Harmonic Series: Jan 27, 2016

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots$$

Type partial sums into calc. & see how large you get...

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{100}\approx 5.18\ldots$$

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1000}\approx 7.48\ldots$$

$$1+\frac{1}{2}+\cdots\cdots+\frac{1}{3000}\approx 8.58\ldots$$

$$1+\frac{1}{2}+\cdots\cdots+\frac{1}{10000}\approx 9.78$$

$$\cancel{1+\frac{1}{2}+\frac{1}{2}+\frac{1}{4}}$$

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots \qquad \sum_{n\geq 0}\frac{1}{2^n}$$

$$\underbrace{1}_{\uparrow}+\frac{1}{2}+\frac{1}{4}+\cdots\cdots+\frac{1}{2^{100}} \underset{\uparrow}{=} \cancel{\neq} 2$$

include this?

Calculator

$$= \frac{2535301200456458802993406410751}{1267650600228229401496703205376}$$

$$1+\frac{8}{9}+\left(\frac{8}{9}\right)^2+\left(\frac{8}{9}\right)^3+\left(\frac{8}{9}\right)^4+\cdots = \sum_{n\geq 0}{}^{'}\left(\frac{8}{9}\right)^n$$

$$1+\frac{8}{9}+\cdots+\left(\frac{8}{9}\right)^{100} = 9$$

calculator

$$= 8.999938647$$

Q: How do we prove $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ diverges?

A: Compare it to a clearly diverging series.

Remark: Read def$^{\underline{n}}$ of <u>positive series</u> in §10.3.

The partial sums of a positive series are monotone increasing.

Thm: (Comparison Test) For positive series: Suppose there is an $M>0$ so that $0 \le a_n \le b_n$ for $n \ge M$.

i) If $\sum_{n=1}^{\infty} b_n$ converges, so does $\sum_{n=1}^{\infty} a_n$

ii) If $\sum_{n=1}^{\infty} a_n$ ~~converges~~ diverges, so does $\sum_{n=1}^{\infty} b_n$.

Pf in book

Proof of divergence of harmonic series:

Consider $N = 2^m$.

$$S_{2^m} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots + \frac{1}{2^m}$$

$$\geq \frac{1}{2} + \frac{1}{2} + \underbrace{\left(\frac{1}{4} + \frac{1}{4}\right)}_{} + \underbrace{\left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)}_{} + \cdots + \frac{1}{2^m})$$

$$= \underbrace{\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}}_{m+1 \ \frac{1}{2}\text{'s}}.$$

inequality! $= \dfrac{m+1}{2}$.

$$S_{2^m} \geq \frac{m+1}{2}.$$

As $m \to \infty$, $\dfrac{m+1}{2} \to \infty$.

So, $S_{2^m} \to \infty$. So,

Harm. Series diverges.

Basel Problem: (Hard to prove)

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}$$

Def^n: A p-series is an infinite series of the form

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots.$$

Note: $p=1$ is Harmonic, $p=2$ is Basel.

Correct name: $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots=\zeta(p)$, the Riemann Zeta function.

$\underline{p=4}$:

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots = 1+\frac{1}{16}+\frac{1}{81}+\frac{1}{256}+\cdots = \frac{\pi^4}{90}.$$

$\underline{p=26}$:

$$\frac{1}{1^{26}}+\frac{1}{2^{26}}+\frac{1}{3^{26}}+\cdots = \frac{1315862}{11094481976030578125}\cdot\pi^{26}.$$

$\underline{p=3}$: $\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\cdots$ is called Apery's Constant.

<u>Unsolved problem:</u> Is this series of the form $\frac{p}{q}\cdot\pi^3$ for integers p,q?

<u>Thm</u>: A p-series converges if $p>1$, p a real number, and diverges otherwise.

<u>Ex</u>: $\frac{1}{1^{1/2}}+\frac{1}{2^{1/2}}+\frac{1}{3^{1/2}}+\dots=\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots$

~~Converges.~~ diverges. since $p<1$.

Proof is by comparison to a geometric series. I'll put a link to a proof of this on website.

<u>Remark</u>: Read the <u>Limit Comparison Test</u> in §10.3.

Use comparison test on:

$$\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}.$$

main growth.

<u>Idea</u>: I know $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series.

$$\frac{5}{2n^2+4n+3} \leq \frac{5}{2n^2}$$

$$\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3} \leq \sum_{n=1}^{\infty} \frac{5}{2n^2} = \underbrace{\frac{5}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\text{converges.}}$$

So, original series converges.

a

Series

sequence

$$\sum_{n=1}^{\infty} \left(\text{eqn in form} \right)$$

Tests

$\frac{\pi^2}{6}$, Basel

p-series

$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\dots$ converges

$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\dots$

Harmonic, diverges

$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\dots$

!!! Crazy!

Series

$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots$

$\sum_{n=1}^{\infty} a_n$

Sequence

$\{a_n\}=a_1, a_2, \dots$

Geometric

$r=\frac{1}{2}$ $\frac{1}{1-\frac{1}{2}}$

$1+r+r^2+\dots=\frac{1}{1-r}$

Compare to $\frac{1}{2}+\frac{1}{2}+\dots+\frac{1}{2}$

Since $\frac{1}{n}\to 0$

Friday, Jan 29, 2016

- Attendance Sheets at front of room!
  Sign in!!

- Q: A series is alternating if the terms alternate in sign.
  (i) Find an example of a divergent alternating series.
  (ii) Find an example of a convergent alternating series.
  Hint: Consider a geometric series with $-1 < r < 0$....

Sequence

$$(-1)^n$$

$$1, -1, 1, -1, \ldots$$

Series

$$\sum_{n=0}^{\infty} (-1)^n$$

$$\underbrace{1-1+1-1+\cdots}_{\text{Divergent alternating Series.}}$$

Important distinction!

<u>2 topics</u>: • Absolute Convergence
• Alternating Series

<u>Def$^{\underline{n}}$</u>: A series $\underbrace{\sum_1 a_n}_{\text{assume infinite series}}$ <u>converges</u> <u>absdutely</u> if $\sum_1 |a_n|$ converges

<u>Ex</u>: $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} + \cdots$ converges absolutely

because $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ converges.

<u>Thm</u>: Every absolutely convergent series is convergent.

Bad example!

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots \text{ diverges.}$$

BUT...

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots \text{ converges.}$$

Implications: Abs conv $\Longrightarrow$ conv

implies

Conv $\not\Longrightarrow$ Abs. conv.

$= \ln(2)$.

Ex: Does $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\cdots$ converge?

Apply Absolute Conv.

Remark: Abs. conv. works for any sign pattern on terms.

Let's get a better tool for alternating series....

for example

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots$$ ← Alternating harmonic series

Leibniz, a Alt. Series, Test:

Assume $\{a_n\} = \{a_1, a_2, a_3, \ldots\}$ that has all positive terms with

$$a_1 > a_2 > a_3 > a_4 > \cdots > 0.$$

AND: $\lim_{n \to \infty} a_n = 0.$

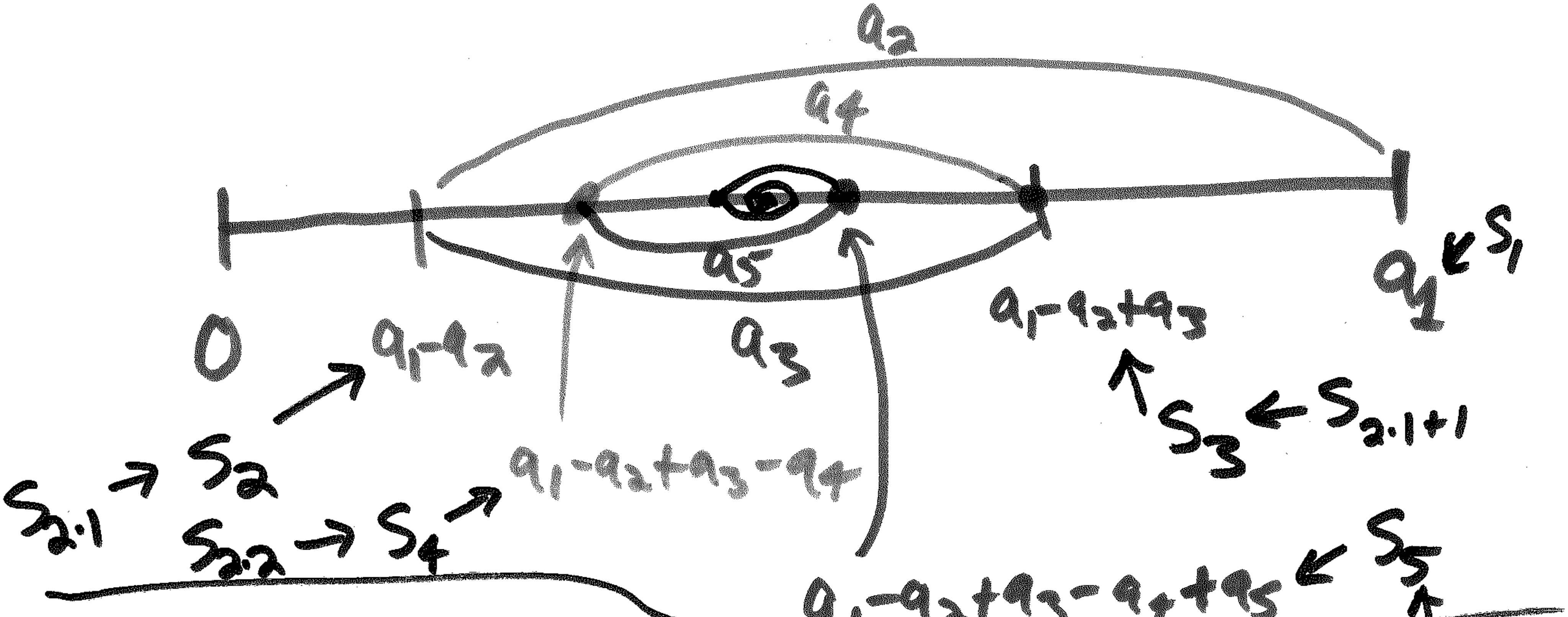
Then $a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$

converges with:

$$0 \leq \sum_{n=1}^{\infty} (-1)^{n-1} a_n \leq a_1 \quad \underline{\text{and}} \quad S_{2N} < \sum_{n=1}^{\infty} (-1)^{n-1} a_n < S_{2N+1}$$

$$a_1 - a_2 + a_3 - a_4 + \cdots - a_{2N}$$

You should believe this!



<u>Def$^{\underline{n}}$</u>: $\sum_1 a_n$ <u>converges conditionally</u> if $\sum_1 a_n$ converges but $\sum_1 |a_n|$ diverges.

<u>Ex</u>: Alt. harmonic Series.

__Q__: Does $\sum_{n=1}^{\infty} \frac{\cos(n)}{2^n}$ converge?

__Hint__:
- First, study abs. convergence.
- Second, comparison test.

$$0 \le \left|\frac{\cos(n)}{2^n}\right| = \frac{|\cos(n)|}{2^n} \le \frac{1}{2^n}.$$

$$\sum_{n=1}^{\infty} \left|\frac{\cos(n)}{2^n}\right| \le \sum_{n=1}^{\infty} \frac{1}{2^n}$$

← converge geometric series.

↑ comparison test

$\Rightarrow$ Original series converges.