<u>Mon, March 7</u>

1. Exam 2 tomorrow!

   ↪ Taylor series for $\frac{1}{1-x}$, e^x, $\ln(1+x)$ provided. You are responsible for <u>all</u> <u>other</u> formulas, etc.

2. Review Session Tonight! See Common Web page for time/location.

<u>Linear Density</u>: Find the total mass of a rod of length 1m when linear density x meters from one end of rod is $\rho(x) = x + x^2 + 2x^3$ kg/m for $0 \le x \le 1$.

$$\int_0^1 \rho(x)\,dx = \int_0^1 (x + x^2 + 2x^3)\,dx$$

$$= \left[\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{2}\right]_0^1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{2} = \frac{4}{3}\,\text{kg}.$$

remember units!

WORK: Suppose we built a square pyramid of stone w/ density $2000\,\text{kg/m}^3$, 146 m high + square base side length 230 m.

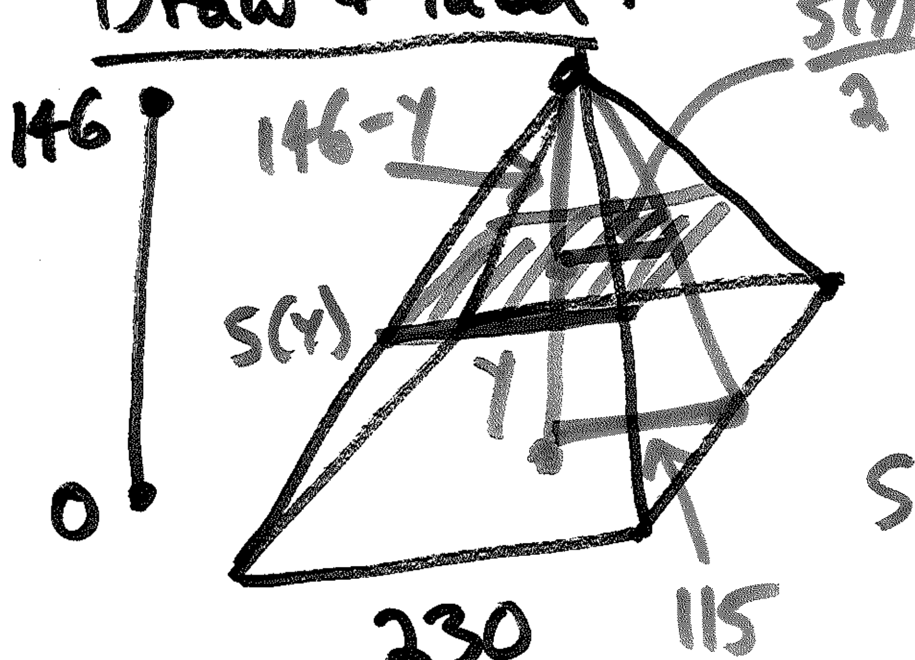
How much work is done against gravity to build the pyramid?

Q: What will the steps be when solving this?

Steps will include:

- Identify directions and signs.
- Find area of cross-section as fn of ht.
- Multiply by density, gravity, distance
- ~~a~~ ID upper + lower bounds.
- Integrate

Draw + label:



↑ pos. direction.

↓ neg. direction.

So, $A(y) = S(y)^2$.

Force against gravity goes up, so I need $a = 9.8$.

Similar triangles to compute $A(y) = s(y)^2$:

$$\frac{115}{146} = \frac{s(y)/2}{146-y} \Rightarrow s(y) = 230 - \frac{230y}{146}.$$

So, $A(y) = \left(230 - \frac{230y}{146}\right)^2$.

Mass of y^{th} layer is density $\cdot$ Volume, i.e.

$$2000 \cdot A(y)\,dy = 2000\left(230 - \frac{230y}{146}\right)^2 dy.$$

So, Force on y^{th} layer is

$$9.8 \cdot 2000\left(230 - \frac{230}{146}y\right)^2 dy$$

Work to raise y^{th} layer is $\underbrace{y}_{\text{distance}} \cdot 19{,}600\left(230 - \frac{230}{146}y\right)^2 dy$.

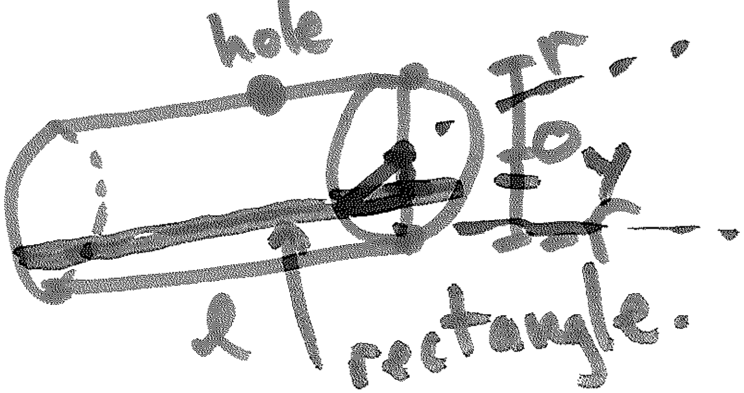
Total work is

$$\int_0^{146} y \cdot 19{,}600\left(230 - \frac{230}{146}y\right)^2 dy =$$

integrate Polynomial after expanding

1,841,773,453,333 J.

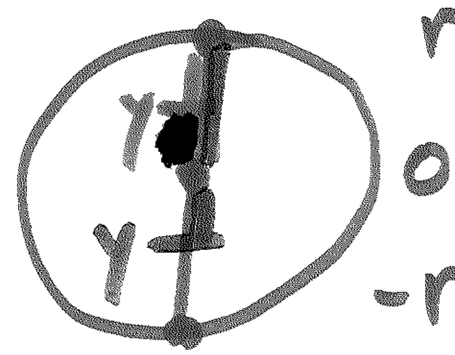
Ex: Horizontal Cylinder, pump out water via hole at top. Density H_2O is $1000\ kg/m^3$.



Pyth. Thm says width $= 2\sqrt{r^2-y^2}$.

$A(y) = 2l\sqrt{r^2-y^2}$

$$\int_{-r}^{r} \underbrace{(r-y)}_{\text{why?}} \cdot 9.8 \cdot 1000 \cdot \ell \cdot 2\sqrt{r^2-y^2}\, dy$$



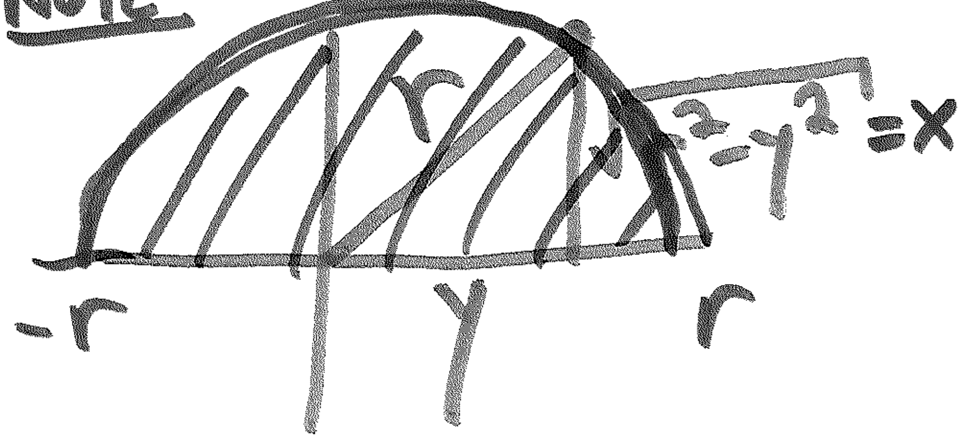
If $y<0$, note $r-y$ is green distance.

If $y>0$, then $r-y$ is orange distance.

$$= 2 \cdot 9800 \ell \int_{-r}^{r} (r-y)\sqrt{r^2-y^2}\, dy$$

$$= 2 \cdot 9800 \ell \left[\underbrace{r\int_{-r}^{r} \sqrt{r^2-y^2}\, dy}_{\text{Area of a semicircle of radius } r.} + \underbrace{\int_{-r}^{r} -y\sqrt{r^2-y^2}\, dy}_{\text{what is this? } =0.} \right]$$

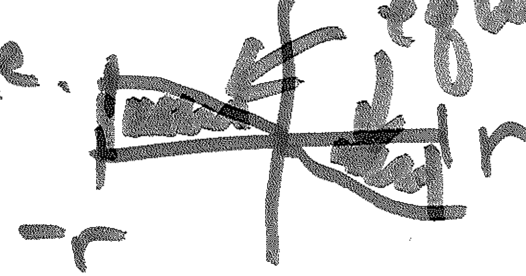
NOTE:



$x^2+y^2=r^2$ is circle of radius r.

NOTE: $-y\sqrt{r^2-y^2}$ is an odd function.

i.e.



equal areas w/opposite signs.

So, integral is 0.

Thus, integral is

$$2\cdot 9800\,l\cdot r\cdot \frac{1}{2}\pi r^2 = 19{,}600\cdot l\cdot\frac{\pi r^3}{2}\text{ J.}$$

$$= 9{,}800\,l\,\pi r^3\text{ J.}$$

Wed, March 9:

1. Watch Numberphile video about Fund. Thm. of Algebra I sent via Canvas.

2. Today we start Partial Fractions!

Ex: $\dfrac{1}{(1-x)(1-y)(1-xz)(1-\cancel{y}z^3)}$

"Bad" partial fractions leads to $2=1$ here....

§7.5: Motivation

$$\int_0^x \frac{1}{1+t}\,dt = \ln(x) \longrightarrow$$ integrate a "rational" function, i.e. $\frac{P(x)}{Q(x)}$ ← polys.

$$\int_0^x \frac{1}{1+t^2}\,dt = \tan^{-1}(x).$$

What is complicated here? $1+t^2$.

Why? Roots of this expression are:

Roots are solns to $1+t^2=0$,

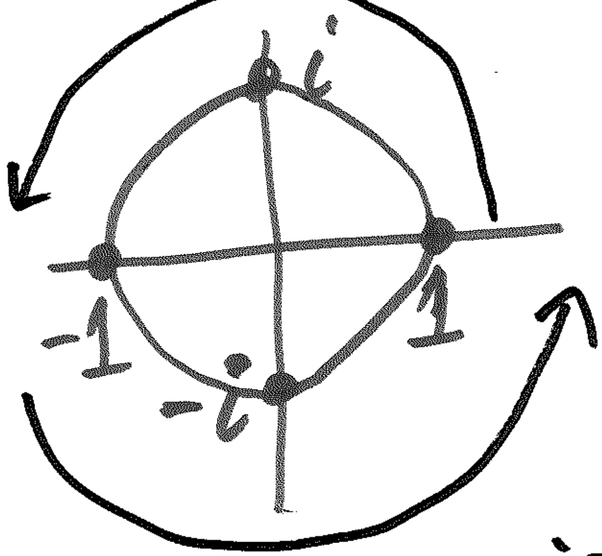
so $t^2=-1$. So, $t=\pm\sqrt{-1}$.

1600's, "negative" #'s called "false" #'s.

+ $\sqrt{-1}$ called "imaginary" unit.

$i = \sqrt{-1}$.

→ <u>See</u> Wikipedia for Complex #'s ←



$i^2 = -1$.

$(-1)^2 = 1$.

$i \cdot x$ is ½ of a 180° rotation, since $i^2 = -1$,

So $i \cdot i x = (-1)x$.

So, it is a 90° rotation.

$(-1) \cdot x$ is a 180° rotation.

<u>NOTE</u>: First time Complex #'s show up in school + history is solving polynomial equations.

Simplex Example:

$$\int \frac{1}{x^2-1}\,dx$$

One option: $-\int \frac{1}{1-x^2}\,dx$, expand $\frac{1}{1-x^2}$ as a geom. series.

Idea of partial fractions: "Undo" finding a common denominator.

Factor $x^2-1=(x+1)(x-1)$

$$\frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)} = \frac{A}{(x+1)} + \frac{B}{(x-1)}.$$

NOTE:

$\left(\frac{3}{5}-\frac{1}{7}\right)\pi$

Make sure you can do this!

Key insight: We can solve for A+B!

There are various ways to do this. I'm going to use systems of equations.

$$\frac{A}{(x+1)} + \frac{B}{(x-1)} = \frac{A(x-1)+B(x+1)}{(x+1)(x-1)} = \frac{1+0\cdot x}{(x+1)(x-1)}.$$

So, numerators are equal:

$$A(x-1)+B(x+1) = (A+B)x + (B-A) = \underset{\sim}{1}.$$
$$= 1 + 0\cdot x.$$

So, $A+B=0 \leftarrow$ coeff of x

$-A+B=1 \leftarrow$ coeff of constant.

From *high school* we see 2nd eqn gives $B=1+A$, + plugging this into 1st eqn give $A+(1+A)=0$, so $A=-\frac{1}{2}$.

So, $B=\frac{1}{2}$.

So, $\displaystyle\int \frac{1}{x^2-1}\,dx = \int\left(\frac{-\frac{1}{2}}{x+1} + \frac{\frac{1}{2}}{x-1}\right)dx$

$\displaystyle = -\frac{1}{2}\int \frac{1}{x+1}\,dx + \frac{1}{2}\int\frac{1}{x-1}\,dx$

$\displaystyle = -\frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C.$

<u>Ex</u>: $\displaystyle\int \frac{x+5}{x^2+x-2}\,dx.$

<u>Step 1</u>: Factor denominator.

<u>Step 2</u>: Rewrite w/ unknown numerators to "undo" common denominator.

<u>Step 3</u>: Solve for unknowns

<u>Step 4</u>: Compute integral.

Step 1: $x^2+x-2 = (x-1)(x+2)$.

Step 2: $\dfrac{x+5}{x^2+x-2} = \dfrac{x+5}{(x-1)(x+2)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x+2)}$

Step 3: Compare numerators,

$$x+5 = A(x+2) + B(x-1).$$

$$\Rightarrow \quad x+5 = (A+B)x + (2A-B)$$

Solve for A + B.

$$\begin{cases} A+B=1 & \text{coeff's of } x \\ 2A-B=5 & \text{constant terms} \end{cases}$$

Idea: Set $B=1-A$, plug into 2nd eqn. Solve for A, then for B.

$A=2$, $B=-1$.

Step 4: $$\int \frac{x+5}{x^2+x-2}\,dx = \int\left(\frac{2}{x-1} + \frac{-1}{x+2}\right)dx$$

$$= 2\ln|x-1| - \ln|x+2| + C.$$

General Strategy needs a few definitions to describe.

Def$^{\underline{n}}$: If $P(x)$ & $Q(x)$ are polynomials with degree of P $\underset{\text{strictly}}{\wedge}$ less than degree of Q, we say

$\frac{P(x)}{Q(x)}$ is **proper** rational function.

Ex: $\frac{x+5}{x^2+x-2}$ proper, $\frac{x^3-1}{x^2+1}$ not proper.

Fundamental Thm of Algebra: (real version)

Every polynomial $P(x)$ with real coefficients can be factored uniquely (up to a constant multiple) as a product of real linear polynomials and real quadratic polynomials where quadratics only have non-real complex roots/zeros.

Ex: $1-x^2 = (1-x)(1+x) = \left(\frac{1}{2}-\frac{x}{2}\right)(2+2x)$.

$$1-x^3 = (1-x)\underbrace{(1+x+x^2)}_{\text{has 2 complex roots.}}$$

irreducible over the reals.

Friday, March 11

(Fun)

Q: What is $\underbrace{1+2+3+4+5+6+7+\cdots}_{\text{infinite series}} = ?$

Suggested answer #1: ∞

" " #2: $\sum_{n=1}^{\infty} n = \frac{-1}{12}$.

Both are correct!

Back to partial fractions:

Ex: Evaluate $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$ using partial fractions.

Step 1: Factor denominator:

$$2x^3+3x^2-2x = x(2x^2+3x-2)$$
$$= x(2x-1)(x+2)$$

Step 2:

$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$
$$= \frac{A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)}{x(2x-1)(x+2)}.$$

Step 3: **Solve for A, B, C:**

Set $x^2+2x-1 = A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1)$

$$= (2A+B+2C)x^2 + (3A+2B-C)x - 2A$$

Set coefficients equal:

$$\underbrace{\begin{aligned} 2A+B+2C &= 1 \\ 3A+2B-C &= 2 \\ -2A &= -1 \end{aligned}}_{\text{linear system of eqns} \to \text{see MA322.}} \leftarrow A = \frac{1}{2}.$$

sub $A=\frac{1}{2}$, get 2 eqns in B, C.

MA214 involves derivatives

Solving gives $A = \frac{1}{2}$, $B = \frac{1}{5}$, $C = \frac{-1}{10}$.

Step 4: Do the integral:

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx = \int \left(\frac{1}{2 \cdot x} + \frac{1}{5} \cdot \frac{1}{2x-1} - \frac{1}{10} \cdot \frac{1}{x+2} \right) dx$$

$$= \frac{1}{2} \ln|x| + \frac{1}{10} \ln|2x-1| - \frac{1}{10} \ln|x+2| + C.$$

u-sub here

Ex: (Repeated factors):

If $(x-a)^M$ shows up in a factorization,

use $$\frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \frac{A_3}{(x-a)^3} + \cdots + \frac{A_M}{(x-a)^M}.$$

To evaluate $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$, we observe that the integrand is not proper:

Degree of num > Degree of den!

Do long division to obtain

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1+\frac{4x}{x^3-x^2-x+1}.$$

"polynomial long division"

Our integral is

$$\int x+1+\frac{4x}{x^3-x^2-x+1}\,dx =$$

$$\frac{x^2}{2}+x+\underbrace{\int\frac{4x}{x^3-x^2-x+1}\,dx}_{\text{Focus here, proper integrand.}}+C$$

<u>Step 1</u>: factor $x^3-x^2-x+1=(x-1)^2(x+1)$.

<u>Step 2</u>: $\dfrac{4x}{x^3-x^2-x+1}=\dfrac{A}{(x-1)}+\dfrac{B}{(x-1)^2}+\dfrac{C}{(x+1)}$.

<u>NOTE</u>: If I saw $(x-1)^3$, write $\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{C}{(x-1)^3}+\dfrac{D}{(x+1)}$.

Step 3: Find common den, set coeff's equal, solve for A, B, C.

Exercise to fill in details $\Rightarrow$

$$4x = (A+C)x^2 + (B-2C)x + (-A+B+C)$$

$$A=1,\ B=2,\ C=-1.$$

Step 4: $$\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx = \frac{x^2}{2}+x+\int \frac{4x}{x^3-x^2-x+1}\,dx + C$$

$$= \frac{x^2}{2}+x+\int\left(\frac{1}{x-1}+\frac{2}{(x-1)^2}-\frac{1}{(x+1)}\right)dx + C$$

$$= \frac{x^2}{2}+x+\ln|x-1| - \frac{2}{x-1} - \ln|x+1| + C.$$

via u-sub.

Ex: **Quadratic factors!**

If ax^2+bx+c has only non-real complex roots, we say it is **irreducible**.

If $\frac{P(x)}{Q(x)}$ has $Q(x)$ factor w/ irreducible factors, we do the following: If $(ax^2+bx+c)^M$ is an irreducible factor of $Q(x)$, include in P.F.D.

$$\frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \frac{A_3x+B_3}{(ax^2+bx+c)^3} + \cdots + \frac{A_Mx+B_M}{(ax^2+bx+c)^M}.$$

<u>WARNING</u>: Trig sub might be needed, See examples in book.

Ex: $\int \frac{1-3x+2x^2-x^3}{x(x^2+1)^2}\,dx$, we set

$$\frac{1-3x+2x^2-x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{(x^2+1)} + \frac{Dx+E}{(x^2+1)^2}.$$

Find a common den, & set coeff's equal, get

$$1-3x+2x^2-x^3 = A+(C+E)x+(2A+B+D)x^2 + Cx^3 + (A+B)x^4.$$

$\Rightarrow A=1, B=-1, C=-1, D=1, E=-2.$

So, $\displaystyle\int \frac{1-3x+2x^2-x^3}{x(x^2+1)^2}\,dx =$

$$\int \left(\frac{1}{x} - \frac{x+1}{x^2+1} + \frac{x-2}{(x^2+1)^2}\right)dx \quad \leftarrow \text{u-sub + trig sub.}$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+1| - \tan^{-1}(x) - \frac{1}{2(x^2+1)}$$

$$-2\left(\frac{1}{2}\tan^{-1}(x) + \frac{x}{x^2+1}\right) + C.$$