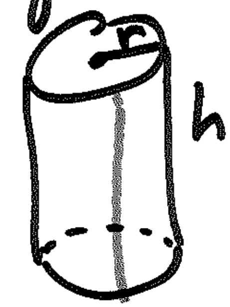
Friday, March 25, 2016 :

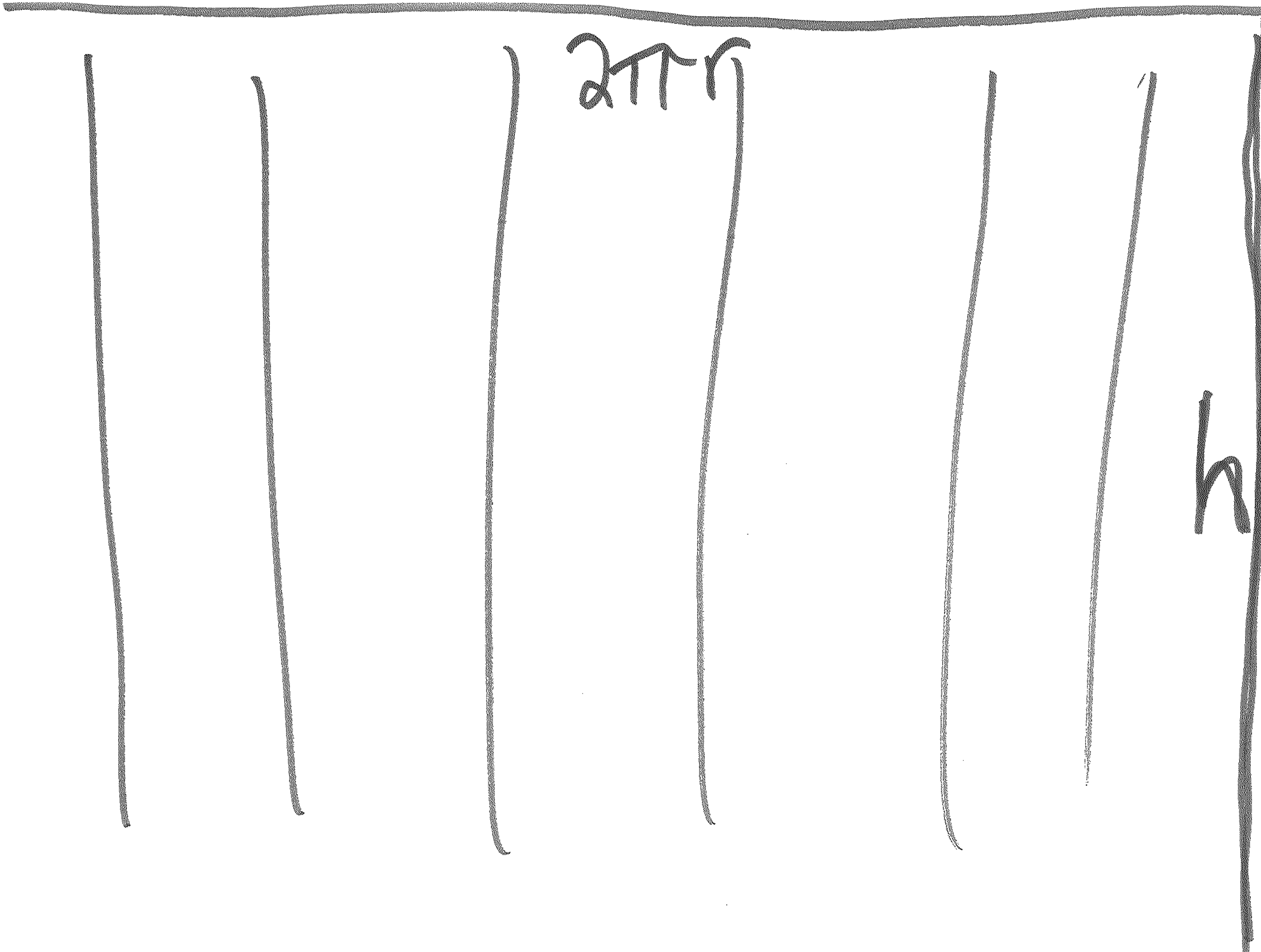
Discuss with neighbors:

1) What is surface area S of an open cylinder of radius r and height h?

top & bottom ↓ open



2) What is surface area S of a circular cone w/ base radius r and slant height ℓ?



2πr
h

1) $S = 2\pi r \cdot h$ after cutting cylinder on red line & laying flat.

2) Cut on bl + flat.

isk is

nt S,

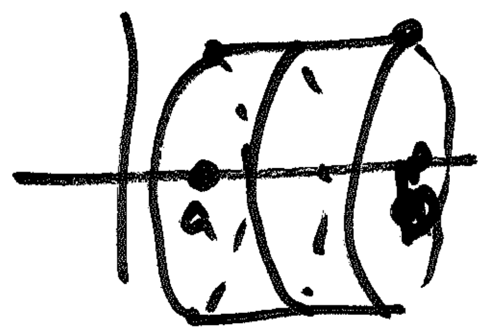
area of sector.

ratio of areas $\left\{ \frac{S}{\pi l^2} = \frac{2\pi r}{2\pi l} \right\}$ ratio of arcs of boundary.

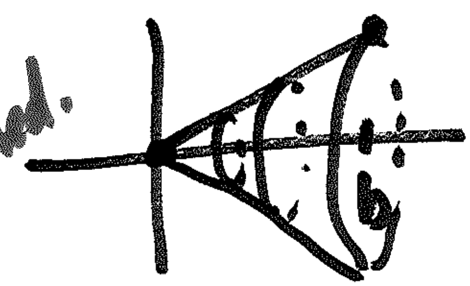
$\Rightarrow S = \pi l r.$

<u>General Problem</u>: Rotate graph of $y=f(x)$ around x-axis, $f(x) \geq 0$ on $[a,b]$. Calculate surface area S of this. (not including disks on end.)

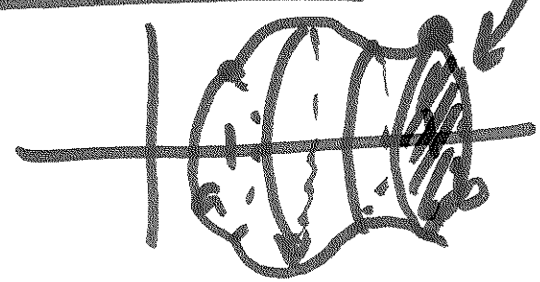
<u>Ex</u>: Cylinder is $f(x)=c$.



<u>Ex</u>: Circular cone is $f(x)=mx$ on $[0,b]$.



<u>General Picture</u>:



disk on end is disregarded.

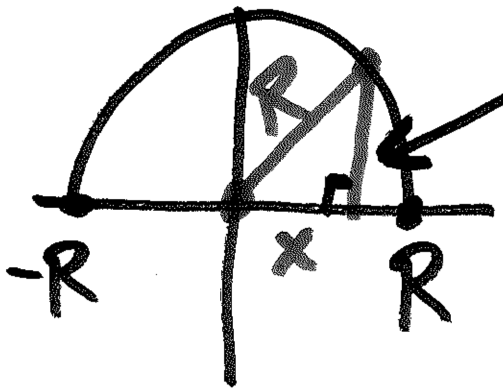
<u>Def$^{\underline{n}}$</u>: (Motivation in text): If $f(x) \geq 0$ and $f'(x)$ exists and is cts on $[a,b]$, the surface area S of the surface of revolution obtained by rotating the graph of f about the x-axis on $[a,b]$ is

$$S := \int_a^b \underbrace{2\pi f(x)}_{\text{circumference}} \underbrace{\sqrt{1+[f'(x)]^2}}_{\text{change in circumference given by curve of } f.} \, dx$$

<u>NOTE</u>:

$\sqrt{1+x^2} \approx \sqrt{x^2} \approx x$ when $x >>> 0$.

Ex: Surface Area of a sphere of radius R.



$$f(x) = \sqrt{R^2 - x^2} \text{ on } [-R, R].$$

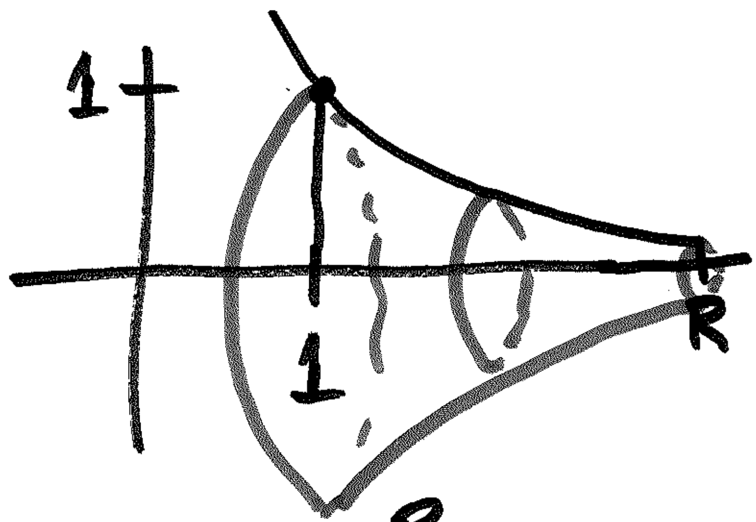
$$f'(x) = \frac{-x}{\sqrt{R^2-x^2}}, \text{ so}$$

$$1 + f'(x)^2 = \frac{R^2}{R^2 - x^2}. \quad \text{So,}$$

$$S = \int_{-R}^{R} 2\pi \sqrt{R^2-x^2} \cdot \sqrt{\frac{R^2}{R^2-x^2}}\, dx = \int_{-R}^{R} \underbrace{2\pi R}_{\text{constant}}\, dx$$

$$= 4\pi R^2.$$

Ex: Let $f(x) = \frac{1}{x}$ on $[1, R]$.



Surface area is:

$$S = \int_1^R 2\pi \cdot \frac{1}{x}\sqrt{1+\left(\frac{-1}{x^2}\right)^2}\,dx = \int_1^R \frac{2\pi}{x}\sqrt{1+\frac{1}{x^4}}\,dx.$$

Since $\sqrt{1+\frac{1}{x^4}} > 1$ on $[1, R]$, $\geq \int_1^R \frac{2\pi}{x}\,dx = 2\pi \ln(R)$

So, I don't know S, but it is $\geq 2\pi \ln(R)$.

So, if $f(x) = \frac{1}{x}$ on $[1, \infty)$, an "infinite" horn or trumpet, then the surface area is

$$\int_1^\infty 2\pi \frac{1}{x} \sqrt{1 + \frac{1}{x^4}}\, dx,$$ which diverges.

<u>But</u>: Volume of this is $\int_1^\infty \frac{\pi}{x^2}\, dx \underset{\uparrow \text{exercise}}{=} \pi$.

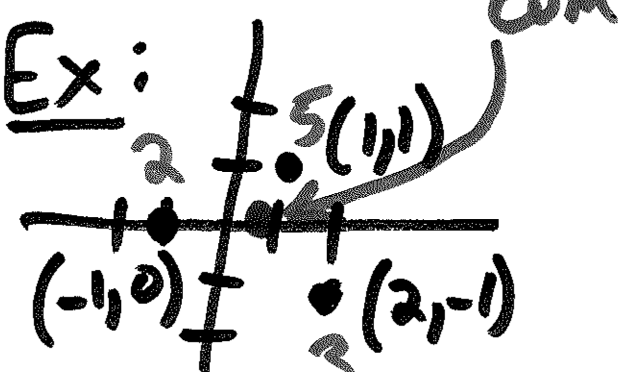
"Painter's Paradox", Toricelli's Trumpet, Gabriel's Horn.

Monday, March 28, 2016

§8.3: COM (Center of Mass)

Given points $(x_1,y_1), (x_2,y_2), \ldots, (x_n,y_n)$ in $\mathbb{R}^2$, place a particle of mass m_i at the point (x_i,y_i).

Ex:



We say total mass of the particles is $M = m_1 + \cdots + m_n$.

The moments of the particles are

$$M_y = m_1x_1 + m_2x_2 + \cdots + m_nx_n.$$

y-axis

$$M_x = m_1y_1 + \cdots + m_ny_n$$

x-axis

$$\text{COM} = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$$

$M = 2+5+3 = 10.$

$M_y = 2\cdot -1 + 5\cdot 1 + 3\cdot 2 = 9$

$M_x = 2\cdot 0 + 5\cdot 1 + 3\cdot -1 = 2$ $\quad \text{COM} = \left(\frac{9}{10}, \frac{2}{10}\right)$

Ex: Same 3 points, use

$(-1,0) : 2$

$(1,1) : 100$

$(2,-1) : 3$

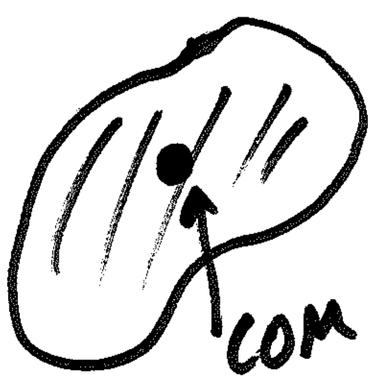
$M = 105$

$M_y = 104$

$M_x = 97$

$COM = \left(\frac{104}{105}, \frac{97}{105}\right)$

Note: If we have a "lamina", i.e. a thinplate, w/ constant density ρ, then COM = centroid.



For constant density, COM is really measuring distribution of Area of the plate.

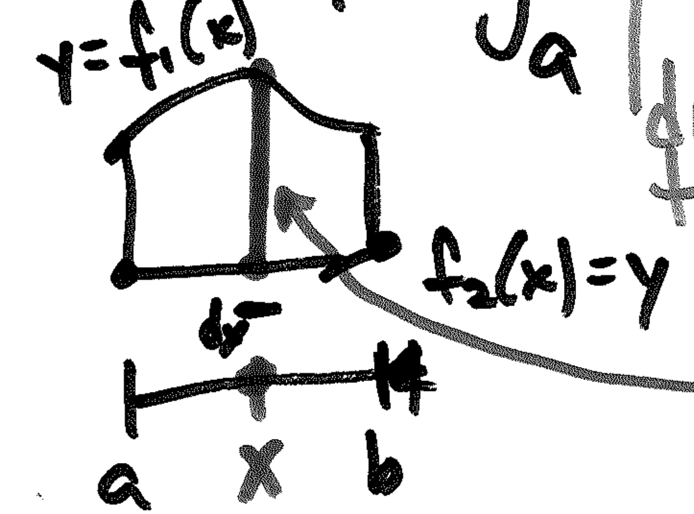
We need $COM = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$ here.

If a lamina of constant density ρ occupies the region between $f_1(x) \geq f_2(x)$ on $[a,b]$, then

$$M = \text{total mass} = \rho \cdot \text{area} = \rho \int_a^b \left[f_1(x) - f_2(x)\right] dx.$$

And

$$M_y = \int_a^b \underbrace{x}_{\text{distance to } y\text{-axis.}} \cdot \underbrace{\rho\big(f_1(x) - f_2(x)\big)\, dx}_{\text{mass of the "infinitely thin" rectangle of height } f_1(x)-f_2(x) \text{ \& width } dx.}.$$



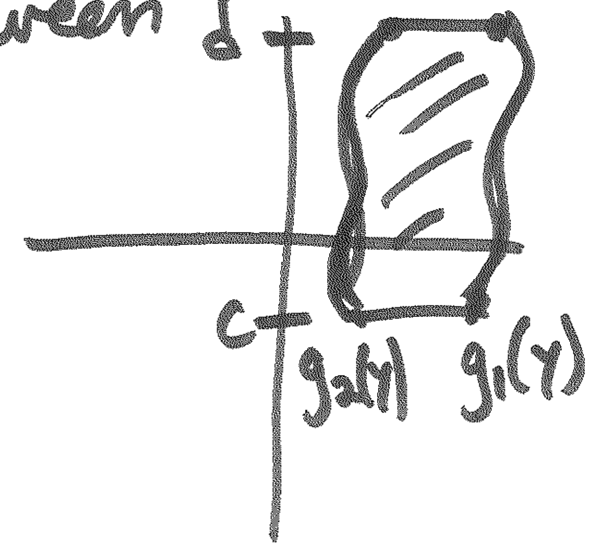
M_x is tougher: There are two options:

If we have same setup as before, $f_1(x) \geq f_2(x)$ on $[a,b]$ bounding lamina,

use $$M_x = \int_a^b \frac{1}{2}\rho\left(f_1(x)^2 - f_2(x)^2\right) dx.$$

$\longrightarrow$ See pg 482-483 for a Riemann Sum derivation.

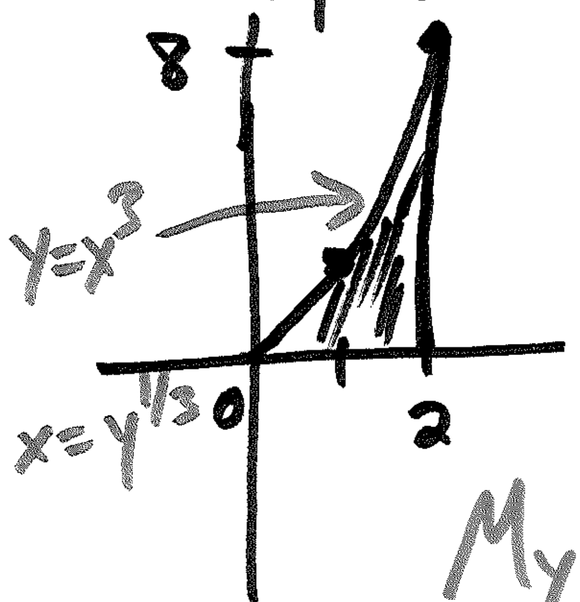
Alternatively, if lamina is between $x = g_1(y) \geq g_2(y)$ on $[c,d]$,



then $$M_x = \int_c^d y\rho\left(g_1(y) - g_2(y)\right) dy,$$

similar to M_y.

Ex: Find COM for lamina defined by $f_1(x) = f(x) = x^3$ on $[0,2]$, bounded below by x-axis, $f_2(x) = 0$ w/ density ρ.



Step 1: Compute total Mass

$$M = \int_0^2 \rho x^3 \, dx = \left[\rho \frac{x^4}{4}\right]\Big|_0^2 = 4\rho.$$

$$M_y = \int_0^2 x \cdot \rho \underbrace{(x^3 - 0)}_{f_1 \geq f_2} \, dx = \int_0^2 \rho x^4 \, dx = \rho \frac{32}{5}.$$

M_x in two ways:

$$M_x = \int_0^2 \frac{1}{2} \rho \left((x^3)^2 - 0^2\right) dx = \rho \cdot \frac{64}{7}.$$

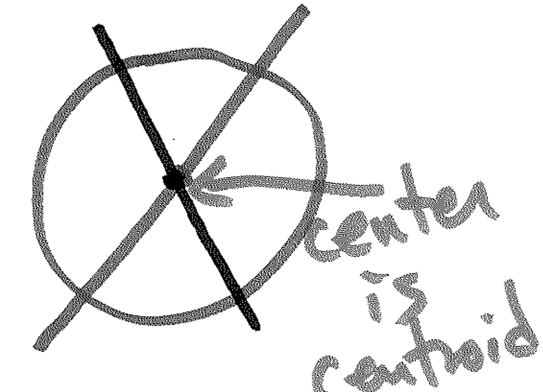
<u>Alternately</u>:

$$M_x = \int_0^8 \rho \cdot y \cdot (2 - y^{1/3})\, dy = \rho \frac{64}{7}.$$

$$\text{So, COM} = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{8}{5}, \frac{16}{7}\right).$$

<u>Symmetry Principle</u>: If a lamina is symmetric with respect to a line, its centroid lies on that line.

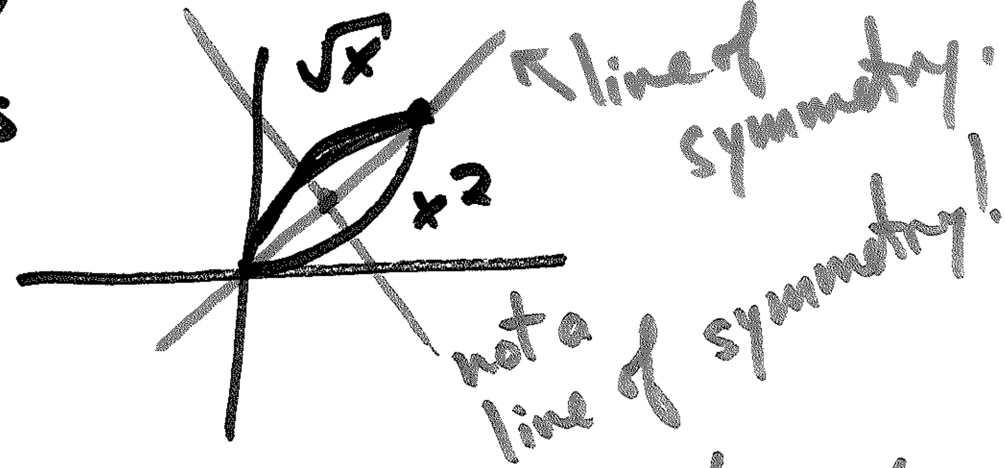
<u>Ex</u>: <u>Circle</u>: Center is centroid



WARNING: CHECK FOR SYMMETRY!

<u>Ex</u>: $f_1(x) = \sqrt{x}$, $f_2(x) = x^2$ on $[0,1]$.

Rough sketch is



See symmetry examples in text.

Wed, March 30:

Q: Why does $y = mx + b$ have a solution set that is a straight line?

1: The rate at which x & y are changing is constant.

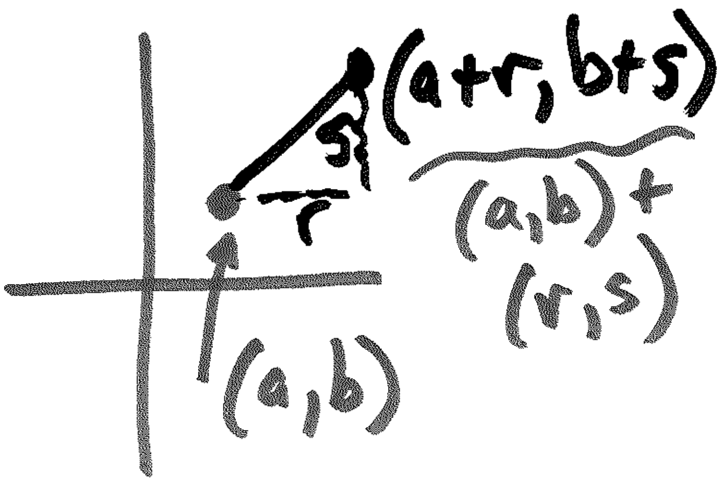
2: derivative is constant.

3: No one knows.....

<u>Idea</u>: We think of a curve in $\mathbb{R}^2$ as being <u>traced</u> over time, + we specify where pen is at time t.

One of reasons it is difficult to consider <u>lines</u> in $\mathbb{R}^2$ is $y=mx+b$ <u>obscures</u> the tracing process.

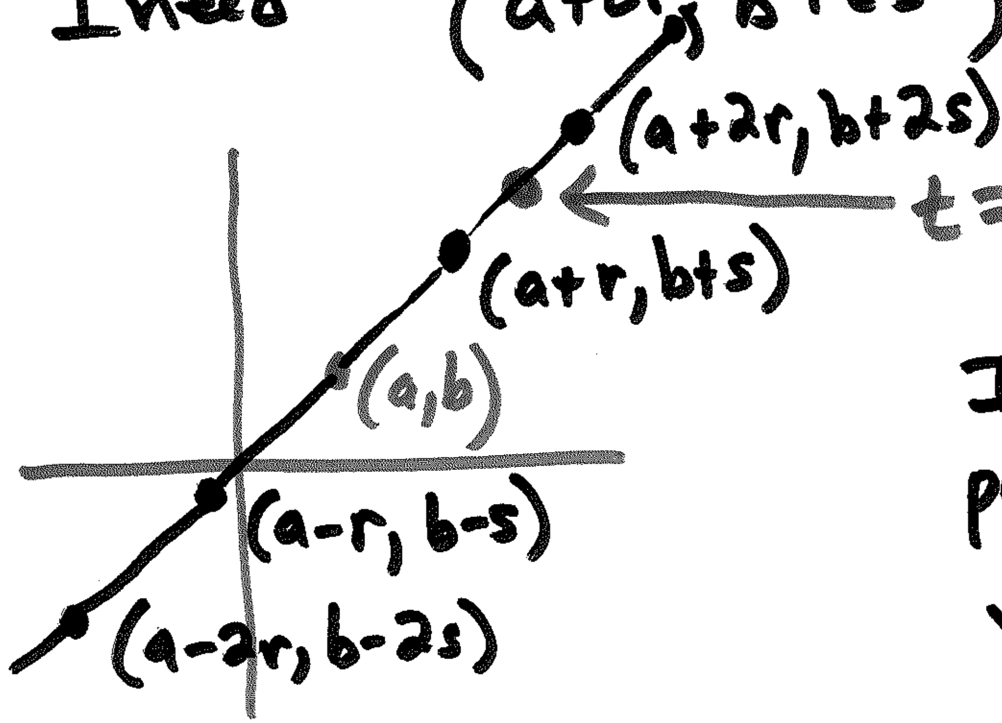
Let's create a line by picking a point + a slope + tracing.



(a,b) = starting point.

I can move in the direction (r,s) having slope m if $\frac{s}{r} = m$.

I need $(a+tr, b+ts)$ with t as a parameter.



$t = \frac{3}{2}$, $\left(a+\frac{3}{2}r, b+\frac{3}{2}s\right)$.

If $m = s/r$, then every point $(a+tr, b+ts)$ satisfies

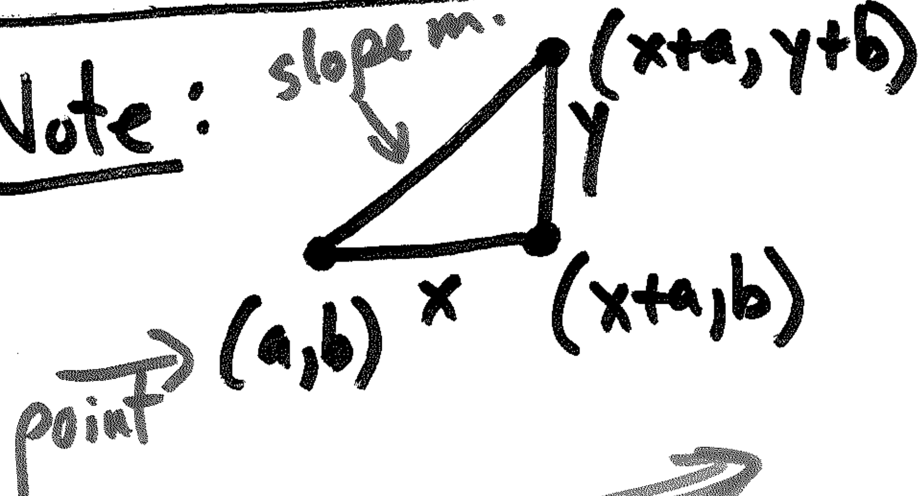
$y - b = m(x-a)$.

Instead of writing $y-b=m(x-a)$ for the point-slope form of a line, we can instead write

$$c(t)=(a+tr,\ b+ts) \leftarrow \text{parametrized curve.}$$

traces the line if $m=\frac{s}{r}$.

Note:

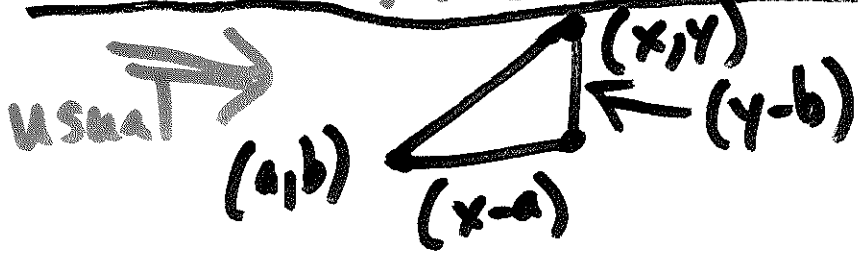


The points $(x+a, y+b)$ w/ slope m from (a,b) satisfy

$$(y+b)-b=m((x+a)-a).$$

Since $\frac{y}{x}=m$.

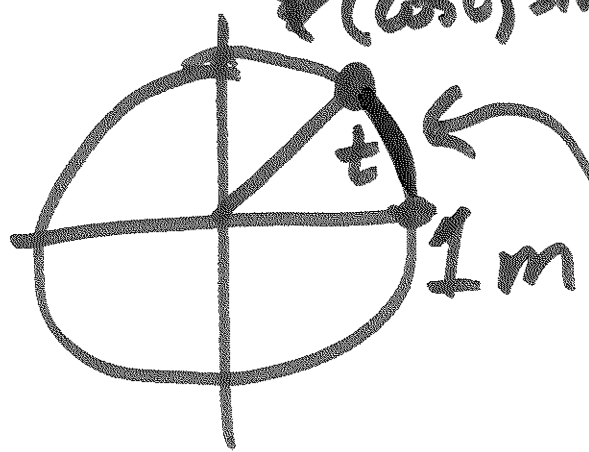
more clear.



$\Rightarrow$ $\frac{y-b}{x-a}=m$. cross multiply to get point-slope form.

Q: How do we trace a circle?

A: It is hard! Idea: you have to specify a parameter!



Suppose that you trace at a rate of 1m/s.

Q: How long to get all the way around?

2π seconds!

In this case, this distance is equal to time it took to cover the distance. So, t radians / sec / meters. equivalent here. Coord's of point are $(\cos t, \sin t)$

These are hard because

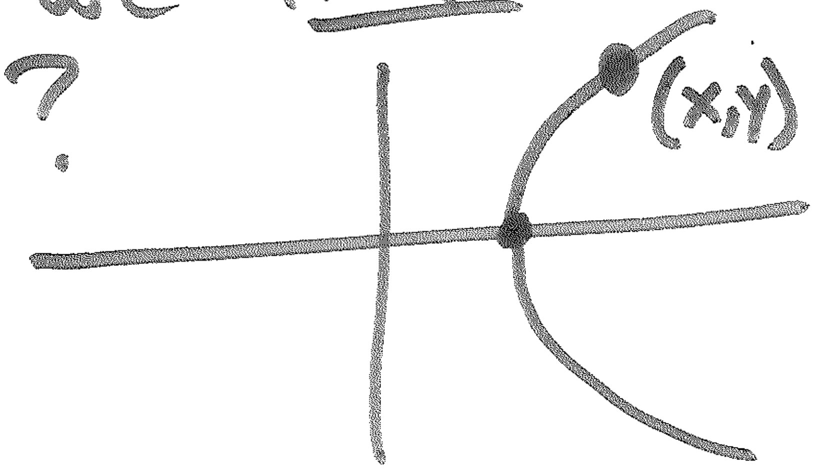
$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots$$

$$+ \quad \cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots$$

Takeaway: Parametrization of unit circle is $t \mapsto (\cos t, \sin t)$.

Q: How do we trace the hyperbola $x^2 - y^2 = 1$?

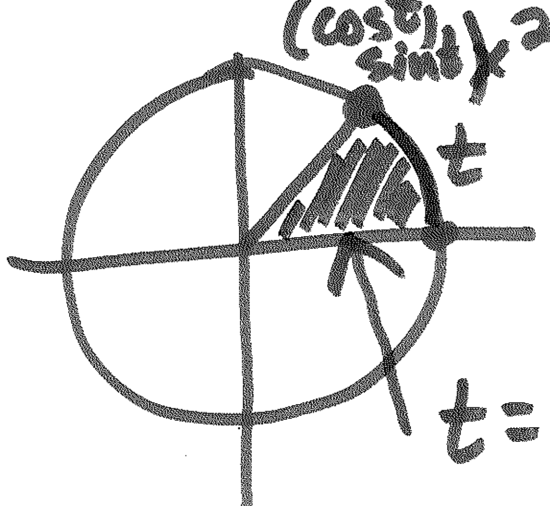
$x > 0$.

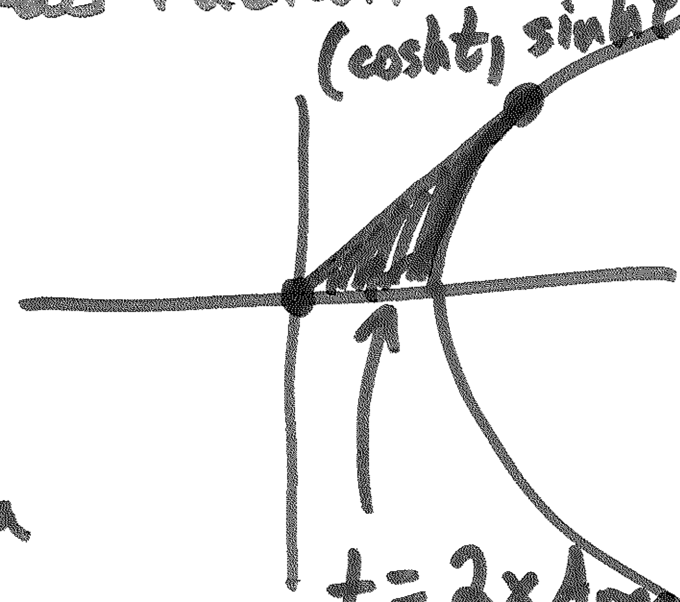


__Again, Hard!__ Call coords

$(\cosh t, \sinh t)$.

__Q__: What is t in this case? For a circle, it was radians.....





$t = 2 \times \text{Area}$ when parametrizing w/ $(\cosh t, \sinh t)$.

$$\frac{t}{2\pi} = \frac{\text{Area}}{\pi}$$

2π → Circumference

π ← area of unit circle

much like $\cos t$, $\sin t$ have power series expansions,

$$\cosh t = \frac{e^t + e^{-t}}{2}, \quad \sinh t = \frac{e^t - e^{-t}}{2}.$$

<u>Check</u>: These satisfy $(\cosh t)^2 - (\sinh t)^2 = 1$.

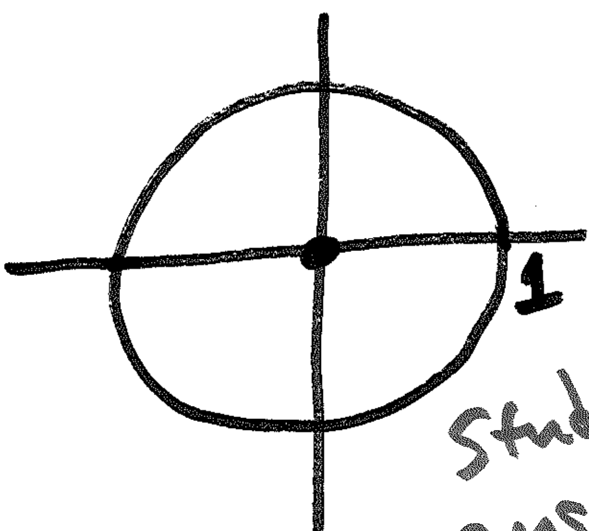
$$x^2 - y^2 = 1.$$

So, $t \mapsto (\cosh t, \sinh t)$ parametrizes the unit hyperbola.

<u>ON FRIDAY</u>: $c(t) = (f(t), g(t))$ for ~~a~~ other f, g.

Friday, April 1, 2016

Motivating Ex: Circle $x^2+y^2=1$ cannot be completely described by a function $y=f(x)$.



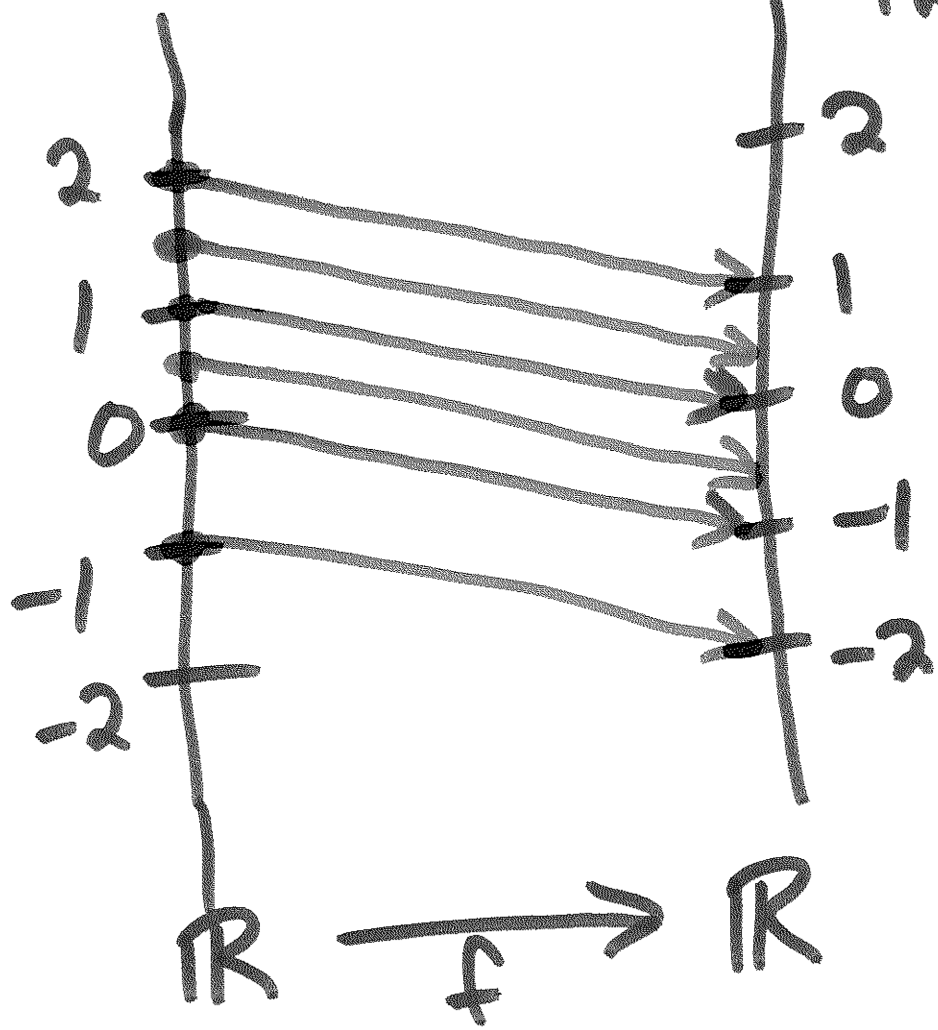
Student answers

- circle is not a function
- Vertical line test
- You only get a semicircle
- No $\sqrt{\cdot}$ of neg. #'s.
- 2 y-values for every x-value.

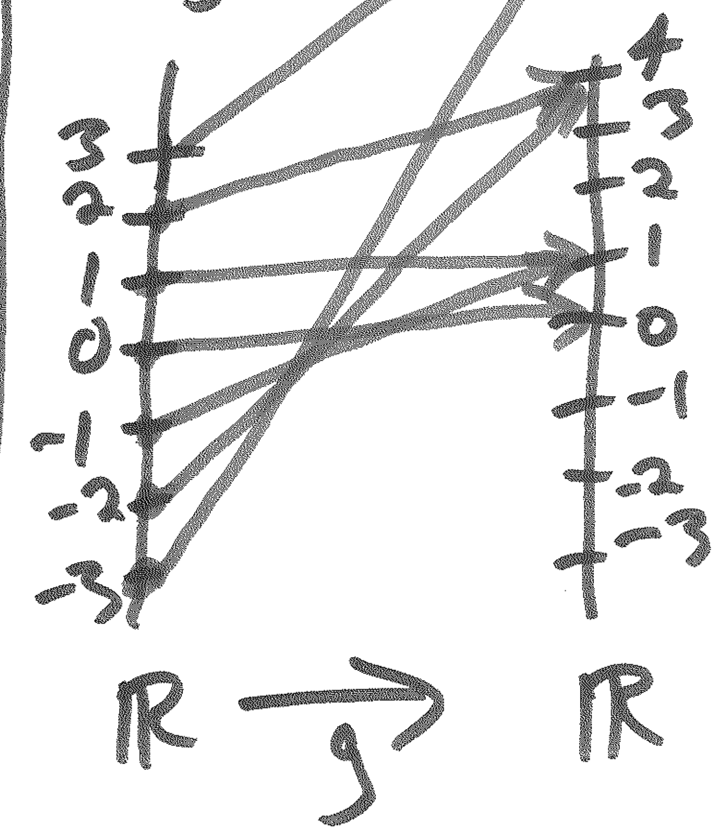
What does the graph of $f(x)$ do?

It Pairs an input w/ an output in a picture.

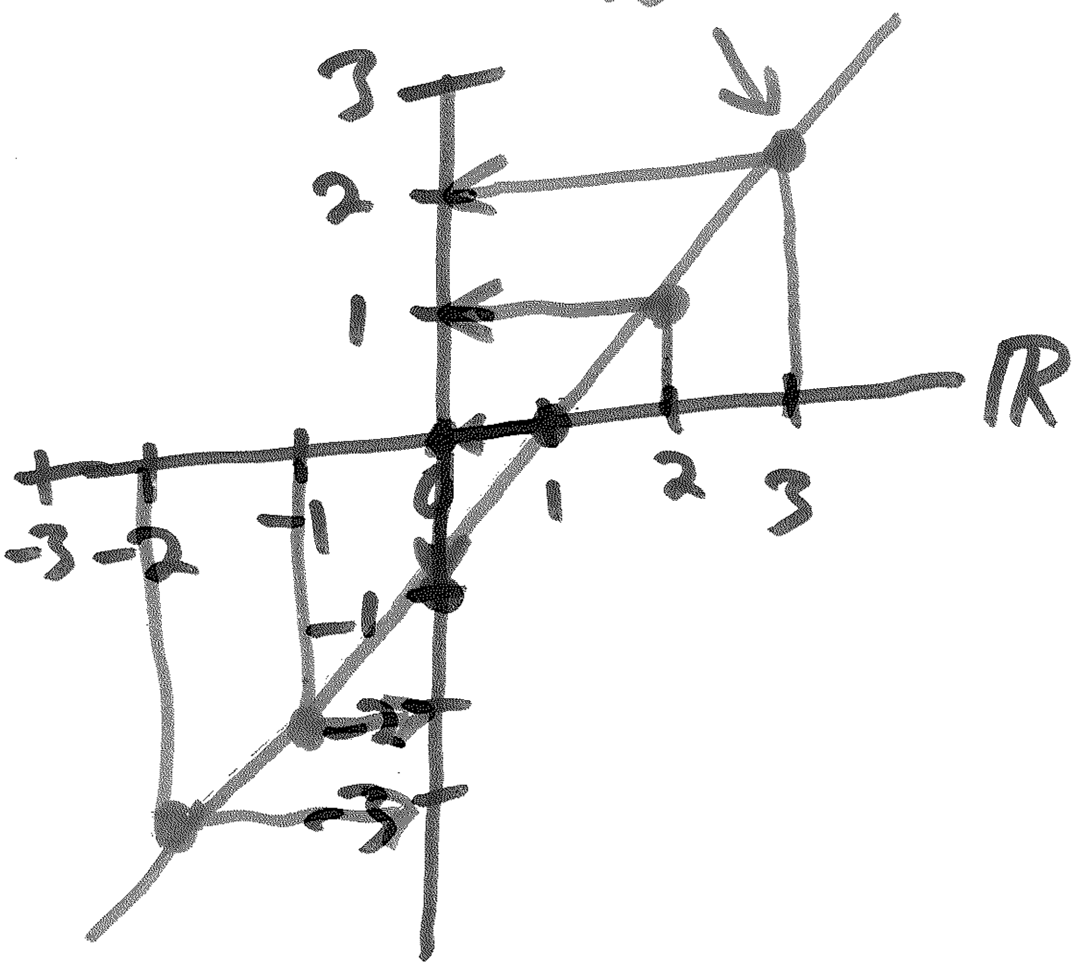
$f(x) = x - 1$. f assigns the input x to the output $x-1$. (Only one output allowed)

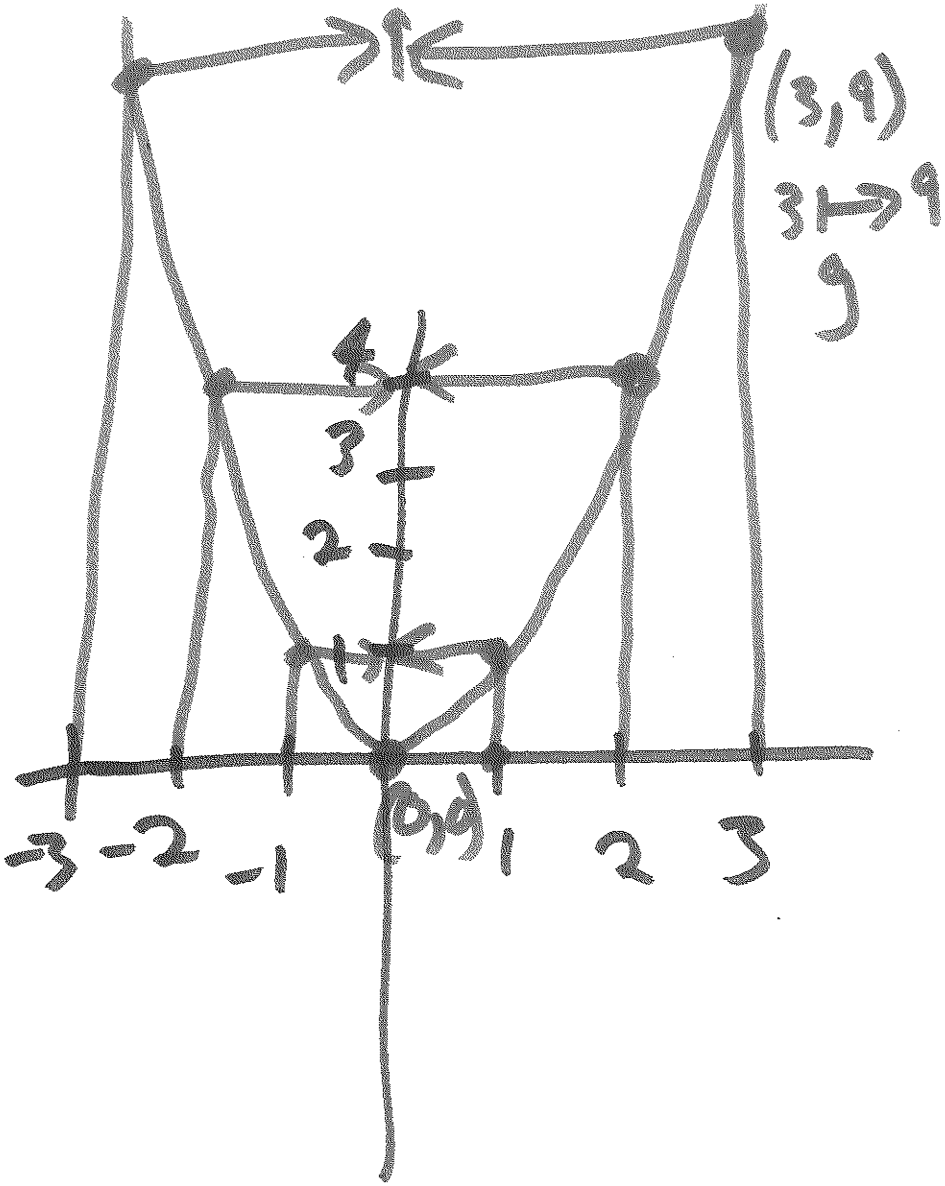


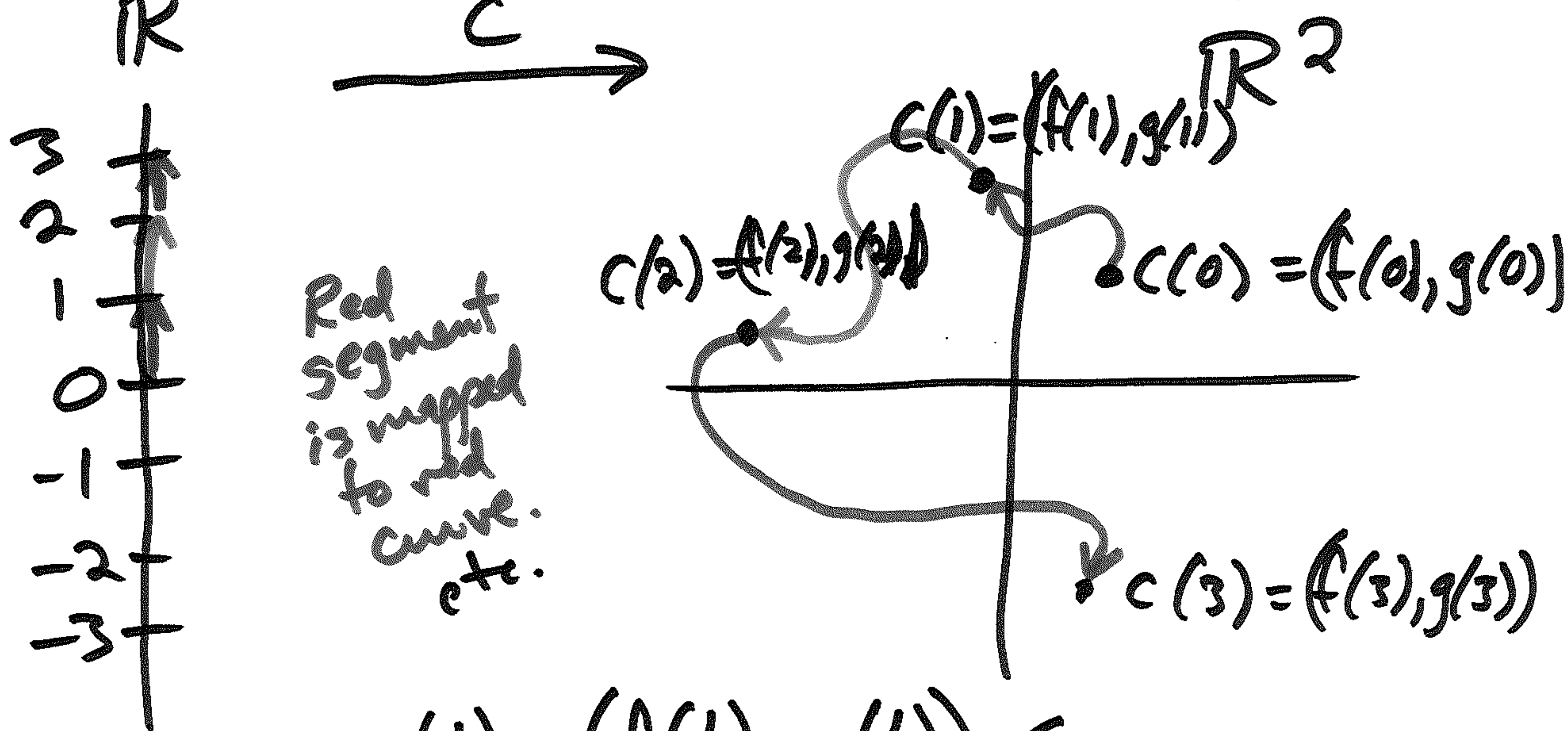
$g(x) = x^2$



$f(x) = x - 1$ $\quad 3 \xmapsto{f} 2$





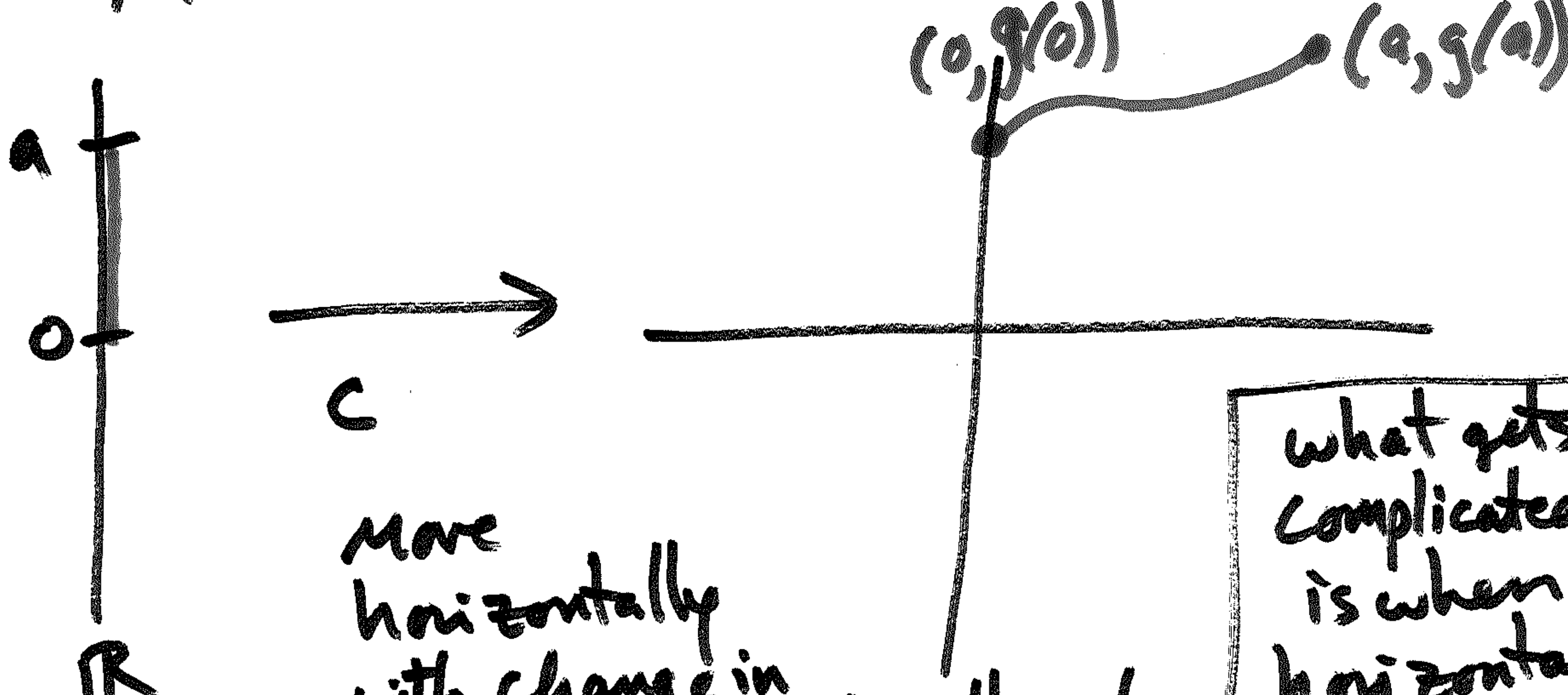


$$c(t) = (f(t), g(t))$$

given an expression of this form, we call the image of c a <u>parametrized curve</u>.

$x = f(t)$, $y = g(t)$ are called <u>parametric eqns.</u>

__Ex__: For $y=g(x)$ a function, we can parametrize the graph of $g(x)$ by setting $x=t$, $y=g(t)$, i.e. $c(t)=(t,g(t))$.



move horizontally with change in t, move vertically w/ change in g.

what gets complicated is when horizontal change is also complicated.

__Ex__: (Line) (See Thm 1, pg 609 of text).

If $r \neq 0$ and s satisfy $m = \frac{s}{r}$, the line of slope m through point (a,b) is given by $x(t) = a + rt$, $y(t) = b + st$.

$$\Rightarrow c(t) = (\underset{\uparrow}{a+rt},\ b+st).$$

this is __not__ t, as it would be for graph of a function.

In this case, image of $c(t)$ is the graph of $y - b = m(x-a)$.

Q: What are similarities/differences between the parametrizations

$$c(t) = (\cos t, \sin t)$$

and $$d(t) = (\sin t, \cos t) \;?$$

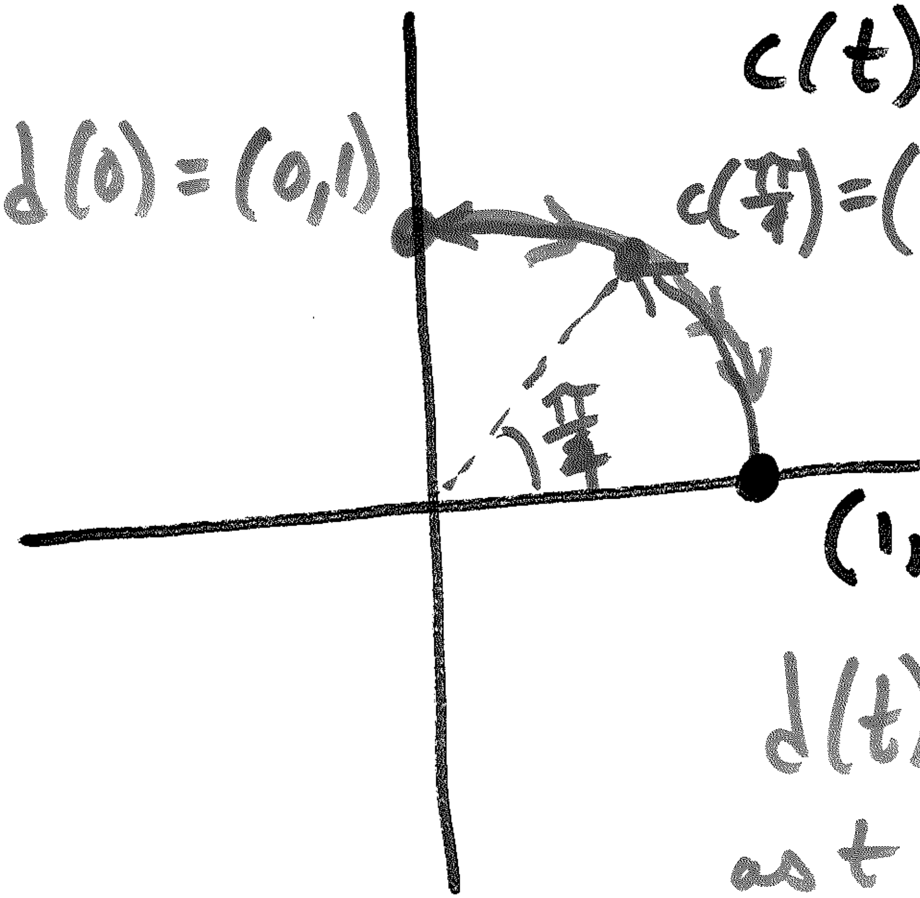
Similarities:
- c and d trace same Circle.
- Same domain & range.

Differences:
- start at different points.
- symmetric along $y=x$.
- different directions.

$c(t) = (\cos t, \sin t)$

$d(0) = (0,1)$

$c(\frac{\pi}{4}) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$



as t increases, move up along arc of circle.

$(1,0) = c(0)$

$d(t) = (\sin t, \cos t)$

as t increases, move over + down along arc of circle.

In both cases, $\sqrt{\sin^2 t + \cos^2 t} = 1$.

$c(t) = (a+rt, b+st).$

(a,b) as y-intercept.

$a=0$

$c(t) = (rt, b+st).$

$r=1, s=m. \quad \frac{s}{r} = m.$

$(1,m)$

$c(t) = (t, b+mt).$

$(\pi, \pi m)$

$\left(\cancel{\frac{m}{10^{10}}}\left(\frac{1}{10^{10}}, \frac{m}{10^{10}}\right)\right)$

__Note__: The same curve admits different parametrizations.

__Ex__: $c(t) = (\underset{x}{t}, \underset{x^2}{t^2})$ & $d(t) = (\underset{x}{t^2}, \underset{x^2}{t^4})$.

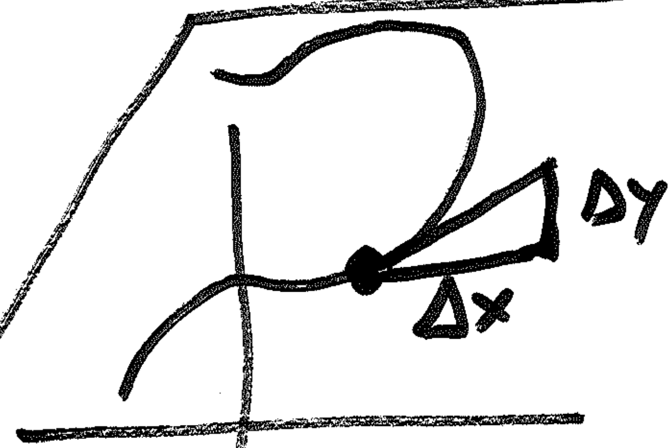
In both cases, image satisfies
$y = x^2$. But....
$c(t)$ parametrizes all of this graph,
$d(t)$ only parametrizes $x \geq 0$.

__Suggestion__: Go to desmos, use sliders to visualize & think about it.

__Extra__: $e(t) = (t^3, t^6)$

__Thm__: Let $c(t) = (x(t), y(t))$ where $x(t), y(t)$ are diff. Assume $x'(t)$ is cts and non-zero. Then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}.$$



$$\frac{\Delta y}{\Delta x} \approx \frac{y'(t)}{x'(t)}.$$

__Ex__: __Cycloid__.

Finish Monday.

See Wikipedia page for Cycloid for a good picture.