4/3/17 MA113

1. REEF today
2. See Canvas announcement for assignments
3. Today: §5.1, Areas + Distances

Reminder: Exam 3 is Tues, April 11. Cover all material since Exam 2, see course calendar for specifics.

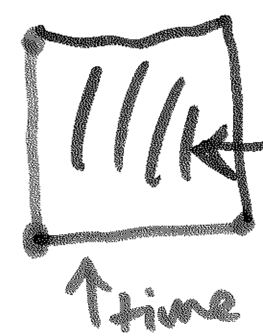
__Distance problem:__ Given a known velocity function, find distance traveled in a certain time interval. Our first goal is to __approximate__ the total dist. traveled.

__Ex:__ If $v(t) = 6$ m/s, then from $t = 5$ s to $t = 7$ s, object moves $\underline{12}$ m.

$6 \cdot \underbrace{(7-5)}_{\text{time}}$

Because w/ constant velocity,

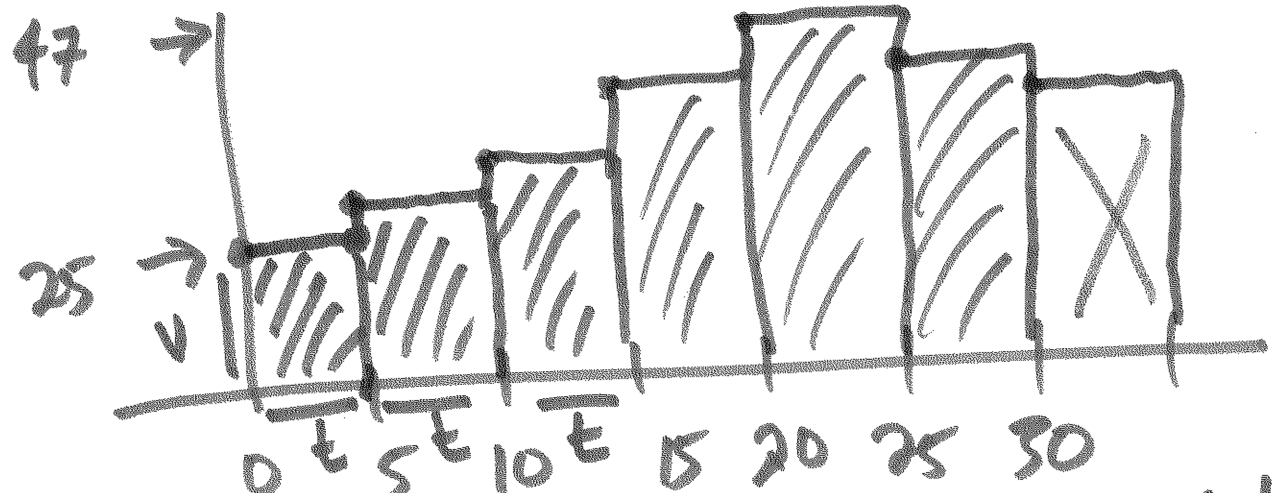
$$\boxed{\text{distance} = \text{velocity} \cdot \text{time}}$$



distance = area.

<u>Ex:</u>

time (s)	0	5	10	15	20	25	30
vel (m/s)	25	31	35	43	47	45	41



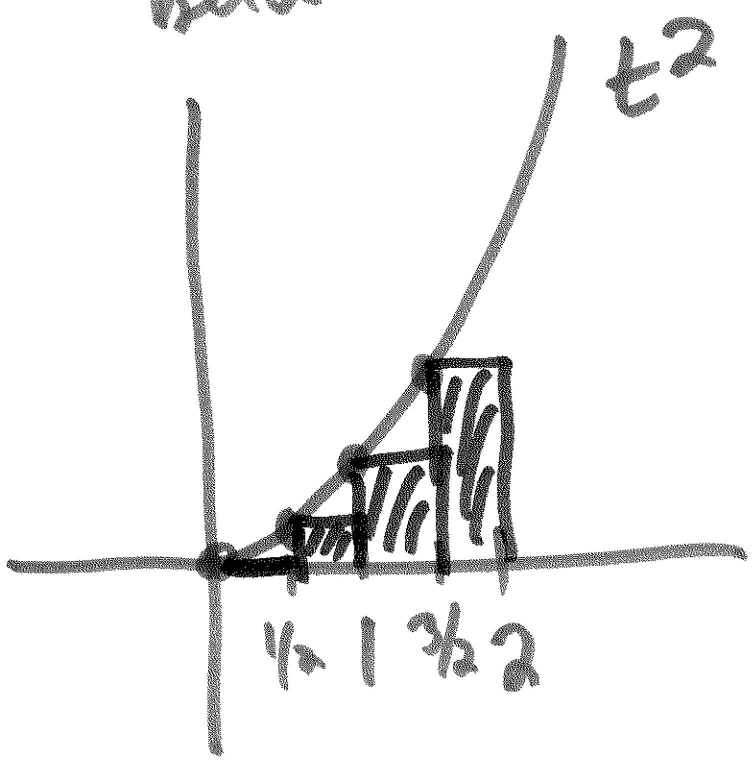
Estimate total dist. traveled by adding up (vel · time) at each point, assume const. velocity on intervals

So, estimate tot. dist. traveled between $t=0, t=30$

w/ $\left[25(5-0) + 31(10-5) + 35(15-10) + 43(20-15) + 47(25-20) + 45(30-25)\right]$ m.

Ex: You model object in motion, in model velocity is $v(t) = t^2$ m/s.

Estimate total distance traveled between $t = 0$ s and $t = 2$ s.



Sample some points & do what we did before.

Measure velocity at 0 sec, $\frac{1}{2}$ s, 1 s, 1.5 sec.

$$\left[0\cdot\left(\tfrac{1}{2}-0\right) + \left(\tfrac{1}{2}\right)^2\left(1-\tfrac{1}{2}\right) + 1^2(1.5-1) + \left(\tfrac{3}{2}\right)^2(2-1.5)\right] \text{m}$$

= estimate of dist. traveled.

We <u>chose</u>: <u>1</u>: length of time to sample at, i.e. number of sample points.

<u>2</u>: sample points are upper left corner of rectangles.

<u>Q</u>: what happens when we make different choices?
→ desmos...

Let's write this precisely: we will frequently use "sigma" notation:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{\underset{\uparrow\, 2^7}{128}}.$$

$$= \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7}$$

All that matters is

1. pattern
2. where to start, where to end.

"Add $\frac{1}{2^n}$ for n starting at 0, ending at 7."

$$\Rightarrow \sum_{n=0}^{7} \frac{1}{2^n}.$$

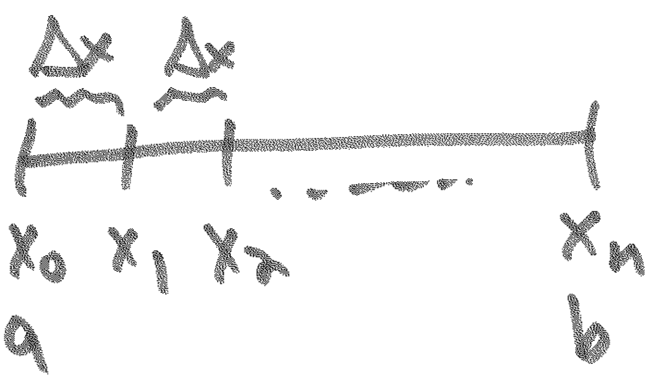
"Σ" means sum, lower value is start point, upper value is end pt, pattern is next to Σ.

name of index can be different, eg.

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n.$$

Given a cts fn $f(x)$ on $[a,b]$, break interval into n equal pieces of length

$$\Delta x = \frac{b-a}{n}.$$



$x_0 = a$

$x_1 = a + \Delta x$

$x_2 = a + \Delta x + \Delta x = a + 2\Delta x$

$x_3 = a + 3\Delta x$

$\vdots$

$x_n = a + n\Delta x = a + n\left(\frac{b-a}{n}\right) = a + b - a = b.$

<u>Def$^{\underline{n}}$/Thm</u>: The <u>area</u> A of the region S that lies between x-axis and the graph of $f(x)$ on $[a,b]$, w/ $f(x)$ is cts, is

$$A = \lim_{n\to\infty} \left[f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x \right]$$

$$= \lim_{n\to\infty} \underbrace{\sum_{i=0}^{n-1} f(x_i)\Delta x}_{\text{Call this } L_n \text{ for left endpt.}}$$

You can also compute

$$A = \lim_{n\to\infty} \left[f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x \right]$$

$$= \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} f(x_i)\Delta x}_{\text{Call this } R_n \text{ for right endpt.}}.$$

Note: See Ex 3 in §5.1 for midpoint version, M_n.

REEF. 2

4/5/17 MA 113

1 REEF today

2 §5.2 today

3 Reminder: Exam 3 next week! Tuesday.

4 Thursday (tomorrow) evening ~~[illegible]~~
a review session @ The study.
(Canvas announcement later today)

Ex: Find area between x-axis and $y=x^2$ on $[0,t]$ using R_n, i.e. right-hand sums.

Goal: $A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left(f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x\right)$

$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x.$

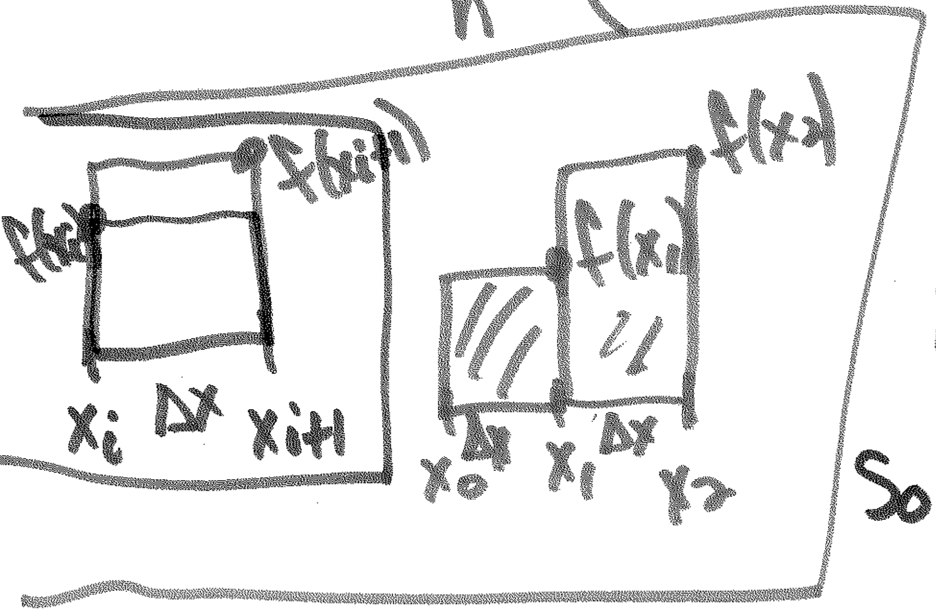
$\Delta x = \frac{t-0}{n} = \frac{t}{n}.$

$x_0 = 0,\ x_1 = 0+\Delta x = 0+\frac{t}{n} = \frac{t}{n},\ x_2 = 0+2\Delta x = \frac{2t}{n},$

$x_3 = 0+3\Delta x = \frac{3t}{n}, \dots$ etc $x_i = \frac{it}{n}.$

$$R_n = \underbrace{\left(\frac{t}{n}\right)^2}_{(x_1)^2} \cdot \underbrace{\frac{t}{n}}_{\Delta x} + \underbrace{\left(\frac{2t}{n}\right)^2}_{(x_2)^2} \cdot \underbrace{\frac{t}{n}}_{\Delta x} + \underbrace{\left(\frac{3t}{n}\right)^2}_{(x_3)^2} \cdot \underbrace{\frac{t}{n}}_{\Delta x} + \dots + \underbrace{\left(\frac{nt}{n}\right)^2}_{(x_n)^2} \cdot \underbrace{\frac{t}{n}}_{\Delta x}$$

$$= \frac{t^3}{n^3}\left(1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2\right).$$



<u>Fact ("High School Gap"):</u>

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

So, $R_n = \frac{t^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$.

So, $\text{Area} = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{t^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$

$$= t^3 \cdot \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} = t^3 \cdot \frac{2}{6} = \frac{t^3}{3}.$$

to the limit...

__Notes:__ Read def^n 2 in §5.2 for full def^n of definite integral; short version:

Given a function $f(x)$ on $[a,b]$, write

$$\int_a^b \underbrace{f(x)}_{\text{integrand}}\,\underbrace{dx}_{\text{differential}} = \lim_{n\to\infty} R_n \left[= \lim_{n\to\infty} L_n = \lim_{n\to\infty} M_n\right]$$

if the limit exists.

integral

bounds or limits of integration

integrand

differential.

This is called the __definite integral__ of $f(x)$ on $[a,b]$.

__Def^n:__ $\sum_{i=1}^{n} f(x_i)\Delta x$ is an example of a __Riemann Sum.__

Thm: $\int_a^b f(x)\,dx$ exists if

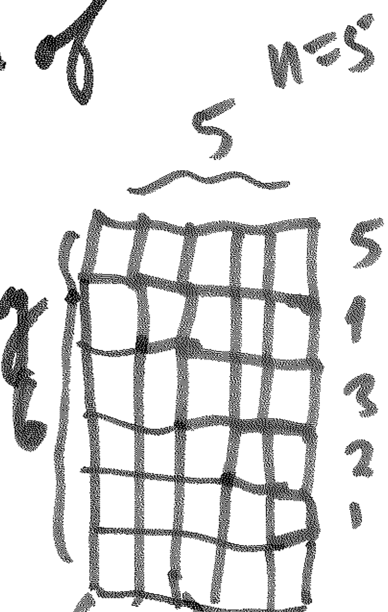
- f is cts on $[a,b]$

or

- f has a finite number of jump discontinuities.

These are okay

Ex:



Useful tools:

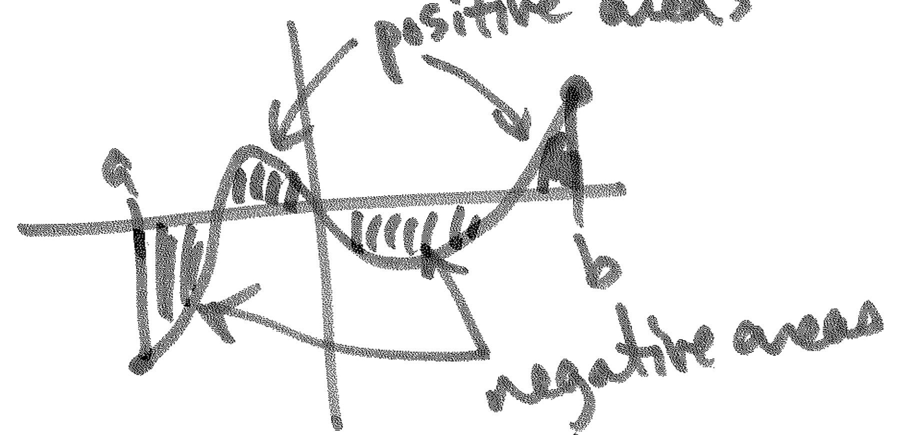
- $1+2+3+4+\cdots+n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $1^2+2^2+3^2+\cdots+n^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- $1^3+2^3+3^3+\cdots+n^3 = \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$.

REEF: 6

NOTE: Read §5.2 for more complicated summation examples.

Comment: Areas are "signed".

Ex:



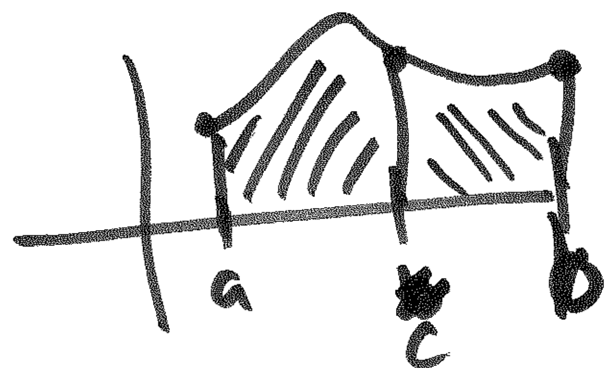
Properties of integrals: (A) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(B) $\int_a^a f(x)\,dx = 0$.

(C) $\int_a^b c\cdot f(x) \pm g(x)\,dx = c\int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$.

__Ex:__ $\int_0^3 x^3 - 6x\,dx = \int_0^3 x^3\,dx - 6\int_0^3 x\,dx$

(D) $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$



(E) If $f(x) \geq 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq 0$.

__NOTE:__ There are a few more less-used but important properties in §5.2, so read the entire section!