Wed, Feb 24, 2016:

1. No office hours Friday.
2. All Math Meeting TODAY
   5-6:15PM CB 110 (Pizza ...)

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§6.5: Work

Units: Force is measured in Newtons,

$$1N = 1 \frac{kg \cdot m}{s^2}.$$

Energy or Work is measured in Joules,

$$1J = 1 Nm = 1 \frac{kg \cdot m^2}{s^2}.$$

__Idea__: Work is what you do when you apply force over a distance.

If you apply a constant force over a fixed distance, the work required to do so is __defined__ to be

$$\underset{\text{Force}}{F} \cdot \underset{\text{distance}}{d} = W.$$

__Recall__: $F = \underset{\text{mass}}{m} \cdot \underset{\text{accel.}}{a}$

__Ex__: To lift a 3kg box 2 meters high, we have $d = 2m$, $F = 3kg \cdot (\underset{\text{accel. gravity}}{-g}) = -3 \cdot 9.8 \frac{kg\,m}{s^2}$

So, Work to do so is $F \cdot d = -6 \cdot 9.8 \frac{kg \cdot m^2}{s^2} = -58.8J$

Q: What happens if our force is varying?

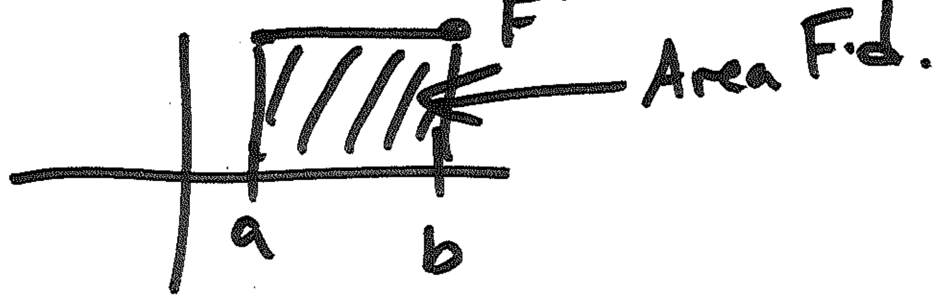
Def$^{\underline{n}}$: The work performed in moving an object along the x-axis from a to b by applying a force of magnitude $F(x)$ at point x is

$$W = \int_a^b F(x)\,dx.$$

NOTE: See book for Riemann Sum motivation.

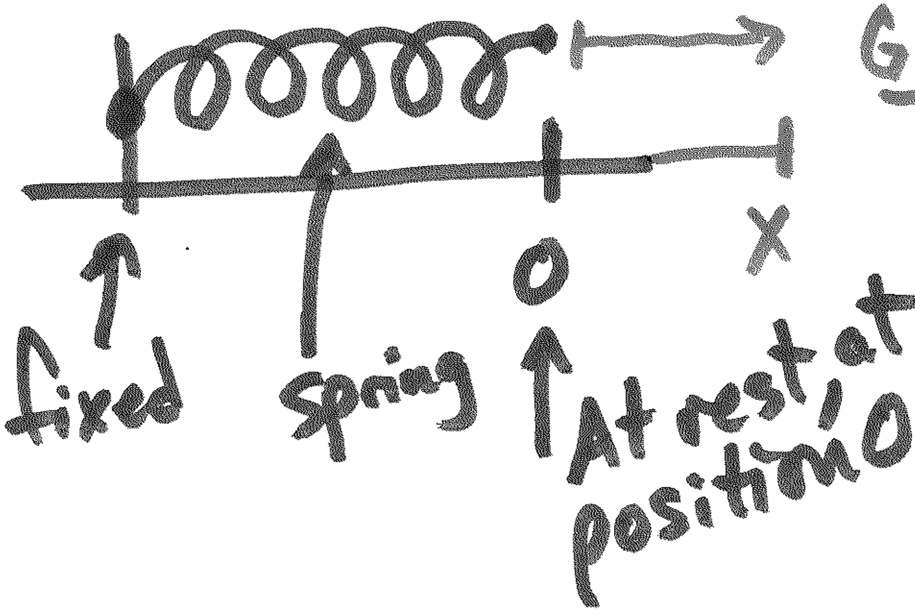
Two motivations:

① If F is constant, $d = b - a$, $F \cdot d$ is area under $y = F$ on $[a,b]$.



② At each moment x, your work is $F(x) \cdot (\text{tiny distance}) = F(x) \cdot dx$. Adding all these tiny amounts of work is done w/ integral.

__Ex__: Hooke's Law + Springs.



__Goal__: Pull or displace the spring a length of x from resting position. How do we compute the amount of work required to do this?

Hooke's Law says each spring has a __constant__ k so that when stretched to point x, the restoring force is $-kx$.

This is an example of __linearity__, further explored in MA322.

Compute Work in Joules required to stretch or compress a spring w/ $K = 800\,N/m$

1. Stretch to 12cm past equilibrium.

Set up integral: Note: Watch units! Convert cm to m.

$$\int_0^{.12} \underbrace{800x}_{\text{force}} \cdot \underbrace{dx}_{\substack{\text{distance} \\ \text{(tiny)}}} = \left[800\frac{x^2}{2}\right]_0^{.12} = 5.76\,J.$$

2. Stretch from 5cm to 15cm past equilibrium?

$$\int_{.05}^{.15} 800x\,dx = \left[800\frac{x^2}{2}\right]_{0.05}^{.15} = 8\,J.$$

Watch out for signs!

Gravity:

is this dist a pos or neg dist?

$9.8\ \frac{m}{s^2}$

Force to lift up is against gravity, so ~~negative~~ change sign.

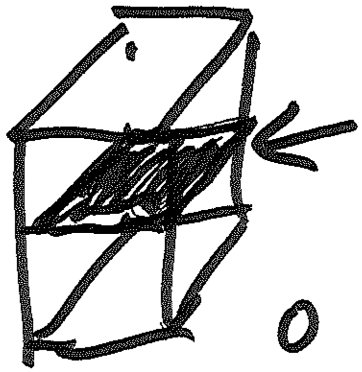
Spring:

stretched

to pull the spring, force is kx.

restoring force is $-kx$.

__Ex:__ To lift up or pump full an object, each layer of the object has to work against gravity.



Mass at layer y has to work against gravity at height y.

Layer has area $A(y)$, so volume of y^{th} layer is area · width

$$= A(y) \cdot \underbrace{dy}_{\text{tiny}}.$$

Mass of layer is density · volume, so

$$\text{mass} = (\text{density}) \cdot A(y) \cdot dy.$$

Force on y^{th} layer is accel · mass, so

$$\underset{\text{grav. accel.}}{g} \cdot (\text{density}) \cdot A(y) \cdot dy$$

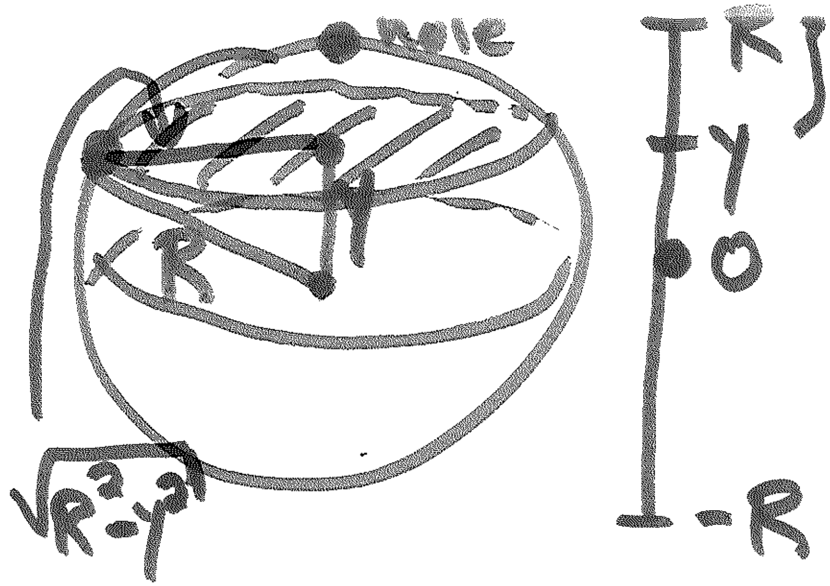
Work for layer at a given height is force · dist, so

$$\begin{pmatrix}\text{distance}\\ \text{moved}\end{pmatrix} \cdot g \cdot (\text{density}) \cdot A(y) \cdot dy .$$

To compute total work in this case, <u>sum</u> the work for all layers:

$$\text{Total Work} = W = \int_a^b \begin{pmatrix}\text{dist.}\\ \text{moved}\end{pmatrix} \cdot g \cdot (\text{density}) \cdot A(y) \cdot dy .$$

<u>Ex</u>: (Ex 3 in §6.5) A spherical tank of radius R meters is filled w/ water. To pump water <u>out</u> of a small hole at top, how much work is required? Water density is $1000\ kg/m^3$.



Moves y^{th} layer of water up a distance of $R-y$.

Area of y^{th} layer is

$$A(y) = \pi\left(\sqrt{R^2-y^2}\right)^2 = \pi(R^2-y^2).$$

I have oriented ↑ as positive, so the acceleration I need to move it is $9.8\ m/s^2$, since $g = 9.8\ m/s^2$.

$$\text{So, } W = \int_{-R}^{R} (R-y)\cdot 9.8 \cdot 1000 \cdot \pi(R^2-y^2)\,dy.$$

Work it out: $= \frac{39,200}{3} \pi R^4$ J.

Friday, February 26th: Integration Techniques

Evaluate $\int x e^{x^2}\, dx$

We can use u-substitution (calc I)

$$u = x^2$$
$$du = 2x\, dx$$

$$\int x e^{x^2}\, dx = \int e^u \tfrac{1}{2}\, du$$
$$= \tfrac{1}{2} e^u + C$$
$$= \tfrac{1}{2} e^{x^2} + C.$$

(review!)

What about $\int x e^x \, dx$?

<u>Note</u>: u-sub. will not work.

<u>Integration by Parts Formula</u> (IBP)

$$\int \underline{u(x) \cdot v'(x) \, dx} = u(x) \cdot v(x) - \int u'(x) v(x) \, dx$$

This formula arises from integrating the product rule.

$$\int \left(u(x) \cdot v(x)\right)' = \int u'(x) v(x) + u(x) \cdot v'(x)$$

$$u(x)\cdot v(x) = \int u'(x)v(x)\,dx + \int \underline{u(x)\cdot v'(x)}\,dx$$

$$u(x)\cdot v(x) - \int u'(x)v(x)\,dx = \int u(x)\cdot v'(x)\,dx.$$

We can rewrite as

$$\int u\,dv = uv - \int v\,du$$

Ex: Evaluate $\int \underline{x}\cdot \underline{e^x\,dx}$

$u = x \qquad dv = e^x\,dx$

$du = 1\cdot dx \qquad v = \int e^x\,dx = e^x$

By IBP formula,

$$\int xe^x\,dx = x\cdot e^x - \int e^x\,dx$$
$$= xe^x - e^x + C.$$

<u>Ex:</u> Evaluate $\int \underline{x^3}\cdot\underline{\ln(x)}\,\underline{dx}$.

$u = \ln(x) \qquad dv = x^3\,dx$

$du = \frac{1}{x}\,dx \qquad v = \int x^3\,dx = \frac{1}{4}\cdot x^4$

using IBP

$$\int x^3\ln(x)\,dx = \ln(x)\cdot\frac{1}{4}x^4 - \int \frac{1}{4}x^4\cdot\frac{1}{x}\,dx$$

$$= \frac{\ln(x)\cdot x^4}{4} - \frac{1}{4}\int x^3\,dx$$

$$= \frac{\ln(x)\cdot x^4}{4} - \frac{1}{4}\cdot\frac{x^4}{4} + C$$

$$= \frac{\ln(x)\cdot x^4}{4} - \frac{x^4}{16} + C$$

<u>Remark:</u> If we had chosen $u = x^3$. Then $dv = \ln(x)dx$ so $v = \int \ln(x)dx = ?$

<u>Tips for choosing "u"</u>

- pick u so that du is simpler than u.
- pick dv so that $v = \int dv$ is computable

- LIPET is an acronym for choosing which function should be "u":

start → **L**og

Inverse Trig.

Polynomial

Exponential

Trig.

interchangeable (Exponential, Trig.)

<u>Ex:</u> Evaluate $\int x^2 \sin(x)\,dx$.

$u = x^2$ $\qquad$ $dv = \sin(x)\,dx$

$du = 2x\,dx$ $\qquad$ $v = \int \sin(x)\,dx$

$\qquad\qquad\qquad = -\cos(x)$

$$= -x^2\cos(x) - \int -\cos(x)\cdot 2x\,dx$$

$$= -x^2\cos(x) + \int \underline{2x}\,\underline{\cos(x)\,dx}$$

$u = 2x \qquad dv = \cos(x)\,dx$

$du = 2\,dx \qquad v = \int \cos(x)\,dx = \sin(x)$

using IBP,

$$= -x^2\cos(x) + 2x\sin(x) - \int 2\sin(x)\,dx$$

$$= -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C$$

__Ex:__ Evaluate $\int \underline{e^x} \, \underline{\cos(x)\,dx}$.

$u = e^x \qquad dv = \cos(x)\,dx$

$du = e^x\,dx \qquad v = \int \cos(x)\,dx = \sin(x)$

$$= e^x \sin(x) - \int \underline{\sin(x)} \cdot \underline{e^x} \, \underline{dx}$$

$u = e^x \qquad dv = \sin(x)\,dx$

$du = e^x\,dx \qquad v = \int \sin(x)\,dx = -\cos(x)$

$$\int e^x \cos(x)\,dx = e^x \sin(x) - \left(-e^x \cos(x) - \int -\cos(x) e^x \,dx \right)$$

$$\int e^x \cos(x)\,dx = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\,dx$$

$$+\int e^x \cos(x)\,dx \qquad\qquad +\int e^x \cos(x)\,dx$$

$$\frac{2\int e^x \cos(x)\,dx}{2} = \frac{e^x \sin(x) + e^x \cos(x)}{2} + 0 + C$$

$$\int e^x \cos(x)\,dx = \frac{e^x}{2}\left(\sin(x) + \cos(x)\right) + C$$

__Note:__ Look for "wrap around" IBP.

Remark: IBP applies to definite integrals.

Ex: $\displaystyle\int_0^1 \arctan(x)\,dx$

$u = \arctan(x) \qquad dv = 1 \cdot dx$

$du = \dfrac{1}{1+x^2}\,dx \qquad v = \displaystyle\int dx = x$

$$\int_0^1 \arctan(x)\,dx = x\arctan(x)\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$$

now use u-sub

$u = 1+x^2$

$du = 2x\,dx$

$$= x\arctan(x)\Big|_0^1 - \int_{u(0)}^{u(1)} \frac{1}{2}\cdot\frac{1}{u}\,du$$

$$= x\arctan(x)\Big|_0^1 - \frac{1}{2}\ln(|1+x^2|)\Big|_0^1$$

$$= 1\cdot\arctan(1) - \frac{1}{2}\ln(1+1^2) - \left(0 - \frac{1}{2}\ln(1)\right)$$

$$= \frac{\pi}{4} - \frac{\ln(2)}{2}.$$