MONDAY, Feb 1, 2016 | EAVES LECTURE TODAY 4PM, CB114

0. **Reminder**: Exam 1 is Tuesday, Feb 9.

1. Alternate Exam assignments will be arranged this week, watch your email if you submitted a form last week.

2. **With your neighbors:**

Q: Why is $.9999... = 1$? (Don't compute, just state in words.)

Q: What are the **specific** **examples** of infinite series that you keep in mind to make sense of this Material in Ch. 10? (Remember our discussion about concept maps from Friday...)

when discussing series, say

"the sum of $\frac{1}{2^n}$" to mean "starting at..."

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$$

or... $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$

§10.5: Ratio + Root Tests

<u>Idea:</u> We need the terms of $\sum_{n=1}^{\infty} a_n$ to go to zero "fast enough" to converge.

<u>Compare:</u>

- $\sum \frac{1}{n}$ diverges, so $\frac{1}{n}$ does not go to zero "rapidly", even though $\frac{1}{n} \to 0$ as $n \to \infty$.
- $\sum \frac{1}{n^2}$ converges, so $\frac{1}{n^2} \to 0$ "faster" than $\frac{1}{n} \to 0$.

__Ratio Test:__ For $a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$,

if $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists and equals ρ, then:

i) if $\rho < 1$, $\sum_1^{\infty} a_n$ converges absolutely.

ii) if $\rho > 1$, $\sum_1^{\infty} a_n$ diverges.

iii) if $\rho = 1$, inconclusive (series could converge __or__ diverge.)

__Idea of proof:__ Compare series to a geometric series! See book.

Ex: Consider $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots = \sum_{n=1}^{\infty} \frac{1}{n}$.

[We know divergent.]

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{1}{n+1}}{\frac{1}{n}}\right| = \lim_{n\to\infty} \left|\frac{n}{n+1}\right| = 1.$$

So, since $\rho = 1$, Ratio Test not conclusive!

Ex: Consider $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots = \sum_{n=0}^{\infty} \frac{1}{2^n}$.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}}\right| = \lim_{n\to\infty} \left|\frac{1}{2}\right| = \frac{1}{2} < 1.$$

Cancel $1/2$'s.

By ratio test, converge absolutely.

Ex: Geom. Series. $1 + r + r^2 + r^3 + \cdots = \sum_{n=0}^{\infty} r^n$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{r^{n+1}}{r^n}\right| = \lim_{n\to\infty}|r| = \underbrace{|r| < 1}_{\text{ratio test}}$$

Cancel r^n's.

$\Rightarrow$ abs. conv.

Ex: Recall notation: $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$.

AND: $0!$ is defined to be 1.

$0!$	$1!$	$2!$	$3!$	$4!$	$5!$	$\cdots$
$?$	1	$2\cdot 1$	$3\cdot 2\cdot 1$	$4\cdot 3\cdot 2\cdot 1$	$5\cdot 4\cdot 3\cdot 2\cdot 1$	$\cdots$

$\boxed{1}$

Ex: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$

$\displaystyle = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots$

Fact: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} = e \approx 2.71\ldots$

Reason: $\displaystyle e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

Ratio Test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} \right| = \lim_{n\to\infty} \left| \frac{1}{n+1} \right| = 0 < 1.$$

So, $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$ converges abs. by ratio test.

__Root Test__: For $\sum_{n=1}^{\infty} a_n$, if the limit

$\lim_{n \to \infty} \sqrt[n]{|a_n|} = L$ exists, then:

i) If $L < 1$, series conv. abs.

ii) If $L > 1$, series diverges

iii) If $L = 1$, inconclusive.

Proof is by comparing to a geom. series.

Ex: $\displaystyle\sum_{n=0}^{\infty}\left(\frac{5n-3n^3}{7n^3+2}\right)^n$ ← n^{th} power is clue to use root test

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{5n-3n^3}{7n^3+2}\right)^n\right|} =$$

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{5n-3n^3}{7n^3+2}\right|^n} = \lim_{n\to\infty}\left|\frac{5n-3n^3}{7n^3+2}\right| =$$

$\underbrace{\ldots\ldots}_{\text{You do algebra!}} = \frac{3}{7} < 1$. So, by root test, series conv. abs.

§10.6: Power Series

(Feb 3, 2016

Recall: $1+r+r^2+r^3+\cdots = \dfrac{1}{1-r}$ when $|r|<1$.

Idea: Instead of replacing r w/ a number, substitute a function.

Ex: $r=\frac{x}{2}$ gives a function of x

$$f(x) = 1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \frac{x^4}{16} + \cdots = \frac{1}{1-\frac{x}{2}}$$

$$= \frac{2}{2-x}$$

→ graph partial sums & compare to $f(x)$ ←

$f(x)$ converges when $|r|<1$. So, if $r=\frac{x}{2}$, what does this mean for x?

This means $\left|\frac{x}{2}\right|<1$, so $|x|<2$.

Q: What values of x make the following true?

$$\frac{1}{1-\frac{x-1}{3}} = \frac{3}{4-x} = 1+\frac{(x-1)}{3}+\frac{(x-1)^2}{3^2}+\frac{(x-1)^3}{3^3}+\ldots$$

A: $|r|<1$ with $r=\frac{x-1}{3}$, so $\left|\frac{x-1}{3}\right|<1$.

We write $|x-1|<3$. ← we will call this the radius of convergence.

From geom. series we investigated

$$\sum_{n=0}^{\infty} \frac{1}{3^n}(x-1)^n.$$

Def$^{\underline{n}}$: A <u>power series</u> with center c is an infinite series of the form

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

<u>Remark</u>: The "best" power series are Taylor series...

__Thm 1, §10.6:__ (Radius of Convergence)

Every power series $F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$

has a __radius of convergence__ R,

where R is either ∞ or a non-neg. real number $R \geq 0$, and $F(x)$ __converges__ absolutely! when $|x-c| < R$.

Further, $F(x)$ diverges if $|x-c| > R$.

__Note:__ when $|x-c| = R$, you have to check case by case. i.e. $x = c+R$, or $x = c-R$

Two main tools for detecting radius of cnv. (R of C).

I) Geom Series

II) Ratio Test.

<u>Ex</u>: Find the R. of. C. for

$$\sum_{n=1}^{\infty} \frac{5}{6^n}(x-4)^n$$ using geom. series.

<u>Hint</u>: Write r as a function of x ...

Then use $|r|<1$ ...

$\sum_{n=1}^{\infty} \frac{5}{6^n}(x-4)^n$

(1). $r = \left(\frac{5x-20}{6}\right)^n$ true if 5^n

(2). $r = (x-4)$

(3). $r = \left(\frac{x-4}{6}\right)$

true if 6 w/ no exponent

$\sum_{n=1}^{\infty} 5 \cdot \frac{1}{6^n} \cdot (x-4)^n$

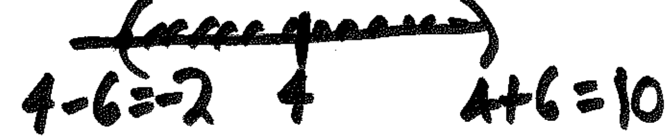
$= 5 \sum_{n=1}^{\infty} \left(\frac{x-4}{6}\right)^n$

geom series w/ $c=5$, $r = \frac{x-4}{6}$.

when is $|r|<1$?

Need $\left|\frac{x-4}{6}\right| < 1$, so $|x-4| < 6$.

So R.o.C. is 6.



NOTE: ON Friday, Ratio Test example!