Wed, Jan 20, 2016 | 0. Quiz in recitation tomorrow!

1. Lecture notes are posted on my 114 site.
2. Attendance sign-in sheets start Friday; they will be on desk by doors.
3. Remember: Our first big Q is "How do we add up infinitely many items?"

Before class: Talk with your neighbors: do you believe it is possible to add up infinitely many values and get a finite answer? Why or why not?

Examples: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots = 1$

or $\frac{9}{10}+\frac{9}{10^2}+\frac{9}{10^3}+\frac{9}{10^4}+\cdots = .\overline{9} = 1$

Technical results + tools about limits of seque-nces

__Thm4, §10.1:__ If $f(x)$ is continuous, and $\lim_{n\to\infty} a_n = L$, then

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L).$$

__Ex:__ $\lim_{n\to\infty} \left(2+\frac{4}{n^2}\right)^{1/3}$

what is a_n?
what is f?
what is $\lim_{n\to\infty} a_n$?

$$\left.\begin{aligned} a_n &= 2+\frac{4}{n^2} \\ a_n &= \frac{4}{n^2} \end{aligned}\right\}$$ both of these are reasonable choices.

use this

$f=(x)=x^{1/3}$

$\lim_{n\to\infty} \underbrace{2+\frac{4}{n^2}}_{a_n} = 2$

So, $\lim_{n\to\infty} \left(2+\frac{4}{n^2}\right)^{1/3} = 2^{1/3}$

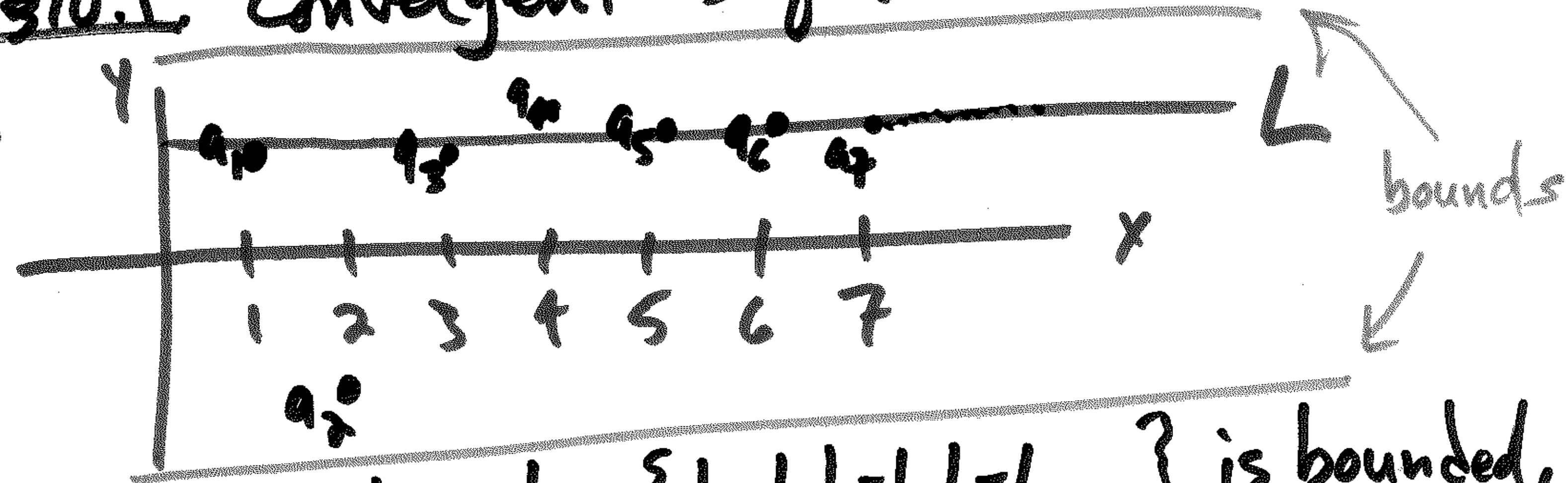
Def$^{\underline{n}}$: A sequence $\{a_n\}$ is:

bounded from above (below) if there is a number M such that $a_n \leq M$ ($M \leq a_n$) for all n.

If $\{a_n\}$ is bounded from both above + below, say it is <u>bounded</u>. Otherwise, it is <u>unbounded</u>.

<u>Thm 5 in §10.1</u>: Convergent sequences are bounded.

<u>Idea</u>:



<u>NOTE</u>: Converse not true! $\{1,-1,1,-1,1,-1,\ldots\}$ is bounded, but does not converge.

<u>Ex</u>: $a_n = \frac{1}{n-3}$ for $n = 4, 5, 6, \ldots$

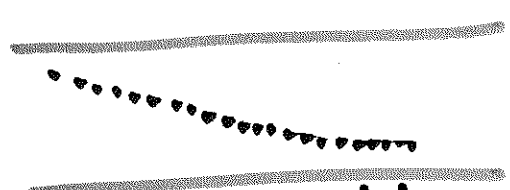
<u>Def$^{\underline{n}}$</u>: A sequence $\{a_n\}$ is <u>monotonic</u> if $\{a_n\}$ is always <u>either</u> (i) increasing or (ii) decreasing.

This is monotonically decreasing.

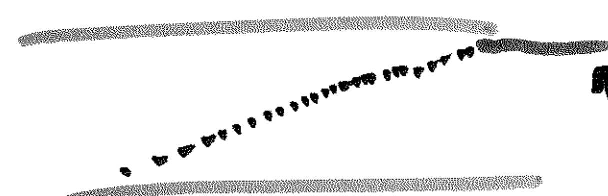
AND

bounded from below by zero.

<u>Thm 6 in §10.1</u>: If $\{a_n\}$ is a bounded, monotonic sequence, then $\{a_n\}$ converges.



dec, bdd



↑ limit might not be your bound...

inc, bdd

This example is super, duper important!

→ This is used to prove almost everything in this course ←

A <u>geometric sequence</u> is: for real number r, and real number $c>0$,

the sequence is $c, cr, cr^2, cr^3, cr^4, \ldots$

<u>Ex</u>: $c=9$, $r=\frac{1}{10}$, $9, \frac{9}{10}, \frac{9}{100}, \frac{9}{1000}, \frac{9}{10000}, \ldots$

$c=72$, $r=\frac{1}{100}$ $72, \frac{72}{100}, \frac{72}{(100)^2}, \ldots$

A <u>finite geometric series</u> is a partial sum of the form

$c+cr+cr^2+cr^3+\cdots+cr^N$, for some N.

A <u>geometric series</u> is $c+cr+cr^2+cr^3+cr^4+\cdots$

Thm: • $c + cr + cr^2 + cr^3 + \cdots + cr^N = c\left(\dfrac{1-r^{N+1}}{1-r}\right)$

• If $c \neq 0$, and $|r| < 1$, then

$$\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + cr^3 + \cdots = \frac{c}{1-r}.$$

Pf: Look in text, §10.2, or look at similar triangle "proof w/out words" link on my website.

Ex: $c = 9$, $r = \frac{1}{10}$.

$$9 + \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots = \frac{\overset{c}{9}}{1 - \underbrace{\tfrac{1}{10}}_{r}} = \frac{9}{\frac{9}{10}} = 10.$$

Subtract 9 from both sides, we get $.\overline{9} = 1$.

Exercise: $c = 72$, $r = \frac{1}{100}$. $\Rightarrow \frac{8}{11} = .\overline{72}$.

Mon, Jan 25, 2016

0. SIGN IN FOR ATTENDANCE AT SHEETS ON FRONT TABLE!!

1. See email from UK Canvas system for schedule change.
2. No office hours Fri, Jan 29. Make-up hour on Mon, Feb 1, from 12-12:50.
3. Section 019: TA Change: Your new TA is Yaowei Zhang, starting tomorrow.

$\boxed{4:}$ With your neighbors, compute partial sums for $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots = \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n$. How does this compare to the formula $\frac{1}{1-\frac{1}{3}}$?

Use calculator or phone.

Geom. Series

$$c + cr + cr^2 + cr^3 + \cdots + cr^N = c\left(\frac{1-r^{N+1}}{1-r}\right)$$

$[c > 0, \ -1 < r < 1]$

Pf by convincing example: $N = 5$

$$(c + cr + cr^2 + cr^3 + cr^4 + cr^5)(1-r) =$$

$$c + \cancel{cr} + \cancel{cr^2} + \cancel{cr^3} + \cancel{cr^4} + \cancel{cr^5} - \cancel{cr} - \cancel{cr^2} - \cancel{cr^3} - \cancel{cr^4} - \cancel{cr^5} - cr^6 =$$

$$c - cr^6 = c(1-r^6)$$

Divide both sides by $1-r$, I get the formula.

$$\underbrace{c + cr + cr^2 + \cdots}_{\text{limit of partial sums.}} = \frac{c}{1-r}.$$

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(c + cr + cr^2 + \cdots + cr^N\right) =$$

$$\lim_{N\to\infty} \frac{c\left(1-r^{N+1}\right)}{1-r} = c\left[\lim_{N\to\infty} \frac{1}{1-r} - \underbrace{\lim_{N\to\infty} \frac{r^{N+1}}{1-r}}_{\to 0 \text{ since } -1<r<1}\right]$$

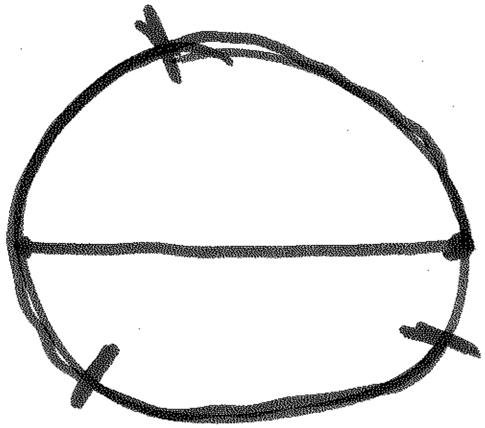
$$= \frac{c}{1-r}.$$

$$cr + cr^2 + cr^3 + \cdots = \frac{cr}{1-r}$$

multiply by r

$$c + cr + cr^2 + \cdots = \frac{c}{1-r}$$

$$\pi = \frac{C}{D}$$



3 a little bit ← Nasty + complicated

$\approx \frac{1}{7}$

$$\pi \approx 3\tfrac{1}{7}$$

Telescoping Series : "Leibniz Series"

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots = 1.$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Fact: $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$

$$S_N = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{N(N+1)} =$$

$$= \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{N-1}} - \cancel{\frac{1}{N}}\right) + \left(\cancel{\frac{1}{N}} - \frac{1}{N+1}\right)$$

$$= \frac{1}{1} - \frac{1}{N+1}.$$

$$\underbrace{\lim_{N\to\infty} \Big(\underbrace{1 - \frac{1}{N+1}}_{S_N}\Big) = 1 - 0 = 1.}$$

Thm 1 in §10.2: If $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ converge,

Then so do: These also converge.

- $\sum_{n=1}^{\infty} (a_n \pm b_n)$
- $c\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} c a_n$

Q: What is a_n, b_n for this series?

$$\sum_{n=0}^{\infty} \frac{2+3^n}{5^n} = \frac{2+3^0}{5^0} + \frac{2+3^1}{5^1} + \frac{2+3^2}{5^2} + \dots$$

$$a_n = \frac{2}{5^n} = 2\left(\frac{1}{5}\right)^n, \quad b_n = \frac{3^n}{5^n} = \left(\frac{3}{5}\right)^n$$

NOTE: $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 2\left(\frac{1}{5}\right)^n = 2\sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^n$

$$a_1+a_2+a_3+\cdots = \frac{2}{5}+\frac{2}{5^2}+\frac{2}{5^3}+\cdots =$$

$$2\Big(\underbrace{\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\cdots}_{\frac{1}{5}\big(\underbrace{1+\frac{1}{5}+\frac{1}{5^2}+\cdots}_{=\frac{1}{1-1/5}}\big)}\Big) = 2\cdot\frac{\frac{1}{5}}{1-\frac{1}{5}}$$

NOTE: Similarly add up b_n geom. series, then add the total sum.

Ex: $1-1+1-1+1-1+\cdots$ diverges.

Partial sum Sequence is $1,0,1,0,1,0,\ldots$ ← no limit

Divergence Test: If $a_n \not\to 0$, then $a_1+a_2+a_3+\cdots$ diverges.

→ See book for a proof. ←

Ex: $\sum_{n=1}^{\infty} \frac{7n}{8n+2} = \frac{7}{8+2} + \frac{7\cdot 2}{8\cdot 2+2} + \frac{7\cdot 3}{8\cdot 3+2} + \cdots$

$\lim_{n\to\infty} \frac{7n}{8n+2} = \lim_{n\to\infty} \frac{7}{8+\frac{2}{n}} = \frac{7}{8}$. So, diverges.

divide by n num + den

Harmonic Series:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n}$$

Thm: The harmonic series diverges.