MA113 4/19/17

$\boxed{1}$ REEF today

$\boxed{2}$ Today is § 5.5 on the substitution method.

$\boxed{3}$ Discuss w/ your neighbors:

What is $\int x e^{x^2} dx$?

$\boxed{4}$ Please fill out your online course evaluations!!!
Currently, section [redacted] response rates are between 0% and 15%!!!

§5.5: Substitution, aka reversing the chain rule.

Note: $\int f(x)\,dx = F(x)+C$ means $F(x)$ is an antiderivative for $f(x)$.

We need ways to "undo" differentiation rules.

$\boxed{\text{NOTE: } \frac{d}{dx}(cf) = c\frac{d}{dx}f \qquad \int cf\,dx = c\int f\,dx}$

Today → (I) Chain Rule ⟶ Substitution

(II) Product Rule ⟶ Integration by Parts (Calc II).

Today's Q: If you hand me $f(x)$, how can I determine if $f(x)$ is the derivative obtained from $F(x)$ using chain rule?

Substitution: If $u=g(x)$ is diff, and if f is cts, then
$$\int f(g(x))\cdot g'(x)\,dx = \int \underbrace{f(u)\,du}.$$

Ex: $\displaystyle\int \overbrace{e^{x^2}}^{e^u}\cdot\overbrace{2x\,dx}^{du} = \int e^u\,du = e^u + C = e^{x^2}+C.$

If $u=x^2$, $\frac{du}{dx}=2x \Rightarrow du = 2x\,dx$

NOTE: $u=g(x)$
$\Rightarrow \frac{du}{dx}=g'(x)$
$\Rightarrow du = g'(x)\,dx.$

Ex: $\displaystyle\int xe^{x^2}\,dx = \int \frac{2}{2}xe^{x^2}\,dx = \frac{1}{2}\underbrace{\int 2xe^{x^2}\,dx}_{\text{Same integral as before.}}$

Observation: You will often need to introduce constants by multiplying + dividing.

Ex: $\int \sqrt{3x+2}\,dx = \frac{1}{3}\int \sqrt{3x+2}\cdot 3\,dx = \frac{1}{3}\int \sqrt{u}\cdot du = (*)$

mult. & divide by 3

$u = 3x+2$

$\frac{du}{dx} = 3 \Rightarrow du = 3\,dx.$

$(*) = \frac{1}{3}\int u^{1/2}\,du = \frac{1}{3}\frac{u^{3/2}}{3/2} + C = \frac{2}{9}(3x+2)^{3/2} + C.$

Step 1: change variable to u, identify du.

Step 2: Integrate w/ u.

Step 3: Change back to x.

Note:

$$\int x^a\,dx = \frac{x^{a+1}}{a+1} + C$$

unless $a = -1$.

Ex: $\displaystyle\int \tan(x)\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \underbrace{\frac{1}{\cos x}}_{\frac{1}{u}} \cdot \underbrace{-\sin x\,dx}_{du} = ⊛$

$u = \cos x \Rightarrow \dfrac{du}{dx} = -\sin x$

$\Rightarrow du = -\sin x\,dx.$

$$⊛ = -\int \frac{1}{u}\,du = -\ln|u| + C$$

$$= -\ln|\cos x| + C.$$

$$= \ln|\cos x|^{-1} + C$$

$$= \ln|\sec x| + C.$$

__Rmk__: usually, in books,

$\int \tan x\,dx = \ln|\sec x| + C.$

Ex: $\int \sqrt{1+x^2} \cdot x^5\,dx = \frac{1}{2}\int \underbrace{\sqrt{1+x^2}}_{\sqrt{u}} \cdot \underbrace{x^2}_{(u-1)} \cdot \underbrace{x^2}_{(u-1)} \underbrace{2x\,dx}_{} = \circledast$

close to du... mult & div by 2 to fix it.

$u = 1+x^2. \Rightarrow u-1 = x^2.$

$du = 2x\,dx.$

$$\circledast = \frac{1}{2}\int \sqrt{u} \cdot (u-1)(u-1) \cdot du$$

do algebra

$$= \frac{1}{2}\int \left(u^{5/2} - 2u^{3/2} + u^{1/2}\right) du$$

$$= \frac{1}{2}\left[\frac{u^{7/2}}{7/2} - 2\frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2}\right] + C$$

$$= \frac{1}{2}\left(\frac{(1+x^2)^{7/2}}{7/2} - 2\frac{(1+x^2)^{5/2}}{5/2} + \frac{(1+x^2)^{3/2}}{3/2}\right) + C.$$

See book for examples.

Rule for definite integrals: $u = g(x)$

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Ex: $$\int_0^3 e^{5x}\,dx = \frac{1}{5}\int_0^3 \underbrace{e^{5x}}_{e^u} \cdot \underbrace{5dx}_{du} = \frac{1}{5}\int_{\underset{5\cdot 0}{\overset{\parallel}{0}}}^{\overset{3\cdot 5}{\underset{\parallel}{15}}} e^u\,du = ⊛$$

change bounds too!

$u = 5x$, $du = 5dx$

$$⊛\ \frac{1}{5}\left[e^u\right]_0^{15} = \frac{e^{15} - e^0}{5}.$$

<u>Ex</u>: $\displaystyle\int_{\pi}^{2\pi} \sin(\pi x+1)\,dx = \frac{1}{\pi}\int_{\pi^2+1}^{2\pi^2+1} \sin(u)\,du = \left[\frac{-\cos u}{\pi}\right]_{\pi^2+1}^{2\pi^2+1}$

$u = \pi x+1,\quad du = \pi\,dx$

$$= \frac{-\cos(2\pi^2+1) + \cos(\pi^2+1)}{\pi}$$

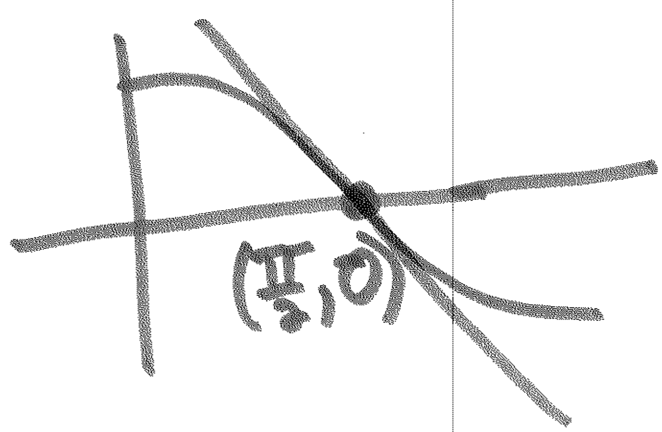
MA 113   4/21/17

1. REEF today
2. See Canvas announcement about REEF scores.
3. Today + Monday: §3.10, Linear Approximations and Handout on Higher-Order Approximations (See website for MA 113 for handout.)
4. Discuss with your neighbors: What is the equation for the tangent line to the graph of $f(x)$ at $x=a$?
5. Final Exam: 8:30 PM – 10:30 PM, Monday May 1.
   50% material since exam 3, 50% previous material.
   Same format as usual.

Linear Approximation:

Idea: We can approximate $f(x)$ by its tangent line near a point $(a, f(a))$.

Ex: $f(x) = \cos(x)$, pt $(a, f(a)) = \left(\frac{\pi}{2}, 0\right)$.



$f'\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1$.

pt-slope form of tangent line says

$y - 0 = -1\left(x - \frac{\pi}{2}\right)$

$\Rightarrow$ tangent line is $y = \frac{\pi}{2} - x$.

In general, tangent line has eqn

$y - f(a) = f'(a)(x-a)$.

rename as $L(x)$ / move to right

Def$^{\underline{n}}$: The <u>linear approximation</u> of $f(x)$ at a is

$$f(x) \approx L(x) = f(a) + f'(a)(x-a).$$

<u>Ex</u>: Approximate $\sqrt{2}$ using $f(x) = \sqrt{1+x}$.

<u>Note</u>: $f(1) = \sqrt{2}$. I know $f(0) = \sqrt{1} = 1$.

Linear approx at $a=0$ is

tangent to $f(x)$ at 0 $\rightarrow$ $L(x) = f(0) + f'(0)(x-0) = 1 + \frac{1}{2}x.$

NOTE: $f'(x) = \frac{1}{2\sqrt{1+x}}$ by chain rule.

$$\sqrt{2} = f(1) \approx L(1) = 1 + \tfrac{1}{2}(1) = \tfrac{3}{2}.$$

<u>Q</u>: How crappy is this estimate?

Def$^{\underline{n}}$: Given an approximation $f(x) \approx A$, we define

$$\text{error} = f(x) - A$$

and

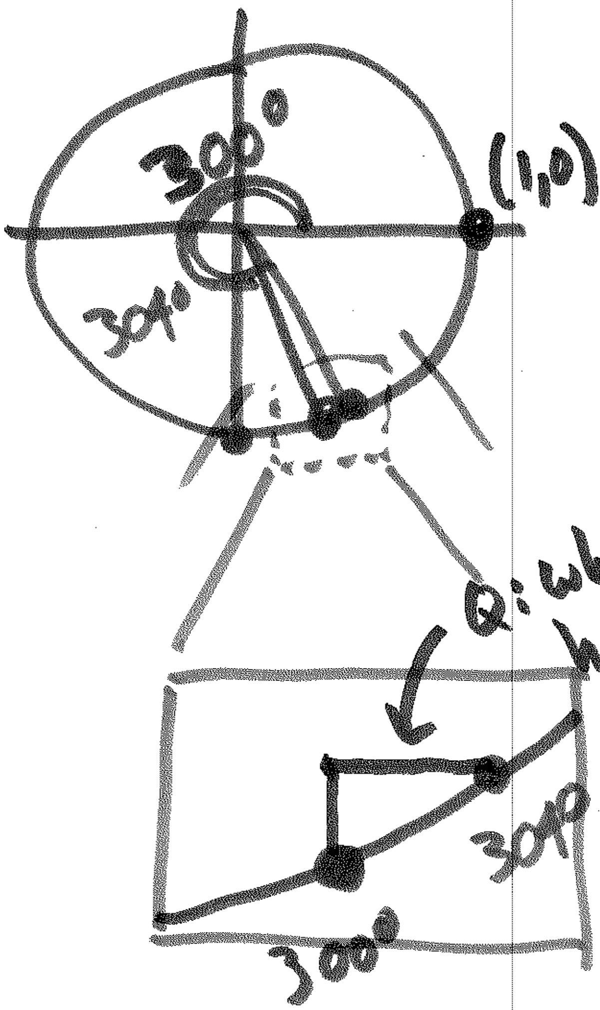
$$\text{percentage error} = \left| \frac{f(x)-A}{f(x)} \right| \cdot 100.$$

NOTE: in real world, we typically don't know $f(x)$. So, in later courses you will learn how to get around this. For now, pretend calculator is an oracle that can evaluate all reasonable functions.

Ex: for $\sqrt{2}$, error is $\sqrt{2} - 1.5 = -0.0857864\ldots$

$$\text{perct. error} = \left| \frac{\sqrt{2} - 1.5}{\sqrt{2}} \right| \cdot 100 = 6.066\ldots\,\%.$$

Ex:



Symbolic version of Q: what is $\cos(304^\circ) - \cos(300^\circ)$?

Settle for approximate answer....

Step 1: Switch to radians.

$$300^\circ = \frac{5\pi}{3} \text{ radians.}$$

$$304^\circ - 300^\circ = 4^\circ = \frac{\pi}{45} \text{ radians.}$$

$$\Rightarrow 304^\circ = \frac{5\pi}{3} + \frac{\pi}{45} \text{ radians.}$$

why did I do this?

Because I know $\cos(300^\circ) = \cos\left(\frac{5\pi}{3} \text{ rad}\right) = \frac{1}{2}$.

Set $f(x) = \cos(x)$, use Lin. approx at $x = \frac{5\pi}{3}$ rad.

$$\cos(x) \approx \cos\left(\frac{5\pi}{3}\right) + \left(-\sin\left(\frac{5\pi}{3}\right)\right) \cdot \left(x - \frac{5\pi}{3}\right)$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{5\pi}{3}\right).$$ for x near $\frac{5\pi}{3}$ rad...

$\boxed{\text{REEF: } 8}$

<u>Plan:</u> estimate $\cos(304°) - \cos(300°)$ by using Lin. approx at 304°, actual value at 300°.

So, $\cos(304°) - \cos(300°) = \cos(300° + 4°) - \cos(300°)$

$$= \cos\left(\frac{5\pi}{3} + \frac{\pi}{45}\right) - \cos\left(\frac{5\pi}{3}\right) \approx$$

$$\left[\frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{5\pi}{3} + \frac{\pi}{45} - \frac{5\pi}{3}\right)\right] - \frac{1}{2} = \left(\frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\pi}{45}\right) - \frac{1}{2}$$

$$= \frac{\sqrt{3}}{2} \cdot \frac{\pi}{45} \approx 0.06045....$$

We can do better than Lin. approx!

The <u>quadratic</u> approximation to $f(x)$ at "a" is

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2.$$

<u>Why?</u> Suppose $f(x) \approx b + c(x-a) + d(x-a)^2$

near a.

If we plug in $x=a$, $f(a) \approx b + c(\overset{0}{a-a}) + d(\overset{0}{a-a})^2$
$= b.$

$f'(x) \approx c + 2d(x-a) \leftarrow$ derivative of quadratic.

$\Rightarrow f'(a) \approx c + 2d(a-a) = c.$

$f''(x) \approx 2d \leftarrow 2^{nd}$ der. of quadratic

So, $f''(a) \approx 2d \Rightarrow d \approx \frac{f''(a)}{2}$.

MA 113 4/24/17

1. REEF today
2. Today we discuss the handout on higher-order approximations.
3. See Canvas announcement for assignments this week.
4. We said last class that the quadratic approx. to $f(x)$ near a is

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2.$$

Q: How did we justify this?

$\rightarrow$ Discuss w/ neighbors!

A: Assume $f(x) \approx b + c(x-a) + d(x-a)^2$. repeatedly take derivatives of both sides + evaluate at a, we get these formulas.

__Ex:__ Approximate $\sqrt{2}$ with $f(x) = \sqrt{1+x}$ using quad. approx. at $a = 0$.

__Need:__ $f(0)$, $f'(0)$, $f''(0)$.

$f(0) = \sqrt{1} = 1$, $f'(x) = \dfrac{1}{2\sqrt{1+x}} \Rightarrow f'(0) = \dfrac{1}{2}$. $f''(x) = \dfrac{-1}{4(1+x)^{3/2}}$

$\Rightarrow f''(0) = \dfrac{-1}{4}$.

Near 0,

$$\sqrt{1+x} \approx f(0) + f'(0)\cdot(x-0) + \frac{f''(0)}{2}(x-0)^2$$

$$= 1 + \frac{1}{2}\cdot x + \frac{-1}{8}\cdot x^2.$$

Thus, $\sqrt{2} = \underset{\uparrow\, x=1}{\sqrt{1+1}} \approx 1 + \frac{1}{2} - \frac{1}{8} = 1.375$.

What about higher degree?
Suppose near a, we have

$$f(x) \approx b + c(x-a) + d(x-a)^2 + g(x-a)^3 + h(x-a)^4.$$

__Idea:__ Take derivatives, evaluate at a.

$$f(a) \approx b + \underbrace{c(a-a) + d(a-a)^2 + g(a-a)^3 + h(a-a)^4}_{=0}$$

$$= b. \quad \text{So, set } b = f(a).$$

$$f'(x) \approx c + 2d(x-a) + 3g(x-a)^2 + 4h(x-a)^3.$$

$$\Rightarrow f'(a) \approx c. \quad \text{So, set } c = f'(a).$$

$$f''(x) \approx 2d + 3\cdot 2\cdot g(x-a) + 4\cdot 3\cdot h(x-a)^2$$

$$\Rightarrow f''(a) \approx 2d. \quad \text{So, set } d = \frac{f''(a)}{2}.$$

$$f'''(x) \approx 3\cdot 2\cdot g + 4\cdot 3\cdot 2\cdot h(x-a).$$

$$\Rightarrow f'''(a) \approx 3\cdot 2\cdot g. \quad \text{So, set } g = \frac{f'''(a)}{3\cdot 2}.$$

$$f^{(4)}(x) \approx 4\cdot 3\cdot 2\cdot h$$

$\Rightarrow f^{(4)}(a) \approx 4\cdot 3\cdot 2\cdot h$. So, set $h = \dfrac{f^{(4)}(a)}{4\cdot 3\cdot 2}$.

We see a pattern!....

Def$^{\underline{n}}$: Let $f(x)$ be infinitely diff. on an open interval I, with a in I. The <u>$n^{\underline{th}}$ Taylor polynomial</u> for $f(x)$ is

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3\cdot 2}(x-a)^3 + \cdots\cdots + \frac{f^{(n)}(a)}{n(n-1)\cdots 3\cdot 2}(x-a)^n.$$

$$\boxed{\begin{array}{l}\underline{\text{NOTE:}}\\ n! = n(n-1)\cdots 2.\end{array}}$$

<u>Ex</u>: $f(x) = e^x$. Let's approximate e^x at $a=0$.

$f'(x) = e^x$, $f''(x) = e^x$, $f'''(x) = e^x$, ..., $f^{(n)}(x) = e^x$.

So, I know $f^{(n)}(0) = e^0 = 1$, for any n.

So, at $a=0$, Let's compute $T_6(x)$ for $f(x) = e^x$.

$$T_6(x) = 1 + 1 \cdot x + \frac{1}{2}x^2 + \frac{1}{3 \cdot 2}x^3 + \frac{1}{4 \cdot 3 \cdot 2}x^4 + \frac{1}{5 \cdot 4 \cdot 3 \cdot 2}x^5 + \frac{1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}x^6.$$

<u>claim</u>: near 0, $T_6(x) \approx e^x$.

So, $e = e^1 \approx T_6(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720}$.

$$= \frac{1957}{720} \approx 2.7180\overline{5}.$$

wolframalpha $\Rightarrow$ $e \approx 2.718281....$

REEF: 4

Ex: If $f(x) = \sin x$, then at $a = 0$,

$f(0) = \sin 0 = 0$, $f'(0) = \cos 0 = 1$, $f''(0) = 0$, $f'''(0) = -1$, repeats etc.

So, $$T_{2n+1}(x) = x - \frac{x^3}{3\cdot 2} + \frac{x^5}{5\cdot 4\cdot 3\cdot 2} - \frac{x^7}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} + \cdots \pm \frac{x^{2n+1}}{(2n+1)!}.$$

Thm: If $f(x)$ is:

- any polynomial
- e^x
- $\sin x$ or $\cos x$

Then $\lim_{n\to\infty} T_n(x) = f(x)$ for any choice of a and any x.

If $f(x) = (1+x)^k$ for a real # k,

Then $\lim_{n\to\infty} T_n(x) = f(x)$ for $a=0$ and x in $(-1, 1]$.

<u>Ex</u>: <u>NOTE</u>: e^{-x^2} is e^x with $-x^2$ subbed in.

So, if $e^x \approx T_n(x)$, Then $e^{-x^2} \approx T_n(-x^2)$.

Approximate $\int_{-1/2}^{1/2} e^{-x^2}\,dx$.

Use $e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{3\cdot 2} + \frac{x^4}{4\cdot 3\cdot 2}$.

So, $e^{-x^2} \approx 1 + (-x^2) + \frac{(-x^2)^2}{2} + \frac{(-x^2)^3}{3\cdot 2} + \frac{(-x^2)^4}{4\cdot 3\cdot 2}$

$= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24}$.

So, $\int_{-1/2}^{1/2} e^{-x^2}\,dx \approx \int_{-1/2}^{1/2}\left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24}\right)dx$

$$= \frac{1785491}{1935360} \approx 0.9225627\ldots$$

Wolframalpha $\Rightarrow \int_{-1/2}^{1/2} e^{-x^2}\,dx \approx 0.9225620128\ldots$