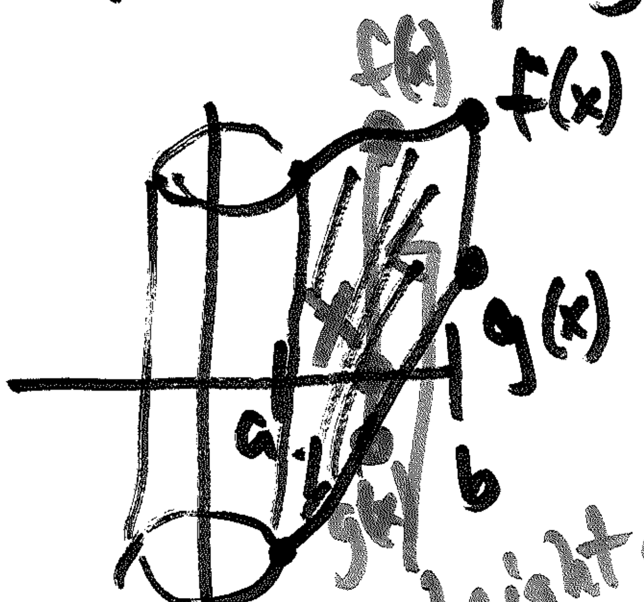
$$V = \int_0^1 2\pi x \sqrt{x}\, dx =$$

$$2\pi \int_0^1 x^{3/2}\, dx = \frac{4\pi}{5}.$$

power rule.

Thm: If f, g are cts on $[a,b]$ with $g(x) \le f(x)$ on $[a,b]$, then volume obtained when rotating about y-axis the region bounded by g + f on $[a,b]$ is

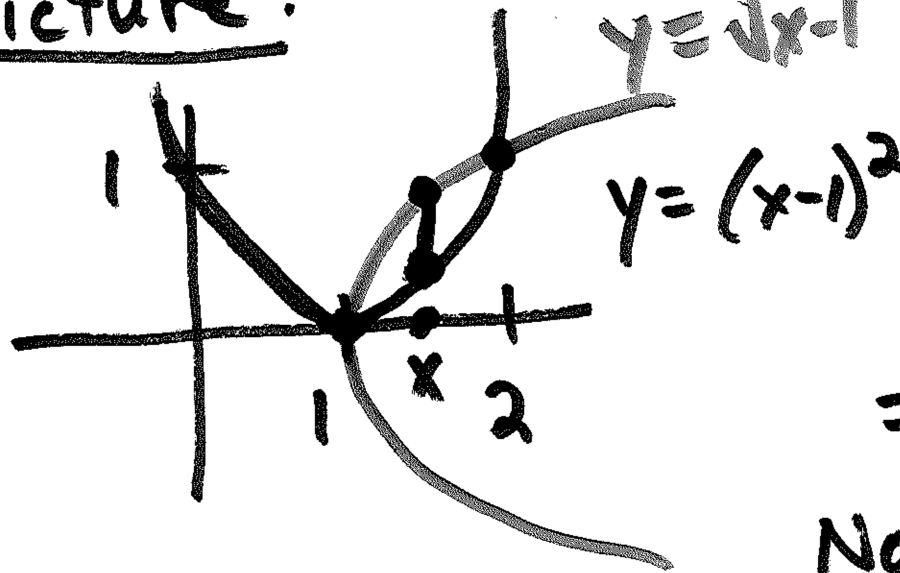
$$V = \int_a^b 2\pi x \left(f(x) - g(x)\right) dx$$



height of shell is $f(x) - g(x)$.

Ex: Find volume when rotating about y-axis with $\sqrt{x-1}$, $(x-1)^2$ as my functions.

Picture:



Check intersection points:

$$\sqrt{x-1} = (x-1)^2$$

$$\Rightarrow \quad x-1 = (x-1)^4$$

NOTE: $x-1$ is a line, $(x-1)^4$ is degree 4 polynomial, so there could be more than 2 intersection points.

In this case, sufficient to graph & observe points of intersection are ~~[illegible]~~ $(1,0)$ and $(2,1)$.

Check which is greater:

From picture, $\sqrt{x-1} \geq (x-1)^2$ on $[1,2]$.

So, my volume is

$$V = \int_1^2 2\pi x\left(\sqrt{x-1} - (x-1)^2\right) dx$$

$$= \int_1^2 2\pi x\sqrt{x-1}\, dx - \int_1^2 2\pi x(x-1)^2\, dx$$

$$= 2\pi \underbrace{\int_1^2 x\sqrt{x-1}\, dx}_{\text{This needs extra work}} - 2\pi \underbrace{\int_1^2 x(x-1)^2\, dx}_{\text{This is more straightforward.}} = \star$$

For left-hand integral, recall u-substitution.

Set $u = x-1$, so $u+1 = x$

and $du = dx$.

$$\star = 2\pi \int_0^1 (u+1)\sqrt{u}\, du - 2\pi \int_1^2 x(x-1)^2\, dx$$

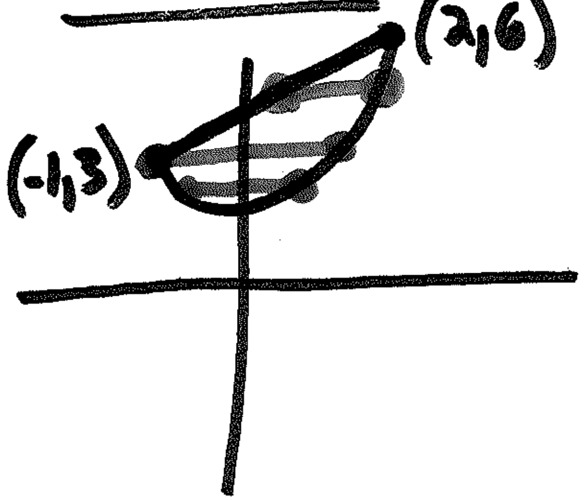
$$= 2\pi \int_0^1 u\sqrt{u}\, du + 2\pi \int_0^1 \sqrt{u}\, du - 2\pi \int_1^2 x(x-1)^2\, dx$$

$$= \frac{29\pi}{30}.$$

Exercise by power rule.

$\underline{\text{Ex}}$: $y = x^2 + 2$, $y = x + 4$, rotate about x-axis.

Picture:



Find endpoints by solving
$x^2 + 2 = x + 4$ for x using
quad. form.

WARNING: At $y = 3$, we change how we describe height of our shells.

fixed y —— length = (x-coord on $x^2+2=y$) − (x-coord on $x+4=y$)

fixed y —— length = 2(x-coord on $x^2+2=y$)

This ends up being

$$y = x^2 + 2 \Rightarrow x = \sqrt{y-2}$$

$$y = x + 4 \Rightarrow x = y - 4.$$

Using shells

$$V = \underbrace{\int_2^3 2\pi y \left(\overbrace{\sqrt{y-2} - \left(-\sqrt{y-2}\right)}^{= 2\sqrt{y-2}}\right) dy}_{\text{blue section}} + \int_3^6 2\pi y \left(\sqrt{y-2} - (y-4)\right) dy$$

$$= \int_2^3 2\pi y \cdot 2\sqrt{y-2}\, dy + \int_3^6 2\pi y \left(\sqrt{y-2} - y + 4\right) dy$$

$$= \frac{162\pi}{5}$$

Use u-sub.