Monday, March 21, 2016

→ SIAM student talk Tues 5-6PM

§7.6: Improper Integrals

Ex: $\int_1^R \frac{1}{x^4}\,dx = \int_1^R x^{-4}\,dx = \overbrace{\frac{1}{3} - \underbrace{\frac{1}{3R^3}}_{\substack{R>1 \text{ implies} \\ \frac{1}{3R^3} < \frac{1}{3}.}}}^{\text{so this is positive.}}$.

We define $\int_1^\infty \frac{1}{x^4}\,dx = \lim_{R\to\infty} \int_1^R \frac{1}{x^4}\,dx = \frac{1}{3}$.

since $\frac{1}{3R^3} \to 0$ as $R \to \infty$.

Def$^{\underline{n}}$: Fix a in real #'s and assume f is integrable on $[a,b]$ for all $b>a$.

We define $$\int_a^\infty f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx.$$

Similarly, write $$\int_{-\infty}^a f(x)\,dx = \lim_{R\to-\infty} \int_R^a f(x)\,dx$$ w/ f integrable on $[R,a]$ for all R.

If f is cts on $(a,b]$ but discontinuous at a, define $$\int_a^b f(x)\,dx = \lim_{R\to a^+} \int_R^b f(x)\,dx.$$ Similar for when f is discts at b.

Q: What about $\frac{1}{x^p}$ on $(0,\infty)$?

Reminder: $$\int_1^x \frac{1}{t}\,dt = \ln(x).$$

$p=1$. Important & complicated.

Thm: For $a>0$,

$$\int_a^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{a^{1-p}}{p-1} & \text{if } p>1 \\ \text{diverges} & \text{if } p \le 1 \end{cases}$$

$$\int_0^a \frac{1}{x^p}\,dx = \begin{cases} \text{diverge} & \text{if } p \ge 1 \\ \frac{a^{1-p}}{1-p} & \text{if } p<1 \end{cases}$$

$p-1$ vs $1-p$

opposite conditions.

Ex: **Compute**: $\displaystyle\int_2^\infty \frac{1}{x^7}\,dx$ using i) Thm
ii) limit defⁿ.

Since integral goes to ∞ and $p = 7 > 1$, we get

$$\int_2^\infty \frac{1}{x^7}\,dx = \frac{2^{-6}}{6} = \frac{2^{1-7}}{7-1}.$$

As a limit, $\displaystyle\lim_{R\to\infty}\int_2^R \frac{1}{x^7}\,dx = \lim_{R\to\infty}\left[\frac{x^{-6}}{-6}\right]_2^R$

$$= \lim_{R\to\infty}\left[\frac{x^{-7+1}}{-7+1}\right]_2^R = \lim_{R\to\infty}\left[\frac{R^{-6}}{-6} - \frac{2^{-6}}{-6}\right]$$

$p=7 \quad a=2$

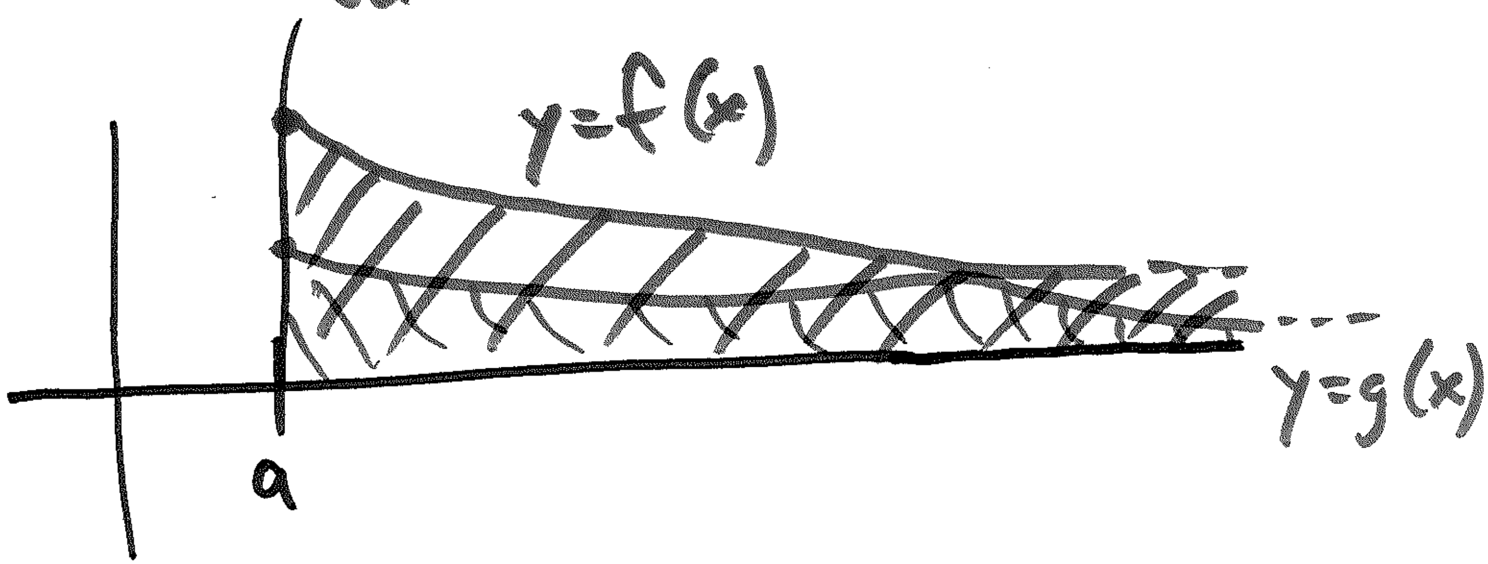
$$= \frac{2^{-6}}{6}.$$

Exercise: Complete other cases.

Thm: (Comparison Test for integrals):
Assume $f(x) \geq g(x) \geq 0$ for $x \geq a$, then

(i) If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges.

(ii) If $\int_a^\infty g(x)\,dx$ diverges, Then $\int_a^\infty f(x)\,dx$ diverges.



Ex: Does $\displaystyle\int_1^\infty \frac{1}{2\sqrt{x}+e^{2x}}\,dx$ converge?

Step 1: Find a comparison function for $\dfrac{1}{2\sqrt{x}+e^{2x}}$.

$$\frac{1}{\sqrt{x}},\ \frac{1}{e^{2x}} \leftarrow \text{Suggestions...}$$

Use $\dfrac{1}{e^{2x}}$.

$$\underbrace{\frac{1}{2\sqrt{x}+e^{2x}}}_{g} \le \underbrace{\frac{1}{e^{2x}}}_{f} \quad \text{for } x \ge 1.$$

$$\int_1^\infty \frac{1}{e^{2x}}\,dx = \lim_{R\to\infty}\int_1^R \frac{1}{e^{2x}}\,dx = \lim_{R\to\infty}\frac{1}{2}\left(e^{-2}-e^{-2R}\right)$$

$$= \frac{1}{2}e^{-2}.$$

Since $\int_1^\infty \frac{1}{e^{2x}}\,dx$ converges, comp. test implies original integral converges.

Application to infinite series:

§10.3: Thm (The Integral Test):

Let $a_n = f(n)$ where $f(x)$ is positive, decreasing, and cts for $x \geq 1$.

(a) If $\int_1^\infty f(x)\,dx$ converges, then $\sum_{n=1}^\infty a_n$ converges.

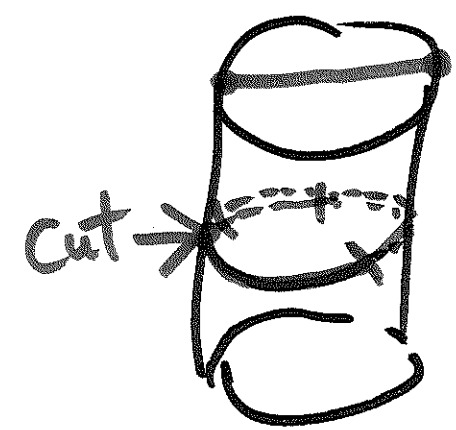
(b) If $\int_1^\infty f(x)\,dx$ diverges, then $\sum_{n=1}^\infty a_n$ diverges.

Ex: $f(x) = \frac{1}{x}$, then $a_n = f(n) = \frac{1}{n}$. So,

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{R \to \infty} \underbrace{\ln(R)}_{\int_1^R \frac{1}{x}dx} = \infty.$$ Thus, $\sum_{n=1}^\infty \frac{1}{n}$ diverges.

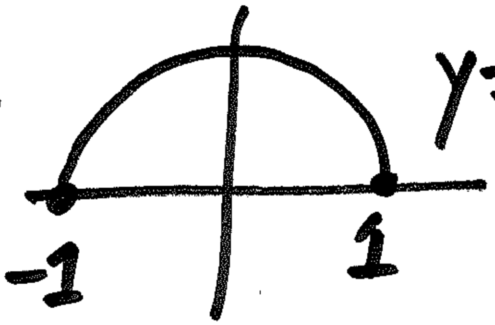
Wed, March 23, 2016:

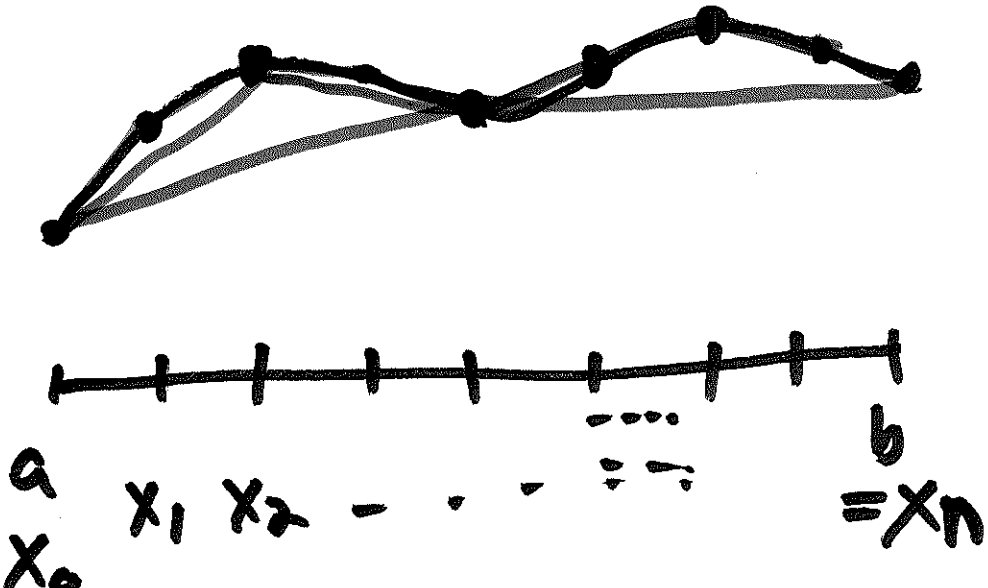
Q: When we define $\pi = \frac{C}{D}$, how do we actually measure C **accurately**? i.e., How do we know $\frac{C}{D}$ makes sense as a ratio?



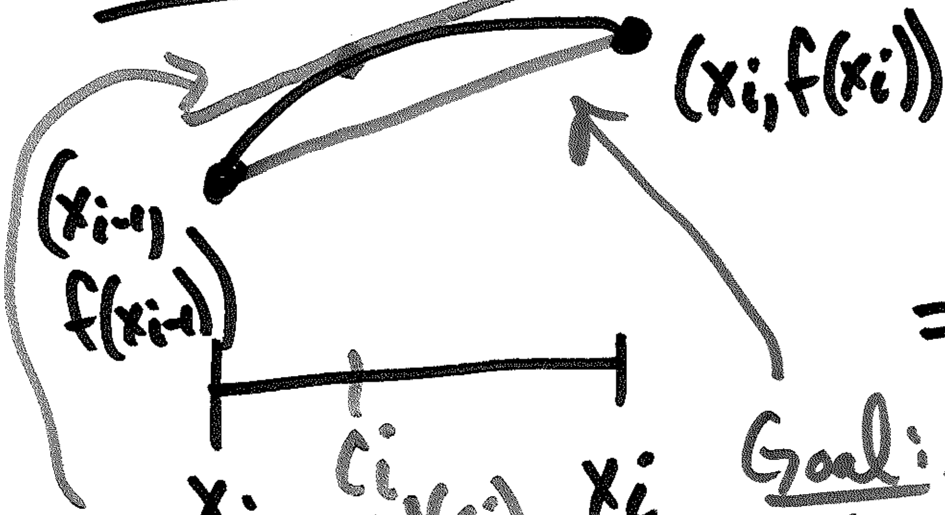
General problem: Given a differentiable curve $y = f(x)$, measure its length.

Ex: $y = \sqrt{1-x^2} = f(x)$





Schematic Picture:



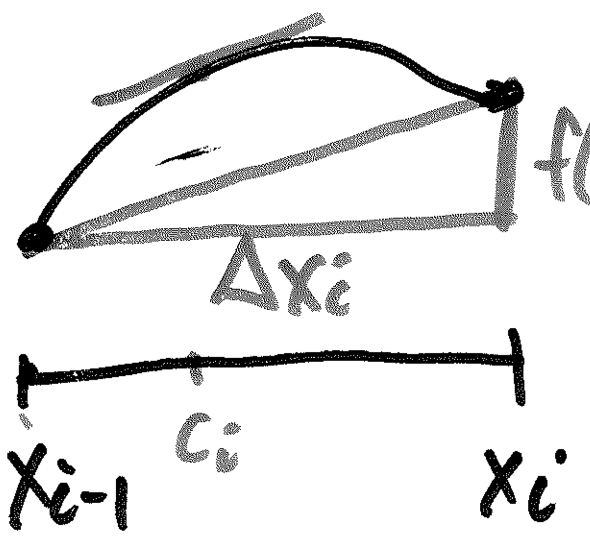
same slope $= f'(c_i)$

Goal: measure length of orange segment.

Mean Value Thm: There is a c_i in (x_{i-1}, x_i) so that $f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1})$

From this, we have setting $x_i - x_{i-1} = \Delta x_i$,

$$f(x_i) - f(x_{i-1}) = f'(c_i)\,\Delta x_i.$$



$f(x_i) - f(x_{i-1}) = f'(c_i)\,\Delta x_i$

So, length of segment is

$$\sqrt{(\Delta x_i)^2 + (f'(c_i)\Delta x_i)^2}$$

$$= \sqrt{1 + [f'(c_i)]^2} \cdot \Delta x_i.$$

Approximate length of curve is

$$\sum_{i=1}^{n} \sqrt{1 + f'(c_i)^2}\,\Delta x_i.$$

Taking limit as $n \to \infty$, we get

Length of curve is

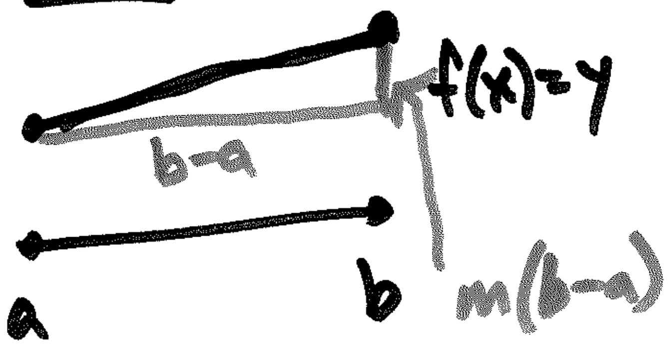
$$\int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

Since $\Delta x_i \to$ "dx".

<u>Def$^{\underline{n}}$</u>: If $f'(x)$ exists and is cts on $[a,b]$, the <u>arc length</u> of $f(x)$ over $[a,b]$ is

$$\int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

__Ex__: $f(x) = mx + c$. i.e. straight line.



using distance formula, length is

$$\sqrt{(b-a)^2 + (m(b-a))^2} = (b-a)\sqrt{1+m^2}.$$

Length is also

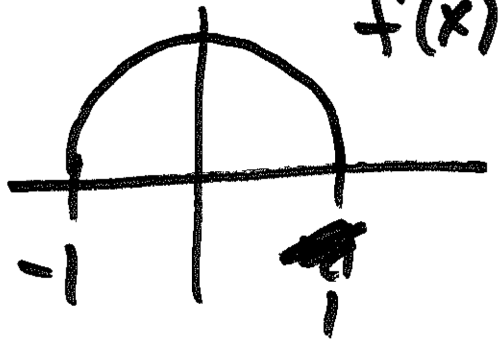
$$\int_a^b \sqrt{1+f'(x)}\,dx = \int_a^b \sqrt{1+m^2}\,dx$$

$$= \sqrt{1+m^2}\int_a^b dx$$

$$= (b-a)\sqrt{1+m^2}.$$

<u>Ex</u>: What about a unit semicircle?

$f(x) = \sqrt{1-x^2}$. Arc length <u>should</u> be π.



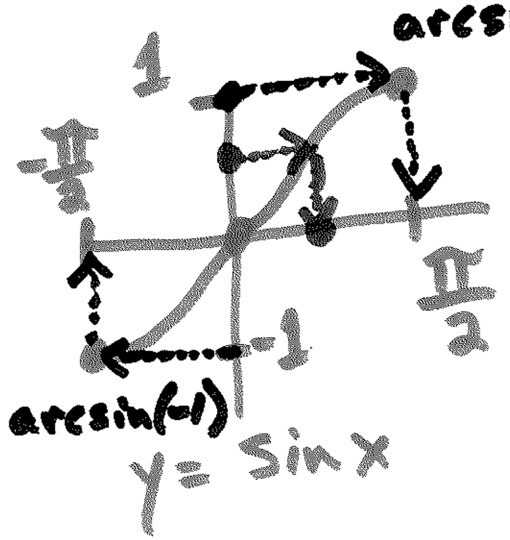
$$f'(x) = \frac{-x}{\sqrt{1-x^2}}$$

So, $$\int_{-1}^{1} \sqrt{1+f'(x)^2}\,dx = \int_{-1}^{1} \sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\,dx =$$

$$\int_{-1}^{1} \sqrt{1+\frac{x^2}{1-x^2}}\,dx = \int_{-1}^{1} \sqrt{\frac{1-x^2+x^2}{1-x^2}}\,dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx.$$

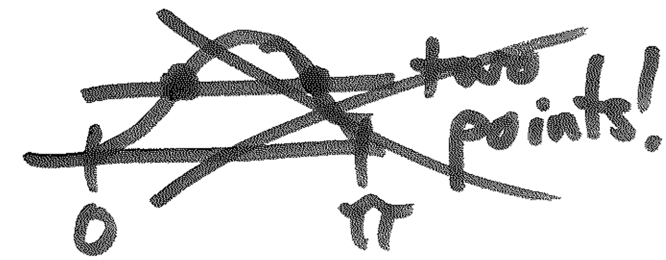
$$= (\arcsin x)\Big|_{-1}^{1} = \arcsin(1) - \arcsin(-1)$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$



domain for arcsinx = $\sin^{-1}x$ is $[-1, 1]$

Range is $[-\frac{\pi}{2}, \frac{\pi}{2}]$.



<u>Ex</u>: Catenary Curve: $f(x) = a\cosh\left(\frac{x}{a}\right)$.

where $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$.

Suppose a chain hangs w/ equation $f(x) = 2\cosh\left(\frac{x}{2}\right)$ between $x=-1$, $x=1$. What is length?

Two options: #1: Find $\frac{d}{dx}\cosh(x)$ in book + memorize it. Also learn hyperbolic trig pyth. identity.

#2: Use $\frac{e^x + e^{-x}}{2}$ + be clever.

#2: $\frac{d}{dx}\left(2\cosh\left(\frac{x}{2}\right)\right) = \frac{d}{dx}\left(2 \cdot \frac{e^{x/2}+e^{-x/2}}{2}\right)$

$= \frac{1}{2}\left[e^{x/2}-e^{-x/2}\right] \left(= 2\sinh\left(\frac{x}{2}\right)\right).$

arc length is $\int_{-1}^{1} \sqrt{1+\left(\frac{e^{x/2}+e^{-x/2}}{2}\right)^2}\,dx \underset{\uparrow \text{ algebra}}{=}$ ← clever!

$\int_{-1}^{1} \frac{e^{x/2}+e^{-x/2}}{4}\,dx \underset{\uparrow \text{ integral}}{=} e^{1/2}-e^{-1/2}.$ ← do this.