

MA13

4/19/17

- 1] REEF today
- 2] Today is § 5.5 on the substitution method.
- 3] Discuss w/ your neighbors:

What is $\int x e^x dx$?

4] Please fill out your online course evaluations!!!
Currently, section response rates are
between 0% and 15%!!

§5.5: Substitution, aka reversing the chain rule.

Note: $\int f(x) dx = F(x) + C$ means $F(x)$ is an antiderivative for $f(x)$.

We need ways to "undo" differentiation rules.

NOTE: $\frac{d}{dx}(cf) = c \frac{d}{dx} f$

$\int cf dx = c \int f dx$

Today \rightarrow (I) Chain Rule \rightarrow Substitution

(II) Product Rule \rightarrow Integration by Parts (Calc II).

Today's Q: If you hand me $f(x)$, how can I determine if $f(x)$ is the derivative obtained from $F(x)$ using chain rule?

Substitution: If $u = g(x)$ is diff, and if f is cts,

$$\int f(g(x)) \cdot g'(x) dx = \int \underbrace{f(u) du.}$$

Then

NOTE:
 $u = g(x)$
 $\Rightarrow \frac{du}{dx} = g'(x)$
 $\Rightarrow du = g'(x) dx.$

$$\int e^u \sqrt{x^2} \frac{du}{dx} \cdot 2x dx = \int e^u du = e^u + C$$

Ex: $\int e^{\sqrt{x^2}} \cdot 2x dx$. $\Rightarrow du = 2x dx$

$$\Rightarrow du = g'(x) dx.$$

$$\text{Ex: } \int x e^{x^2} dx = \int \frac{2}{2} x e^{x^2} dx = \frac{1}{2} \int 2x e^{x^2} dx$$

Same integral as before.

Observation: You will often need to introduce constants by multiplying & dividing.

$$\text{Ex: } \int \frac{x^3}{\sqrt{3x+2}} dx = \frac{1}{3} \int \frac{x^3 \cdot \sqrt{3x+2}}{\sqrt{3x+2}} dx = \frac{1}{3} \int \sqrt{u} \cdot du = \frac{2}{5} u^{5/2} + C$$

mult. & divide by 3

$$u = 3x+2$$

$$\frac{du}{dx} = 3 \Rightarrow du = 3dx$$

$$\textcircled{*} = \frac{1}{3} \int u^{1/2} du = \frac{1}{3} \frac{u^{3/2}}{3/2} + C = \frac{2}{9} (3x+2)^{3/2} + C$$

Step 1: change variable to u, identify du.

Step 2: Integrate w/ u.

Step 3: Change back to x.

Note:

$$\int x^a dx = \frac{x^{a+1}}{a+1} + C$$

unless $a = -1$.

$$\text{Ex: } \int \tan(x) dx = \int \frac{\sin x}{\cos x} dx = \int \frac{1}{\cos x} \cdot (-\sin x) dx = *$$

$$u = \cos x \Rightarrow \frac{du}{dx} = -\sin x$$

$$\Rightarrow du = -\sin x dx$$

$$* = - \int \frac{1}{u} du = -\ln|u| + C = -\ln|\cos x| + C$$

$$= \ln|\cos x|^{-1} + C$$

Remk: usually in books,

$$\int \tan x dx = \ln|\sec x| + C$$

$$= \ln|\sec x| + C$$

Ex: $\int \frac{x}{\sqrt{1+x^2}} \cdot x^p \cdot x^q \cdot x^r \cdot x^s \cdot x^t \cdot x^u \cdot x^v \cdot x^w \cdot x^x \cdot x^y \cdot x^z \cdot x^{\dots} = \textcircled{*}$

close to du = $\sqrt{u} \cdot (u-1)(u-1)$
 mult + div by 2
 to fix it.

$u = 1+x^2 \Rightarrow u-1 = x^2$

$du = 2x \cdot dx$

$\textcircled{*} = \frac{1}{2} \int \sqrt{u} \cdot (u-1)(u-1) \cdot du$

$\rightarrow = \frac{1}{2} \int (u^{5/2} - 2u^{3/2} + u^{1/2}) du$

algebra

$= \frac{1}{2} \left[\frac{u^{7/2}}{7/2} - 2 \frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2} \right] + C$

$= \frac{1}{2} \left(\frac{u^{7/2}}{7/2} - 2 \frac{(1+x^2)^{5/2}}{5/2} + \frac{(1+x^2)^{3/2}}{3/2} \right) + C$

See book for examples.

Rule for definite integrals: $u = g(x)$

$$\int_a^b f(g(x)) \cdot g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

Ex: $\int_0^3 e^{5x} dx = \frac{1}{5} \int_0^3 e^{5x} \cdot 5 dx = \frac{1}{5} \int_0^{15} e^u du = *$

$u = 5x, du = 5 dx$

check bounds too.

$\frac{1}{5} [e^u]_0^{15} = \frac{e^{15} - e^0}{5}$

Ex: $\int_{-\pi}^{2\pi} \sin(\pi x + 1) dx = \frac{1}{\pi} \int_{-\pi}^{2\pi} \sin(u) du = \left[-\frac{\cos u}{\pi} \right]_{-\pi}^{2\pi}$

$u = \pi x + 1, du = \pi dx$

$= \frac{-\cos(2\pi + 1) + \cos(-\pi + 1)}{\pi}$

MA 113

4/21/17

1 REEF today

2 See Canvas announcement about REEF score.

3 Today + Monday: S3.10, Linear Approximations
and Handout on Higher-Order
Approximations (see website for
MA 113 for handout.)

4 Discuss with your neighbors:
What is the equation for the tangent
line to the graph of $f(x)$ at $x=a$?

5 Final Exam: 8:30-10:30, Monday May 1.
PM PM

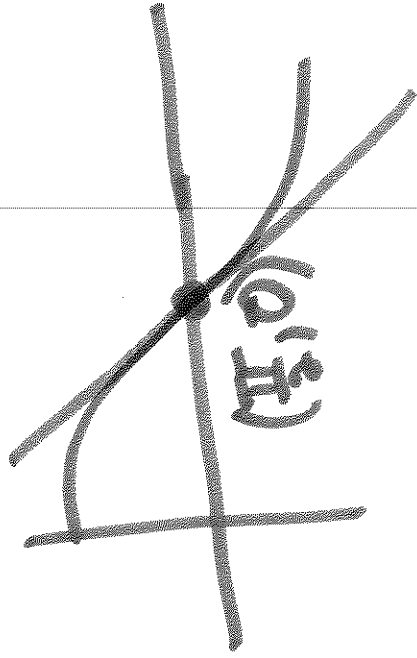
50% material since exam 3, 50% previous material.
Same format as usual.

Linear Approximation:

Idea: We can approximate $f(x)$ by its tangent line near a point $(a, f(a))$.

Ex: $f(x) = \cos(x)$, pt $(a, f(a)) = (\frac{\pi}{2}, 0)$.

$$f'(\frac{\pi}{2}) = -\sin(\frac{\pi}{2}) = -1.$$



pt-slope form of tangent line says

$$y - 0 = -1(x - \frac{\pi}{2})$$

$$\Rightarrow \text{tangent line is } y = \frac{\pi}{2} - x.$$

In general, tangent line has eqn
 $y - f(a) = f'(a)(x - a)$.
constant \rightarrow more to right

Defⁿ: The linear approximation of $f(x)$ at a is

$$f(x) \approx L(x) = f(a) + f'(a)(x-a)$$

Ex: Approximate $\sqrt{2}$ using $f(x) = \sqrt{1+x}$.

Note: $f(1) = \sqrt{2}$. I know $f(0) = \sqrt{1} = 1$.

Linear approx at $a=0$ is

tangent \rightarrow $L(x) = f(0) + f'(0)(x-0) = 1 + \frac{1}{2}x$.

$$\sqrt{2} = f(1) \approx L(1) = 1 + \frac{1}{2}(1) = \frac{3}{2}$$

Q: How crappy is this estimate?

Note:

$$f'(x) = \frac{1}{2\sqrt{1+x}}$$

by chain rule

Defⁿ: Given an approximation $f(x) \approx A$,

we define

$$\text{error} = f(x) - A$$

$$\text{and percentage error} = \left| \frac{f(x) - A}{f(x)} \right| \cdot 100.$$

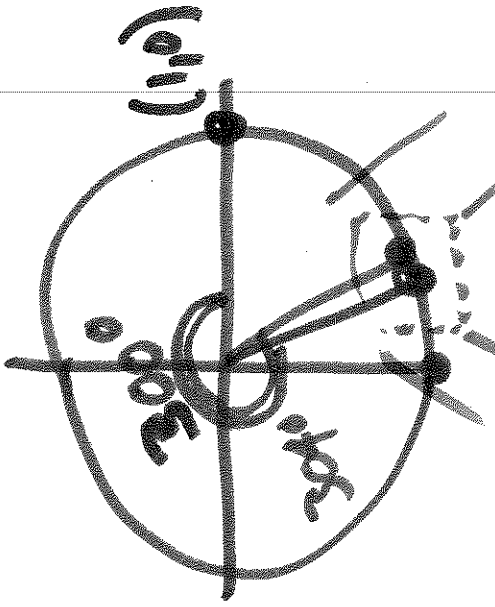
NOTE: in real world, we typically don't know $f(x)$. So, in later courses you will learn how to get around this.

For now, pretend calculator is an oracle that can evaluate all reasonable functions.

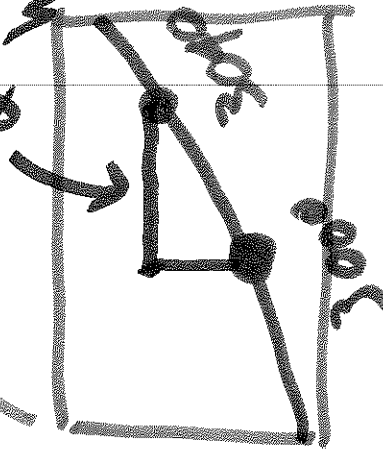
Ex: for $\sqrt{2}$, error is $\sqrt{2} - 1.5 = -0.0857864\dots$

$$\text{pct. error} = \left| \frac{\sqrt{2} - 1.5}{\sqrt{2}} \right| \cdot 100 = 6.066\dots\%$$

Ex:



Q: what is this
 reference
 angle?



~~Problem~~ Symbolic version of Q:
 what is $\cos(304^\circ) - \cos(300^\circ)$?

Settle for approximate answer...

Step 1: Switch to radians.

$$300^\circ = \frac{5\pi}{3} \text{ radians.}$$

$$304^\circ - 300^\circ = 4^\circ = \frac{\pi}{45} \text{ radians.}$$

$$\Rightarrow 304^\circ = \frac{5\pi}{3} + \frac{\pi}{45} \text{ radians.}$$

Why did I do this?

Because I know $\cos(300^\circ) = \cos(\frac{5\pi}{3} \text{ rad}) = \frac{1}{2}$.

Set $f(x) = \cos(x)$, use Lin. approx at $x = \frac{5\pi}{3}$ rad.

$$\cos(x) \approx \cos\left(\frac{5\pi}{3}\right) + \left(-\sin\left(\frac{5\pi}{3}\right)\right) \cdot \left(x - \frac{5\pi}{3}\right)$$

for x near $\frac{5\pi}{3}$

REF:
8

$$= \frac{1}{2} + \frac{\sqrt{3}}{2} \left(x - \frac{5\pi}{3}\right)$$

Plan: estimate $\cos(304^\circ) - \cos(300^\circ)$

by using Lin. approx at 304° , actual value at 300° .

$$\text{So, } \cos(304^\circ) - \cos(300^\circ) \approx \cos(300^\circ + 4^\circ) - \cos(300^\circ)$$

$$= \cos\left(\frac{5\pi}{3} + \frac{\pi}{45}\right) - \cos\left(\frac{5\pi}{3}\right) \approx$$

$$\left[\frac{1}{2} + \frac{\sqrt{3}}{2} \left(\frac{5\pi}{3} + \frac{\pi}{45} - \frac{5\pi}{3}\right)\right] - \frac{1}{2} = \left(\frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\pi}{45}\right) - \frac{1}{2}$$

$$= \frac{\sqrt{3}}{2} \cdot \frac{\pi}{45} \approx 0.06045 \dots$$

We can do better than Lin. approx!

The quadratic approximation to $f(x)$ at "a" is

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2.$$

Why?

$$\text{Suppose } f(x) \approx b + c(x-a) + d(x-a)^2$$

$$\text{near } a. \quad f(a) \approx b + c(a-a) + d(a-a)^2$$

If we plug in $x=a$,

$$= b.$$

$f'(x) \approx c + 2d(x-a) \leftarrow$ derivative of quadratic.

$$\Rightarrow f'(a) \approx c + 2d(a-a) = c.$$

Approx. of quadratic
for $p \ll \lambda \rightarrow p \ll \lambda \approx \frac{p^2}{2\lambda}$
So, $f''(a) \approx p \approx \frac{f''(a)}{2}$

MA 113 4/24/17

1 REEF today

2 Today we discuss the handout on higher-order approximations.

3 See Canvas announcement for assignments this week.

4 We said last class that the quadratic approx. to

$f(x)$ near a is

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2.$$

Q: How did we justify this?

→ Discuss w/ neighbors!

A: Assume $f(x) \approx b + c(x-a) + d(x-a)^2$.
repeatedly take derivatives of both sides + evaluate at a , we get these formulas.

Ex: Approximate $\sqrt{2}$ with $f(x) = \sqrt{1+x}$

using quad. approx. at $a=0$.

Need: $f(0)$, $f'(0)$, $f''(0)$.

$$f(0) = \sqrt{1} = 1, \quad f'(x) = \frac{1}{2\sqrt{1+x}} \Rightarrow f'(0) = \frac{1}{2}, \quad f''(x) = \frac{-1}{4(1+x)^{3/2}} \Rightarrow f''(0) = -\frac{1}{4}.$$

Near 0, $\sqrt{1+x} \approx f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2$

$$= 1 + \frac{1}{2}x + -\frac{1}{8}x^2.$$

Thus, $\sqrt{2} = \sqrt{1+1} \approx 1 + \frac{1}{2} - \frac{1}{8} = 1.375$.
 \uparrow
 $x=1$

What about higher degrees?

Suppose near a , we have

$$f(x) \approx b + c(x-a) + d(x-a)^2 + g(x-a)^3 + h(x-a)^4.$$

Idea: Take derivative, evaluate at a .

$$f(a) \approx \underbrace{b + c(a-a) + d(a-a)^2 + g(a-a)^3 + h(a-a)^4}_{=0}$$

$$= b. \quad \text{So, set } b = f(a).$$

$$f'(x) \approx c + 2d(x-a) + 3g(x-a)^2 + 4h(x-a)^3.$$

$$\Rightarrow f'(a) \approx c. \quad \text{So, set } c = f'(a).$$

$$f''(x) \approx 2d + 3 \cdot 2 \cdot g(x-a) + 4 \cdot 3 \cdot h(x-a)^2$$

$$\Rightarrow f''(a) \approx 2d. \quad \text{So, set } d = \frac{f''(a)}{2}.$$

$$f'''(x) \approx 3 \cdot 2 \cdot g + 4 \cdot 3 \cdot 2 \cdot h(x-a).$$

$$\Rightarrow f'''(a) \approx 3 \cdot 2 \cdot g. \quad \text{So, set } g = \frac{f'''(a)}{3 \cdot 2}.$$

$$f^{(n)}(x) \approx 4.3.2.h$$

$$\Rightarrow f^{(n)}(a) \approx 4.3.2.h. \text{ So, set } h = \frac{f^{(n)}(a)}{4.3.2}$$

we see a pattern....

Defⁿ: Let $f(x)$ be infinitely diff. on an open interval I , with $a \in I$. The n^{th} Taylor polynomial for $f(x)$ is

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n(n-1)\dots 3 \cdot 2}(x-a)^n$$

NOTE:

$$n! = n(n-1)\dots 2.$$

Ex: $f(x) = e^x$. Let's approximate e^x at $a=0$.

$$f'(x) = e^x, f''(x) = e^x, f^{(3)}(x) = e^x, \dots, f^{(n)}(x) = e^x.$$

So, I know $f^{(n)}(0) = e^0 = 1$ for any n .

So, at $a=0$, let's compute $T_6(x)$ for $f(x) = e^x$.

$$T_6(x) = 1 + 1 \cdot x + \frac{1}{2} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \frac{1}{5!} x^5 + \frac{1}{6!} x^6.$$

claim near 0, $T_6(x) \approx e^x$.

$$\text{So, } e^1 \approx T_6(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720}.$$

$$= \frac{1957}{720} \approx 2.718\overline{5}.$$

REF: 4

w/ramulpha $\Rightarrow e \approx 2.718281\dots$

Ex: If $f(x) = \sin x$, then at $a=0$,
 $f(0) = \sin 0 = 0$, $f'(0) = \cos 0 = 1$, $f''(0) = 0$, $f'''(0) = -1$,
 repeats etc.

$$\text{So, } T_{2n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \frac{x^{2n+1}}{(2n+1)!}$$

Thm: If $f(x)$ is:

- any polynomial
- e^x
- $\sin x$ or $\cos x$

Then $\lim_{n \rightarrow \infty} T_n(x) = f(x)$ for any choice of a and any x .

If $f(x) = (1+x)^k$ for a real k ,

then $\lim_{n \rightarrow \infty} T_n(x) = f(x)$ for $a=0$ and x in $(-1, 1]$.

Ex: NOTE: e^{-x^2} is e^x with $-x^2$ subbed in.

So, if $e^x \approx T_n(x)$, then $e^{-x^2} \approx T_n(-x^2)$.

Approximate $\int_{-1/a}^{1/a} e^{-x^2} dx$.

Use $e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!} + \frac{x^{10}}{10!} + \frac{x^{11}}{11!} + \frac{x^{12}}{12!} + \frac{x^{13}}{13!} + \frac{x^{14}}{14!} + \frac{x^{15}}{15!} + \frac{x^{16}}{16!} + \frac{x^{17}}{17!} + \frac{x^{18}}{18!} + \frac{x^{19}}{19!} + \frac{x^{20}}{20!} + \frac{x^{21}}{21!} + \frac{x^{22}}{22!} + \frac{x^{23}}{23!} + \frac{x^{24}}{24!} + \frac{x^{25}}{25!} + \frac{x^{26}}{26!} + \frac{x^{27}}{27!} + \frac{x^{28}}{28!} + \frac{x^{29}}{29!} + \frac{x^{30}}{30!} + \frac{x^{31}}{31!} + \frac{x^{32}}{32!} + \frac{x^{33}}{33!} + \frac{x^{34}}{34!} + \frac{x^{35}}{35!} + \frac{x^{36}}{36!} + \frac{x^{37}}{37!} + \frac{x^{38}}{38!} + \frac{x^{39}}{39!} + \frac{x^{40}}{40!} + \frac{x^{41}}{41!} + \frac{x^{42}}{42!} + \frac{x^{43}}{43!} + \frac{x^{44}}{44!} + \frac{x^{45}}{45!} + \frac{x^{46}}{46!} + \frac{x^{47}}{47!} + \frac{x^{48}}{48!} + \frac{x^{49}}{49!} + \frac{x^{50}}{50!} + \frac{x^{51}}{51!} + \frac{x^{52}}{52!} + \frac{x^{53}}{53!} + \frac{x^{54}}{54!} + \frac{x^{55}}{55!} + \frac{x^{56}}{56!} + \frac{x^{57}}{57!} + \frac{x^{58}}{58!} + \frac{x^{59}}{59!} + \frac{x^{60}}{60!} + \frac{x^{61}}{61!} + \frac{x^{62}}{62!} + \frac{x^{63}}{63!} + \frac{x^{64}}{64!} + \frac{x^{65}}{65!} + \frac{x^{66}}{66!} + \frac{x^{67}}{67!} + \frac{x^{68}}{68!} + \frac{x^{69}}{69!} + \frac{x^{70}}{70!} + \frac{x^{71}}{71!} + \frac{x^{72}}{72!} + \frac{x^{73}}{73!} + \frac{x^{74}}{74!} + \frac{x^{75}}{75!} + \frac{x^{76}}{76!} + \frac{x^{77}}{77!} + \frac{x^{78}}{78!} + \frac{x^{79}}{79!} + \frac{x^{80}}{80!} + \frac{x^{81}}{81!} + \frac{x^{82}}{82!} + \frac{x^{83}}{83!} + \frac{x^{84}}{84!} + \frac{x^{85}}{85!} + \frac{x^{86}}{86!} + \frac{x^{87}}{87!} + \frac{x^{88}}{88!} + \frac{x^{89}}{89!} + \frac{x^{90}}{90!} + \frac{x^{91}}{91!} + \frac{x^{92}}{92!} + \frac{x^{93}}{93!} + \frac{x^{94}}{94!} + \frac{x^{95}}{95!} + \frac{x^{96}}{96!} + \frac{x^{97}}{97!} + \frac{x^{98}}{98!} + \frac{x^{99}}{99!} + \frac{x^{100}}{100!} + \dots$

So, $e^{-x^2} \approx 1 + (-x^2) + \frac{(-x^2)^2}{2} + \frac{(-x^2)^3}{3!} + \frac{(-x^2)^4}{4!} + \dots$

$$= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \frac{x^{10}}{120} + \frac{x^{12}}{720} - \frac{x^{14}}{5040} + \frac{x^{16}}{30240} - \frac{x^{18}}{1632960} + \frac{x^{20}}{79833600} - \frac{x^{22}}{3527136000} + \frac{x^{24}}{154989312000} - \frac{x^{26}}{6026073600000} + \frac{x^{28}}{214748467200000} - \frac{x^{30}}{7526745600000000} + \frac{x^{32}}{263162041600000000} - \frac{x^{34}}{9072853440000000000} + \frac{x^{36}}{316910320000000000000} - \frac{x^{38}}{10992131200000000000000} + \frac{x^{40}}{376372736000000000000000} - \frac{x^{42}}{12978332800000000000000000} + \frac{x^{44}}{428322739200000000000000000} - \frac{x^{46}}{14296543296000000000000000000} + \frac{x^{48}}{457089383040000000000000000000} - \frac{x^{50}}{14530810416000000000000000000000} + \frac{x^{52}}{439722291648000000000000000000000} - \frac{x^{54}}{12992824186240000000000000000000000} + \frac{x^{56}}{359800677235200000000000000000000000} - \frac{x^{58}}{9874817608128000000000000000000000000} + \frac{x^{60}}{256744257811200000000000000000000000000} - \frac{x^{62}}{6418606445280000000000000000000000000000} + \frac{x^{64}}{157965161127040000000000000000000000000000} - \frac{x^{66}}{3796163867033600000000000000000000000000000} + \frac{x^{68}}{88507647831744000000000000000000000000000000} - \frac{x^{70}}{1955173882919040000000000000000000000000000000} + \frac{x^{72}}{41817641541300000000000000000000000000000000000} - \frac{x^{74}}{878165272267200000000000000000000000000000000000} + \frac{x^{76}}{1761132544534400000000000000000000000000000000000} - \frac{x^{78}}{33560508446256000000000000000000000000000000000000} + \frac{x^{80}}{610330160256000000000000000000000000000000000000000} - \frac{x^{82}}{10781942884608000000000000000000000000000000000000000} + \frac{x^{84}}{1870750119232000} - \frac{x^{86}}{307324019116800} + \frac{x^{88}}{4609840281728000} - \frac{x^{90}}{6463776394400} + \frac{x^{92}}{84632645273600} - \frac{x^{94}}{1046326452736000} + \frac{x^{96}}{124632645273600} - \frac{x^{98}}{1446326452736000} + \frac{x^{100}}{164632645273600}$$

$$\text{So, } \int_{-1/a}^{1/a} e^{-x^2} dx \approx \int_{-1/a}^{1/a} \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} \right) dx$$

$$= \frac{1785491}{1935360} \approx 0.9225627\dots$$

$$\text{Wolfram alpha } \Rightarrow \int_{-1/a}^{1/a} e^{-x^2} dx \approx 0.9225620128\dots$$