## Monday, Feb 29

1. Exam 2 next week!
2. Fill out survey for review topics
3. Talk about Cloaking on March 22. see canvas announcement.

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<u>Recall:</u> $\int uv'\,dx = uv - \int vu'\,dx$ ("undoing" product rule)

<u>Problem:</u> $\int \sin^2 x \cos^4 x\,dx$ <u>looks</u> like a good candidate for IBP, but, it doesn't really work here immediately. How to handle it?

General Problem for today:

Compute $\int \sin^m x \cos^n x \, dx$.

<u>One application in pure math:</u> See problem 78 in §7.2 to use $\int_0^{\pi/2} \sin^m x \, dx$ in proof of

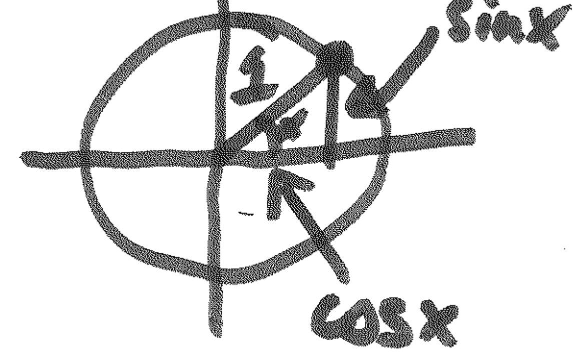
"Wallis formula": $\dfrac{\pi}{2} = \dfrac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdots}$

There a 3 tools we will need.

Tools:

$\boxed{0}$ Pythagorean identities

$\cos^2 x = 1 - \sin^2 x$ , $\sin^2 x = 1 - \cos^2 x$.




$\boxed{1}$ Reduction thm for $\sin^n x$.

$$\int \sin^n x \, dx = \frac{-1}{n}(\sin^{n-1} x)(\cos x) + \frac{n-1}{n}\int \sin^{n-2} x \, dx.$$

complicated!

lower power here.

Idea: If $n$ is even, iterate this to get a long formula w/out "$\int$" signs.

$\boxed{2}$ Reduction thm for $\cos^n x$.

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1}(x) \sin(x) + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

Why is this true? IBP!

<u>Pf of $\boxed{1}$</u>: $\frac{d}{dx}\left((\sin^{n-1} x) \cdot (\cos x)\right)$

$$= (n-1) \sin^{n-2} x - n \sin^n x.$$

some extra steps, including $\cos^2 x = 1 - \sin^2 x$.

Integrate both sides, and $\boxed{1}$ is the result. $\blacksquare$

**Ex**: Evaluate $\int \sin^5 x \, dx$. ← odd exponent, so break off one factor.

$$\int \sin^4 x \cdot \underbrace{\sin x \, dx}_{= -(\cos x)' \cdot dx} = ☆.$$

This suggests u-sub *if* I had a $\cos x$ in integrand.

**NOTE**: $\sin^4 x = (\sin^2 x)^2 = (1-\cos^2 x)^2$.

$u = \cos x$

$du = -\sin x \, dx$

$$☆ = \int (1-\cos^2 x)^2 \sin x \, dx$$

$$= -\int (1-u^2)^2 \, du = -\int 1 - 2u^2 + u^4 \, du =$$

Next pg →

$$-\left(u - \frac{2}{3}u^3 + \frac{u^5}{5}\right) + C =$$

$$\underbrace{-\cos x + \frac{2}{3}\cos^3 x - \frac{\cos^5 x}{5} + C.}_{= \int \sin^5 x\, dx.}$$

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<u>Check</u>: $\int \cos^7 x\, dx = \int (\cos^2 x)^3 \cos x\, dx$

$= \int (1 - \sin^2 x)^3 \cos x\, dx$

1. Use pyth. identity
2. Select u-sub to make.

Then set $u = \sin x$.

<u>NOTE</u>: Even exponents, apply $\boxed{1}$ & $\boxed{2}$ iteratively.

by $\boxed{1}$

<u>Ex</u>: $$\int \sin^4 x\,dx = \frac{-1}{4}\sin^3 x\cos x + \frac{3}{4}\int \sin^2 x\,dx$$

by $\boxed{1}$

$$= \frac{-1}{4}\sin^3 x\cos x + \frac{3}{4}\left[\frac{-1}{2}\sin x\cos x + \frac{1}{2}\int dx\right]$$

$$= \frac{-1}{4}\sin^3 x\cos x - \frac{3}{8}\sin x\cos x + \frac{3}{8}x + C.$$

Case: $\int \sin^m x \cos^n x \, dx$, one exponent is even, one is odd.

Ex: $\int \sin^2 x \underbrace{\cos^7 x}_{\text{odd!}} \, dx = \int \sin^2 x \cos^6 x \underbrace{\cos x \, dx}_{\text{result from } u=\sin x}$

$= \int \sin^2 x (1-\sin^2 x)^3 \cos x \, dx$ ← $u = \sin x$, $du = \cos x \, dx$

$= \int u^2 (1-u^2)^3 \, du$

$= \int u^2 - 3u^4 + 3u^6 - u^8 \, du = \frac{\sin^3 x}{3} - \frac{3}{5}\sin^5 x + \frac{3}{7}\sin^7 x - \frac{\sin^9 x}{9} + C.$

integrate & replace $u=\sin x$

__Challenge__: Check your work by taking derivative of your answer & showing it equals $\sin^3 x \cos^7 x$.

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__Case__: What if both exponents are even?

i.e. $\int (\sin x)^{2k} (\cos x)^{2j}\, dx$.

__Method__:
- Use Pyth. identity on __smaller__ exponent.
- Use $\boxed{1}$ or $\boxed{2}$ iteratively.

**Ex:** $$\int \sin^4 x \underbrace{\cos^2 x}_{\text{smaller exponent}}\, dx = \int \sin^4 x (1 - \sin^2 x)\, dx$$

$$= \int (\sin^4 x - \sin^6 x)\, dx = \underbrace{\int \sin^4 x\, dx}_{\text{we computed this.}} - \underbrace{\int \sin^6 x\, dx}_{\text{Use } \boxed{1} \text{ over + over.}}$$

**NOTE:** If even powers are equal, choose your own adventure.

**Remark**: Read in the book to see how to handle

$$\int \tan^m x \sec^n x \, dx$$

using same strategies, w/ $\tan^2 x + 1 = \sec^2 x$ & other reduction Thms.

Tricks for

$$\int \tan x \, dx \quad +$$

$$\int \sec x \, dx \longleftarrow$$ read in text.

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx \qquad \begin{aligned} u &= \cos x \\ du &= -\sin x\,dx \end{aligned}$$

$$= -\int \frac{1}{u}\,du = -\ln|u| + C$$

$$= -\ln|\cos x| + C = \ln|(\cos x)^{-1}| + C$$

$$= \ln|\sec x| + C.$$

<u>NOTE</u>: <u>See</u> table at end of §7.2 for common integrals.

# Wednesday, March 2, 2016:

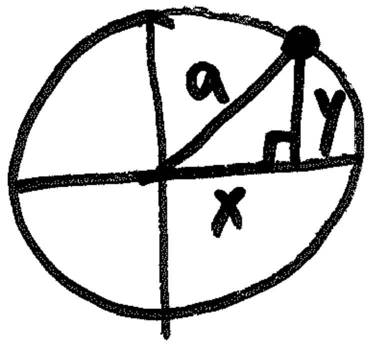
## Reminders:

- Exam 2 is next week.
  - ↳ Read Canvas announcement from today.
- Please fill out survey about review topics → See Canvas announcements for link.

## §7.3: Trig Substitutions

**General idea:** If integrand involving $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, or $\sqrt{x^2-a^2}$, setting $x=$ trig function can be helpful.

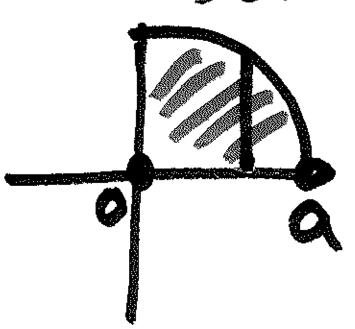
**Ex:**




Pyth. Thm $\Rightarrow x^2+y^2=a^2$.

Assume $a>0$, since radii are positive.

Solving for $y$ leads to $y=\sqrt{a^2-x^2}$.




$$\int_0^a \sqrt{a^2-x^2}\,dx = \text{Area of a quarter of a disk of radius } a.$$

Suppose we try u-sub:

If $u = a^2 - x^2$, $du = -2x\,dx$

Problem!! No $x$ outside the square root.

<u>Alternative technique:</u> Use Pyth. identity for trig functions, by setting

$x = a\sin\theta$, $dx = a\cos\theta\,d\theta$, +

Sub this in.

$$\int_0^a \sqrt{a^2 - x^2}\,dx = \int_0^{\pi/2} \sqrt{a^2 - a^2\sin^2\theta} \cdot a\cos\theta\,d\theta = ☆$$

Scratch work:

If $0 = a\sin\theta$, then $\theta = \sin^{-1}(0) = \arcsin(0)$

same fn, different names

So $\theta = 0$.

If $a = a\sin\theta$, then

$1 = \sin\theta$, so $\theta = \sin^{-1}(1)$.

Thus, $\theta = \frac{\pi}{2}$.

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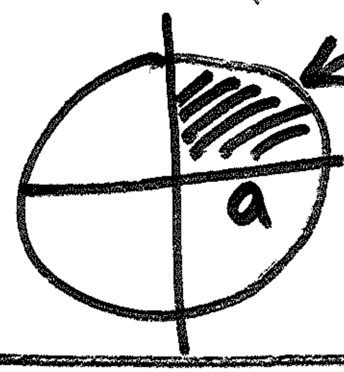
$$\star = \int_0^{\pi/2} a\sqrt{1-\sin^2\theta} \cdot a\cos\theta\, d\theta =$$

$$\int_0^{\pi/2} a^2 \cdot \cos^2\theta\, d\theta = a^2\left[\frac{1}{2}\theta + \frac{1}{2}\cos\theta\sin\theta\right]_0^{\pi/2}$$

$= a^2 \cdot \frac{\pi}{4}$.

~~Why is~~ What does this tell us?




area is $\frac{a^2 \cdot \pi}{4}$, so

area of disk is $\pi \cdot a^2$.

<u>Technique</u>: For $\sqrt{x^2+a^2}$, with $a>0$,

sub $x = a\tan\theta$, $dx = a\sec^2\theta\, d\theta$,

with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ so that

$\sec\theta \geq 0$ and hence can be a

square root.

<u>Ex</u>: Evaluate $\int \sqrt{x^2+2}\,dx$.

Set $a = \sqrt{2}$, so

$$x = \sqrt{2}\tan\theta,$$

$$dx = \sqrt{2}\sec^2\theta\, d\theta,$$

$$\sqrt{x^2+2} = \sqrt{2\tan^2\theta + 2} = \sqrt{2}\sec\theta.$$

Thus,

$$\int \sqrt{x^2+2}\,dx = \int \sqrt{2}\sec\theta\,\sqrt{2}\sec^2\theta\,d\theta$$

$$= 2\int \sec^3\theta\,d\theta. = \star$$

**NOTE:**

<u>Every</u> positive number is a square.

$7 = \sqrt{7}^2.$

$19 = \sqrt{19}^2.$

$\pi = \sqrt{\pi}^2.$

Using reduction thm for $\int \sec^3\theta\, d\theta$, we get

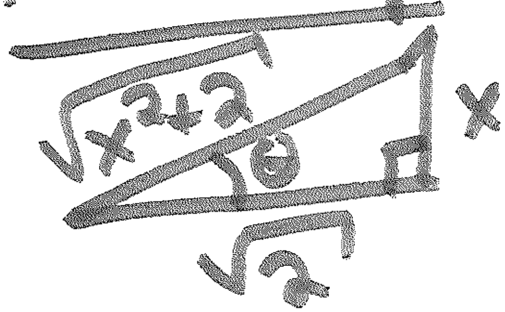
$$\star = 2\left[\frac{\tan\theta\sec\theta}{2} + \frac{1}{2}\int \sec\theta\, d\theta\right]$$

$$= \tan\theta\sec\theta + \ln(\underline{\sec\theta + \tan\theta}) + C.$$

Since $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$,

$\sec\theta + \tan\theta > 0$.

Final step: convert back to x-variable.




$x = \sqrt{2}\tan\theta$, thus $\tan\theta = \frac{x}{\sqrt{2}}$.

By picture, $\sec\theta = \frac{\sqrt{x^2+2}}{\sqrt{2}}$.

$$\int \sqrt{x^2+2}\, dx = \frac{x\sqrt{x^2+2}}{2} + \ln\left(\frac{\sqrt{x^2+2}}{\sqrt{2}} + \frac{x}{\sqrt{2}}\right) + C.$$

BE PRECISE
AND WRITE
MATH LIKE
AN ESSAY!

<u>Technique</u>: For $\sqrt{x^2-a^2}$ with $a>0$, substitute $x = a\sec\theta$, so $dx = a\sec\theta\tan\theta\,d\theta$, where we allow

$0 \le \theta < \frac{\pi}{2}$ if $x > a$ and

$\pi \le \theta < \frac{3\pi}{2}$ if $x \le -a$, so that $a\tan\theta \ge 0$.

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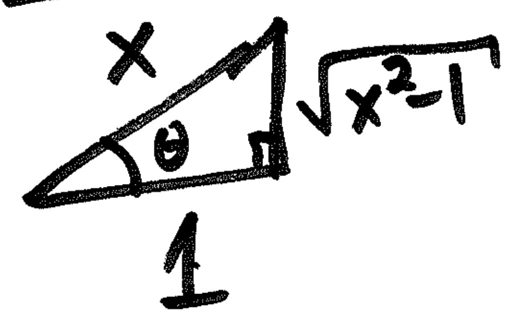
<u>Ex</u>: $\displaystyle\int \frac{dx}{\sqrt{x^2-1}}$.

$$\left.\begin{aligned} a = 1,\ x &= \sec\theta, \\ dx &= \sec\theta\tan\theta\,d\theta, \\ \sqrt{x^2-1} &= \tan\theta. \end{aligned}\right] \text{Input.}$$

So, $\displaystyle\int \frac{dx}{\sqrt{x^2-1}} = \int \frac{\sec\theta \tan\theta}{\tan\theta}\, d\theta = \int \sec\theta \, d\theta$

$= \ln\left|\sec\theta + \tan\theta\right| + C.$

Final Step: back to x's.




$\sec\theta = x$, $\tan\theta = \sqrt{x^2-1}$.

So, final answer is $\ln\left|x + \sqrt{x^2-1}\right| + C$.

## Friday, March 4, 2016

- Exam 2 is Tuesday
- Review Session Monday Night... See MA114 Common Web page for time/room.

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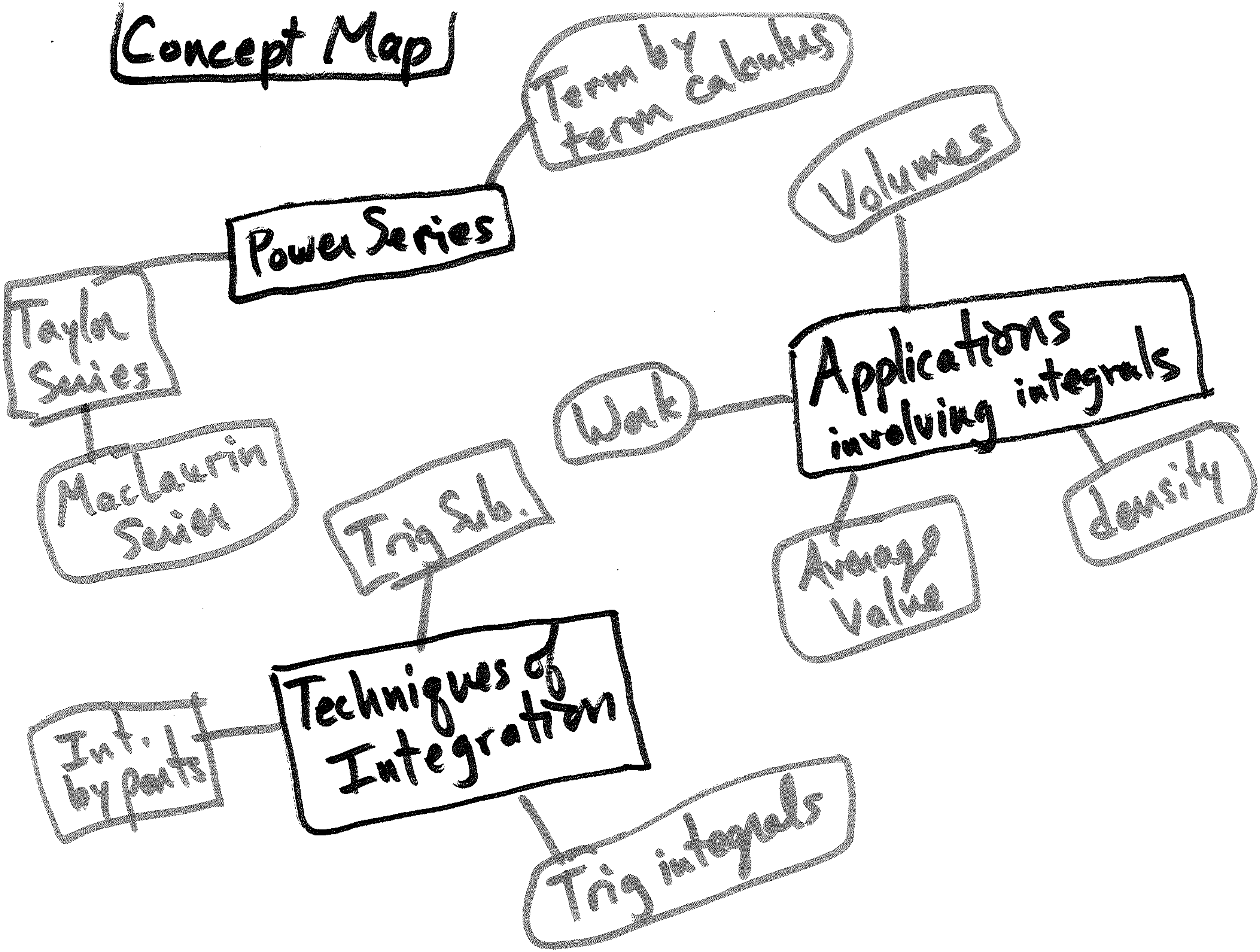
**Task**: Identify key ideas for Exam 2.

Taylor Series

Work

Volume

# Concept Map


Power Series
Term by term calculus
Taylor Series
Maclaurin Series
Applications involving integrals
Volumes
Work
Average Value
density
Techniques of Integration
Trig Sub.
Int. by parts
Trig integrals

**Problem:** If $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ on the interval $(-1,1)$. Show that

$$\frac{f^{(5)}(0)}{5!} = a_5 .$$

First step: We use term-by-term diff.

$$f'(x) = 0 + a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots$$

$$f^{(2)}(x) = 0 + 0 + 2a_2 + 3\cdot 2a_3x + 4\cdot 3a_4x^2 + 5\cdot 4a_5x^3 + \cdots$$

$$f^{(3)}(x) = 3\cdot 2\cdot a_3 + 4\cdot 3\cdot 2a_4\,x + 5\cdot 4\cdot 3a_5x^2 + \cdots$$

$$f^{(4)}(x) = 4\cdot 3\cdot 2a_4 + 5\cdot 4\cdot 3\cdot 2a_5x + \cdots$$

$$f^{(5)}(\cancel{x}) = 0 + 5\cdot 4\cdot 3\cdot 2a_5 + \underbrace{6\cdot 5\cdot 4\cdot 3\cdot 2a_6\overset{=0}{x} + \cdots}_{\text{all have } x=0 \text{ as a factor.}}$$

$$f^{(5)}(0) = 5\cdot 4\cdot 3\cdot 2a_5$$

So, $\dfrac{f^{(5)}(0)}{5\cdot4\cdot3\cdot2\cdot1} = a_5$

$5! \rightarrow 5\cdot4\cdot3\cdot2\cdot1$

center of power series

In general, $\dfrac{f^{(n)}(0)}{n!} = a_n$ for a MacLaurin series.

**NOTE:** $f(x) = a_0 + a_1(x-0) + a_2(x-0)^2 + a_3(x-0)^3 + \cdots$

**Problem**: Find $f^{(14)}(0)$ for $f(x) = x^3 e^{x/2}$.

**Step 1**: Find Maclaurin Series for $f$.

**Step 2**: Identify coefficient of $x^{14}$, i.e. $\dfrac{f^{(14)}(0)}{14!}$. Use that to solve for $f^{(14)}(0)$.

__Step 1:__ Use $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Plug in $z = x/2$.

$$e^{x/2} = 1 + \frac{x}{2} + \frac{1}{2!}\left(\frac{x}{2}\right)^2 + \frac{1}{3!}\left(\frac{x}{2}\right)^3 + \frac{1}{4!}\left(\frac{x}{2}\right)^4 + \dots$$

$$= 1 + \frac{x}{2} + \frac{1}{2! \cdot 2^2} x^2 + \frac{1}{3!\, 2^3} \cdot x^3 + \frac{1}{4!\, 2^4} x^4 + \dots$$

$$x^3 \cdot e^{x/2} = x^3 + \frac{x^4}{2} + \frac{1}{2!\, 2^2} x^5 + \frac{1}{3!\, 2^3} x^6 + \frac{1}{4!\, 2^4} x^7 + \dots$$

End step 1.

**Step 2**: Find coeff of $x^{14}$, it is $\dfrac{1}{11!\,2^{11}}$.

So, $$\frac{f^{(14)}(0)}{14!} = \frac{1}{11!\,2^{11}} \Rightarrow f^{(14)}(0) = \frac{14!}{11!\,2^{11}}$$

$$= \frac{14 \cdot 13 \cdot 12}{2^{11}}.$$