Let's begin with elementary school.

<u>Q</u>: Do you believe that

$$1 = 0.\overline{9} = 0.9999\ldots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000} + \cdots ?$$

Why or why not?

Work & discuss with your neighbors sitting near.

<u>Q</u>: What about

$$\frac{8}{11} = 0.727272\ldots = 0.\overline{72} = \frac{72}{100} + \frac{72}{(100)^2} + \frac{72}{(100)^3} + \cdots ?$$

or $$\frac{4}{7} = 0.\overline{571428} = \frac{571428}{1000000} + \frac{571428}{(1000000)^2} + \cdots ?$$

First big question (from 5th grade):

How can we add infinitely many items together?

- Start w/ sequence of numbers:

$$a_1, a_2, a_3, a_4, \dots = \{a_n\}_{n=1}^{\infty}$$

$$\frac{9}{10}, \frac{9}{100}, \frac{9}{1000}, \dots = \left\{\frac{9}{10^n}\right\}_{n=1}^{\infty}$$

- Create a partial sum:

$$\boxed{a_1,\ a_1+a_2,\ a_1+a_2+a_3 \dots}$$

examples.

$$S_N := a_1 + \dots + a_N = \sum_{n=1}^{N} a_n$$

define

sigma means SUM.

- Create an infinite series:

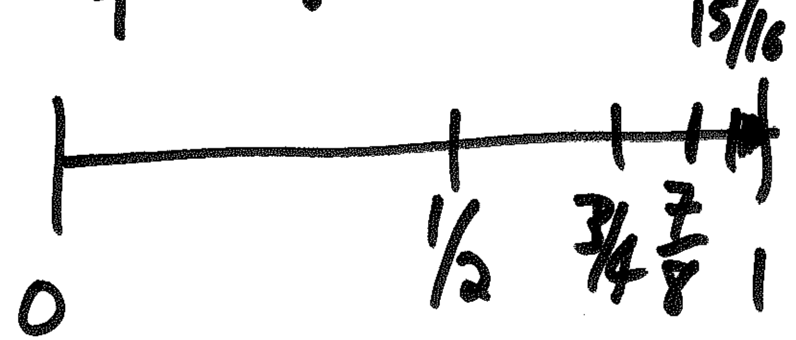
$$a_1 + a_2 + a_3 + \dots = \sum_{n=1}^{\infty} a_n$$

Seq: $\frac{9}{10}, \frac{9}{100}, \frac{9}{1000}, \ldots$

Partial Sum: $\frac{9}{10} + \frac{9}{100} + \cdots + \frac{9}{10^N} = S_N$

Infinite Series: $\frac{9}{10} + \frac{9}{100} + \ldots \ (= .9999\ldots)$

Ex: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = ? \ (= 1)$



① Read 10.1 for recitation next week!

→ Focus on bounded and monotonic sequences – we will discuss in lecture Wed. after you grapple w/ them in recitation (except Math Excel)

② Course Policy Applies to Me + TA's as well as students → Call us on it! We're all needing to improve.

<u>Exercise</u>: Write $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$ as

1) a formula $\left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$

2) a recursion $a_1, a_2, a_3, \dots$

$$a_n = \frac{a_{n-1}}{2} = \frac{1}{2} \cdot a_{n-1}$$

$1, 1, 2, 3, 5, 8, 13, \dots \quad a_n = a_{n-1} + a_{n-2}$

Two Messy Examples:

(A) $\{a_n\}_{n=1}^{\infty} = \{1, -1, 1, -1, 1, -1, \ldots\}$

$1 - 1 + 1 - 1 + 1 - 1 + \cdots\cdots = ?$

$S_1 = a_1 = 1$

$S_2 = a_1 + a_2 = 0$

$S_3 = a_1 + a_2 + a_3 = 1$

$S_4 = a_1 + a_2 + a_3 + a_4 = 0$

$S_5 = a_1 + a_2 + a_3 + a_4 + a_5 = 1$

$\vdots$

$S_N = a_1 + a_2 + \cdots + a_N = \sum_{n=1}^{N} a_n$

$1, 0, 1, 0, 1, 0, \ldots$

bad limiting behavior

(B) $\{a_n\}_{n=1}^{\infty} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\} = \{\frac{1}{n}\}_{n=1}^{\infty}$

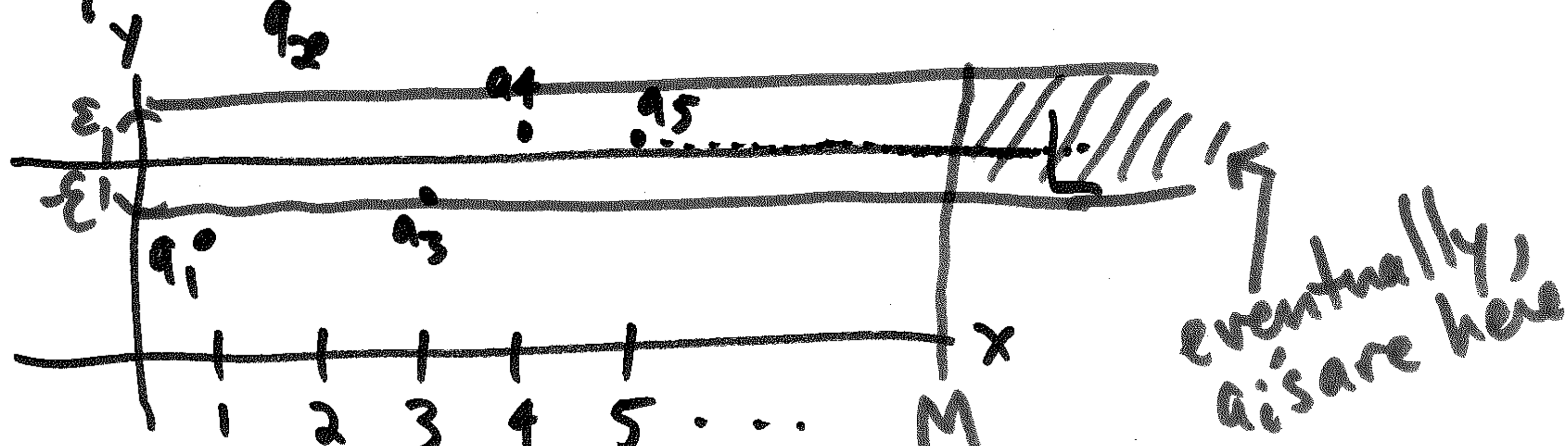
Infinite series: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ ← Super complicated and Subtle.

"Harmonic Series"

$\underline{Q}$: what is a good def$^{\underline{n}}$ of "limiting behavior"?

$\underline{\text{Def}^{\underline{n}}}$: Say that $\{a_n\}_{n=1}^{\infty}$ <u>converges</u> to a limit L, written $\lim_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$, if for every $\varepsilon > 0$ there is a number M so that $|a_n - L| < \varepsilon$ for all $n > M$.

If $\{a_n\}$ has no limit, it <u>diverges</u>. If it grows w/out bound, it <u>diverges to infinity</u>.



Def$^{\underline{n}}$: An infinite series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots$ **converges** to a sum S if the partial sums converge to S, i.e.

$$\lim_{N\to\infty} S_N = S.$$

We write $S = \sum_{n=1}^{\infty} a_n$. If limit does not exist, $\sum_{n=1}^{\infty} a_n$ **diverges**. If limit is infinite, say $\sum_{n=1}^{\infty} a_n$ **diverges to ∞**.

What is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$?

agrees w/ our limit.

$S_1 = \frac{1}{2}$

$S_2 = 3/4$

$S_3 = 7/8$

$S_4 = 15/16$

$S_5 = 31/32$

$\vdots$

$S_N = \frac{2^N - 1}{2^N}$ $\leftarrow$ To "formally" show this, use induction.

$\vdots$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{N \to \infty} S_N = \lim_{N \to \infty} \frac{2^N - 1}{2^N}$$

$$= \lim_{N \to \infty} \frac{2^N}{2^N} - \lim_{N \to \infty} \frac{1}{2^N}$$

$$= 1 - 0 = 1.$$

Why is $\underline{.9999\ldots = 1}$?

This is true!

$$\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots =$$

$$9\left(\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \cdots\right) =$$

$$9\left(\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \cdots\right) =$$

$$9\left(\left(\frac{1}{10}\right)^1 + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \left(\frac{1}{10}\right)^4 + \cdots\right)$$

what if we used $\frac{1}{3}$? or $\frac{7}{8}$?

or 1? $1+1+1+1+1+\cdots$