Mon, Feb 8, 2016:

First item: For HW 3, not on exam: diff. + integration of power series.

Thm 2, §10.6: If $F(x) = \sum a_n(x-c)^n$ with radius of convergence R, then $F'(x)$ and $\int F(x)\,dx$ can be obtained by using the power rule term-by-term on $F(x)$. Further, $F'(x)$ and $\int F(x)\,dx$ both have R. of Conv. R.

Ex: In calc 1, you learned that

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt. \text{ Let's integrate!}$$

$$\int_0^x \frac{1}{1+t}\,dt = \int_0^x \frac{1}{1-(-t)}\,dt =$$

$$\int_0^x (1 - t + t^2 - t^3 + t^4 - \cdots)\,dt =$$

$$\left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5} - \cdots\right)\Big|_0^x =$$

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^3}{4} + \frac{x^5}{5} - \cdots. = \ln(1+x).$$

Because of the integral...

$$\boxed{\begin{array}{c}\text{Result: } x=1 \text{ gives} \\ \ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \\ \text{Alt. Harm. Series.}\end{array}}$$

EXAM REVIEW:

$\boxed{1.}$ Find the radius of convergence for $F(x) = 1 \underbrace{- x - x^2}_{} + x^3 - x^4 - x^5 + x^6 - \cdots$

pattern: $-, -, +, -, -, +, \ldots$

Def$^{\underline{n}}$ of power series: $\sum a_n (x-c)^n$

Example of Power Series: $\sum \frac{x^n}{3^n}$ $\quad \sum \frac{x^n}{n!}$

$\sum x^n$

$$\sum_{n=0}^{\infty} \frac{x^n}{3^n} = \frac{x^0}{3^0} + \frac{x^1}{3^1} + \frac{x^2}{3^2} + \frac{x^3}{3^3} + \cdots$$

$$= 1 + \frac{x}{3} + \frac{x^2}{9} + \frac{x^3}{27} + \cdots$$

start here in your mind.

Recall a power series is:

$$a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} a_n (x-c)^n$$

For $1 - x - x^2 + x^3 - x^4 - x^5 + x^6 - \cdots$

$c = 0 \quad 1 - 1(x-0) - 1\cdot(x-0)^2 + 1\cdot(x-0)^3 - 1\cdot(x-0)^4 - \cdots$

Ratio test ignores signs, so let's use it + not worry about sign pattern.

$$1 - x - x^2 + x^3 \underbrace{- x^4}_{\substack{4^{th} \text{ term} \\ \text{is } -x^4}} - x^5 \underbrace{+ x^6}_{\substack{6^{th} \text{ term} \\ \text{is } x^6}} - \cdots \underbrace{\quad\cdots\quad}_{\substack{n^{th} \text{ term is } \\ \pm x^n \text{ depending} \\ \text{on } n.}}$$

<u>Ratio:</u> $\displaystyle \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \right|$

$= \displaystyle \lim_{n\to\infty} |x| < 1$ ensures by ratio test abs convergence.

So, $|x| < 1$ yields $R = 1$ as R. of. Conv.

Find the int. of conv. for

$$\sum_{n=0}^{\infty} \frac{n^6}{n^8+1}(x-3)^n.$$

$c = 3$. $\quad 0 + \frac{1}{2}(x-3) + \frac{64}{257}(x-3)^2 + \frac{3^6}{3^8+1}(x-3)^3 + \dots$

<u>Use ratio test:</u> $\lim_{n\to\infty} \left| \frac{n+1\text{st term}}{n\text{th term}} \right| < 1$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x-c)^{n+1}}{a_n(x-c)^n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n}(x-c) \right|$$

$$= |(x-c)| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1 \quad \text{converges.}$$

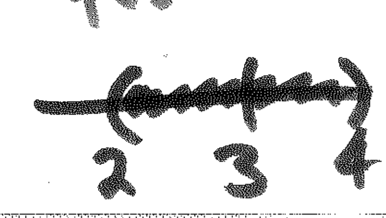
$$\lim_{n\to\infty}\left|\frac{\frac{(n+1)^6}{(n+1)^8+1}(x-3)^{n+1}}{\frac{n^6}{n^8+1}(x-3)^n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)^6}{(n+1)^8+1}}{\frac{n^6}{n^8+1}}(x-3)\right|$$

$$= |x-3|\cdot\lim_{n\to\infty}\left|\frac{(n+1)^6}{(n+1)^8+1}\cdot\frac{n^8+1}{n^6}\right| =$$

$$\underbrace{|x-3|\cdot\lim_{n\to\infty}\left|\frac{(n+1)^6}{n^6}\cdot\frac{n^8+1}{(n+1)^8+1}\right|}_{=1} < 1$$

converges by ratio test.

$$\Rightarrow |x-3| < 1$$



Since $c=3$, $R=1$, we get $(2,4)$

check endpoints!

<u>When $x=2$:</u>

$$\sum_{n=0}^{\infty} \frac{n^6}{n^8+1}(2-3)^n = \sum_{n=0}^{\infty} \frac{n^6}{n^8+1}(-1)^n$$

Alt. series test $\Rightarrow$ conv.

<u>When $x=4$:</u>

$$\sum_{n=0}^{\infty} \frac{n^6}{n^8+1}(4-3)^n = \sum_{n=0}^{\infty} \frac{n^6}{n^8+1}$$

p-series + comp. test $\Rightarrow$ conv.

So, I of C. is $[2,4]$.