MA 113 2/10/17

1. Exam average for all lectures: 69.4%.
   → No curve on Exam 1.

2. Webwork due tonight.

3. Log in to REEF.

4. Note: As we head into "calculus" material, remember to only use what we have done in this class. If you have seen calc before, don't use techniques until we see them!

§2.8: Derivative as a function.

Recall: $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$.

Letting "a" vary, we get a new function f'!

We usually write

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

Ex: If $f(x)=\sqrt{x}$, then $f'(x) = \frac{1}{2\sqrt{x}}$.

why? We must use a limit!

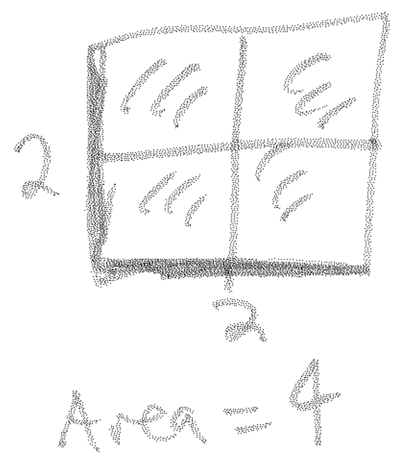
$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \underset{\text{Mult by } \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}}{=} \lim_{h\to 0} \frac{\cancel{x}+h-\cancel{x}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

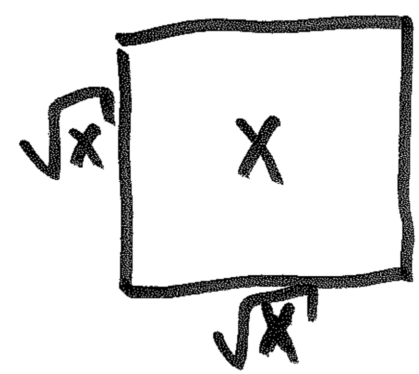
<u>NOTE</u>: $f'(x) = \frac{1}{2\sqrt{x}}$ is a fn w/ domain $(0,\infty)$.

<u>Aside</u>: what is $\sqrt{x}$?

Side length is what?

We can't always easily measure side length, so we give it a name: $\sqrt{x}$.





<u>Similar</u>:

Cube  volume $= x$, side length is "s" with $s \cdot s \cdot s = x$.

Call $s = \sqrt[3]{x}$

NOTATION: If $y = f(x)$, we write any of:

$f'(x)$, $\frac{df}{dx}$, y', $\frac{dy}{dx}$, $\frac{d}{dx} f(x)$, $Df(x)$, $D_x f(x)$.

These all mean the derivative of $f(x)$.

Def^n: A function f is <u>differentiable</u> <u>at "a"</u> if $f'(a)$ exists. $f(x)$ is <u>differentiable</u> on an interval if $f'(x)$ exists for every value in that interval.

REF: 3

Ex: $f(x) = |x^3 - 1|$ is not diff at $x = 1$.

"diff" = differentiable.

$\underline{Q}$: How do we $\underline{\text{show}}$ that $|x^3-1|$ is not diff at $x=1$?

$\rightarrow$ graphing is not enough.

$\rightarrow$ you need to compare right- + left-hand limits in definition of derivative.

derivative at 1 from right ||

$$\lim_{h\to 0^+} \frac{|(1+h)^3-1| - |1^3-1|}{h} = \lim_{h\to 0^+} \frac{|\cancel{1}+3h+3h^2+h^3-\cancel{1}| - |\cancel{1-1}|}{h}$$

$$\lim_{h\to 0^+} \frac{3h+3h^2+h^3}{h} = \lim_{h\to 0^+} 3+\cancel{3h}^{\to 0}+\cancel{h^2}^{\to 0} = \boxed{3}$$

$\underline{\underline{\text{However!}}}$ $$\lim_{h\to 0^-} \frac{|(1+h)^3-1| - |1^3-1|}{h} = \lim_{h\to 0^-} \frac{|3h+3h^2+h^3|}{h}$$

same alg.

$$= \lim_{h\to 0^-} \frac{-3h-3h^2-h^3}{h} = \boxed{-3}$$

not equal!! So lim DNE.

Q: why is $\underbrace{3h+3h^2+h^3}_{=\,(1+h)^3-1\,=\,1+3h+3h^2+h^3-1}$ negative when $h<0$?

$1+h < 1$ if $h<0$ and h close to 0.

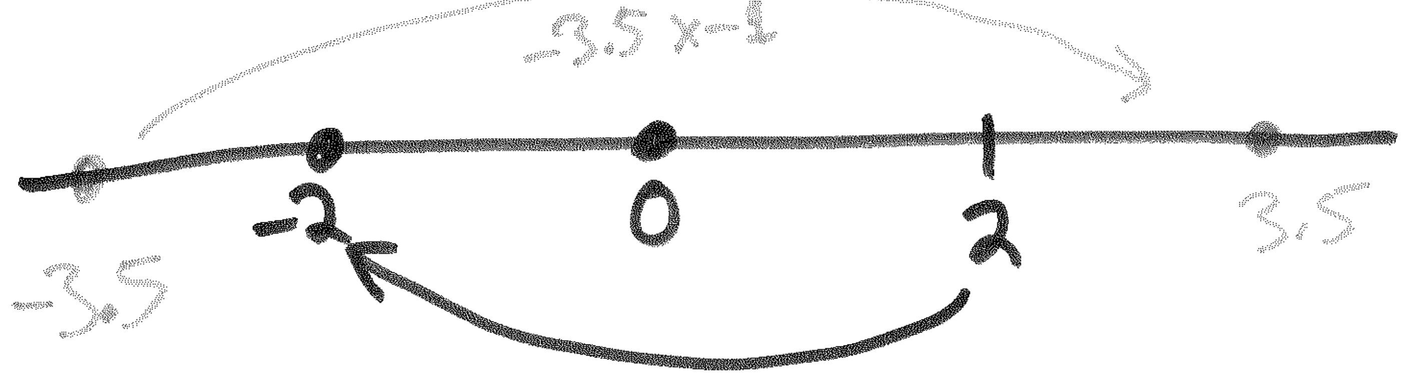
eg. $1+(-0.1)=0.9<1$.

So, $(1+h)^3 < 1^3 = 1$.

Thus, $(1+h)^3 - 1 < 0$.

Review: what is $|x|$?

1: what is mult by -1?

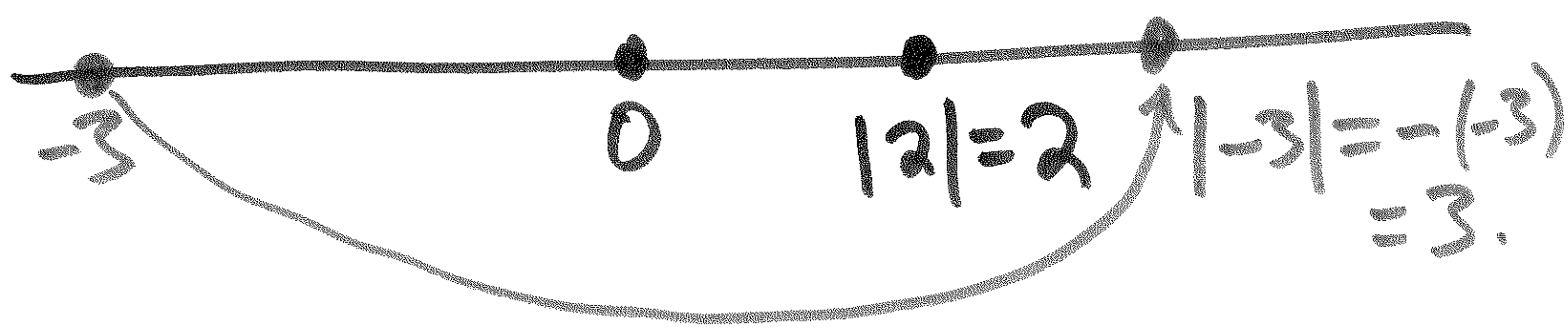


$2\times(-1)$

Multiplying by -1 "flips" the real line.

2: What the def$^{\text{n}}$ of $|x|$?

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



$$|(1+h)^3-1| = -[(1+h)^3-1] = -3h-3h^2-h^3.$$

if $(1+h)^3-1<0$

Read in §2.8 the definitions of

$$f''(x),\ f'''(x),\ f^{(4)}(x),\ \ldots$$

i.e. higher derivatives.

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$\boxed{1}$ Log in to REEF.

② See Canvas announcement for assigments

<u>this week:</u> Webwork B2, B3

Written Assignment #3

Quiz #4

$\boxed{3}$ <u>Theorem:</u> If f is differentiable, then f is continuous.

<u>With neighbors:</u> Discuss why you think this is true.

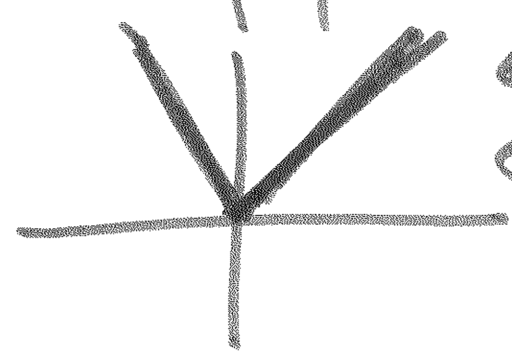
<u>Today:</u> § 3.1, derivatives of polynomials + exponentials.

Note about Thm: Since f differentiable $\Rightarrow$ "zooming in" makes f look like a straight line, we have that f has unbroken graph, hence is cts.

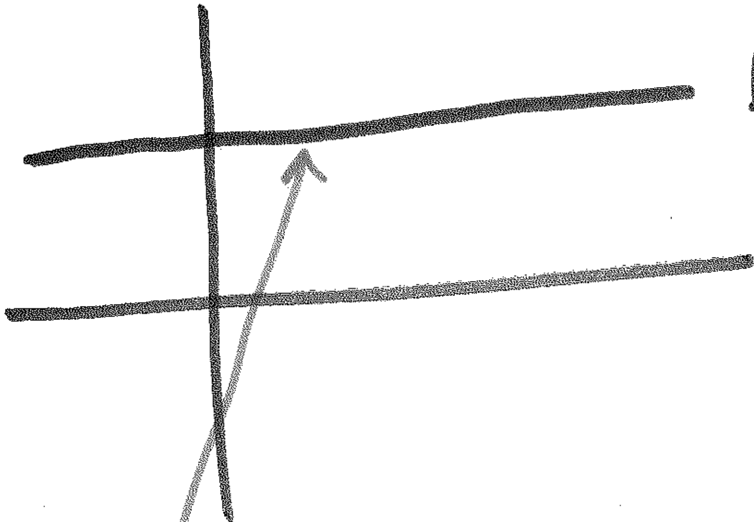
NOTE: See §2.8 for a mathematical proof of this thm.

Note: Cts does not imply diff.

Ex: $y = |x|$.



§3.1: **Q:** what is $\frac{d}{dx}$ of a constant?



$f(x)=C$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{c-c}{h} =$$

$$\lim_{h\to 0} \frac{0}{h} = 0.$$

Thm: $\frac{d}{dx}(c) = 0$ for any constant c.

NOTE: $f(x)=C$ is its own tangent line. So, slope of tangent line to graph of f is 0. This agrees w/ our algebra to the right.

Q: Set $f(x)=x$. What is $f'(x)$?

Algebra:

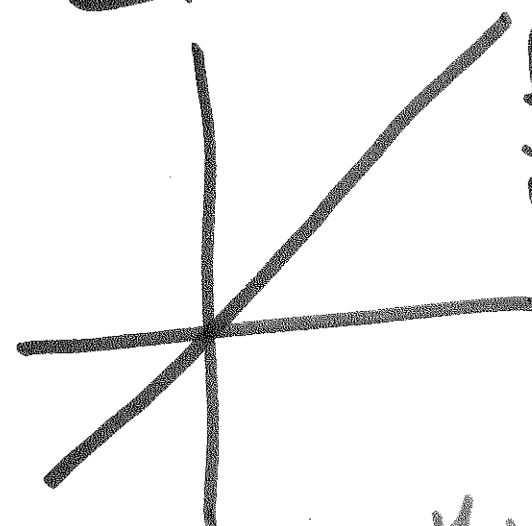
$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{x}+h-\cancel{x}}{h}$$

$$=\lim_{h\to 0}\frac{h}{h}=\lim_{h\to 0}1=1.$$

So, $f'(x)=1$ if $f(x)=x$.

Graph of $f(x)=x$:



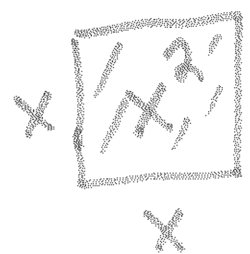
$f(x)=x$ is a straight line w/ slope 1.

Again, in this case the graph of $f(x)$ <u>is</u> the tangent line to any point on the graph. Since $f'(x)=$ slope of tangent line to f at x, we have $f'(x)=1$.

Q: what is $f'(x)$ when $f(x) = x^2$?

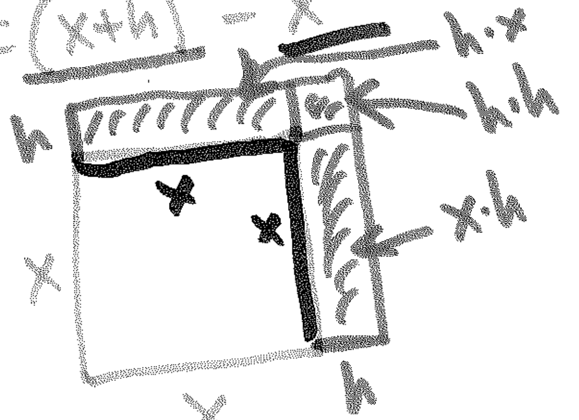
Physical:

x^2 = area of $x \times x$ square.



$f(x+h) - f(x)$

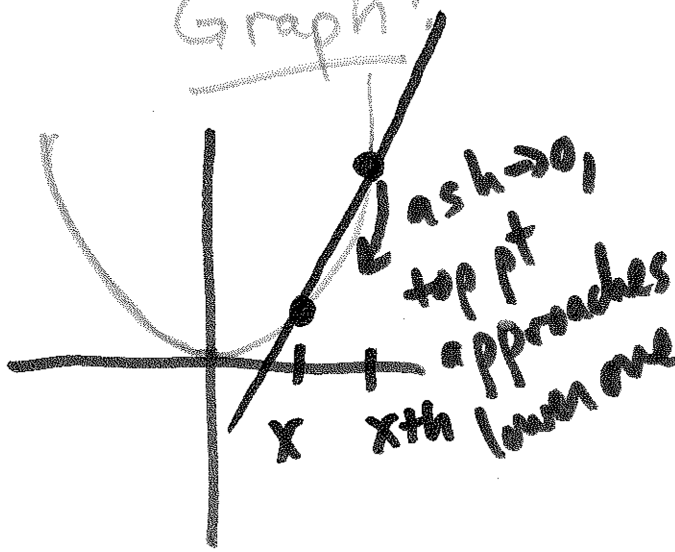
$= (x+h)^2 - x^2$



$x+h$

Total square has area $(x+h)^2$

Graph:



It is not clear how to proceed using only the graph.....

Algebra:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{2x\cancel{h} + h^{\cancel{2}}}{\cancel{h}}$$

$$= \lim_{h \to 0} 2x + h$$

$$= 2x.$$

$$= x + x$$

Power Rule: For any real # n,

$$\frac{d}{dx} x^n = n \cdot x^{n-1}.$$

e.g. $\frac{d}{dx} x^2 = 2x$, $\frac{d}{dx} x^3 = 3x^2$

$\frac{d}{dx} x^4 = 4x^3$, ...

$$\frac{d}{dx} \sqrt[3]{x} = \frac{d}{dx} x^{1/3} = \frac{1}{3} x^{\frac{1}{3}-1} = \frac{1}{3} x^{-\frac{2}{3}}$$

For proof: See §3.1.

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__Exponentials:__ Say $b>0$, $b\neq 1$.

Then set $f(x)=b^x$.

$$f'(x)=\lim_{h\to 0}\frac{b^{x+h}-b^x}{h}=\lim_{h\to 0}\frac{b^x(b^h-1)}{h}$$

$$=b^x\cdot\underbrace{\lim_{h\to 0}\frac{b^h-1}{h}}_{\substack{\text{Exercise: this is}\\ f'(0)\text{ when }f(x)=b^x.}}=b^x\cdot f'(0)=f(x)\cdot f'(0).$$

Thus, __Thm:__ If $f(x)=b^x$, then $f'(x)=f'(0)\cdot f(x)$.

It would be great if $f'(0) = 1$.

<u>Thm</u>: There is a number b satisfying

$$\lim_{h \to 0} \frac{b^h - 1}{h} = 1.$$

We call that number "e".

Thus, $\frac{d}{dx} e^x = e^x$.

<u>Properties of $\frac{d}{dx}$</u>:

- $\frac{d}{dx}(c \cdot f(x)) = c \frac{d}{dx} f(x)$ for any constant.
- $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$.

Ex: $\frac{d}{dx}\left(3e^x + x^4 - x^{1/2} + 1\right)$

$$= \frac{d}{dx}3e^x + \frac{d}{dx}x^4 - \frac{d}{dx}x^{1/2} + \frac{d}{dx}1$$

$$= 3\frac{d}{dx}e^x + \frac{d}{dx}x^4 - \frac{d}{dx}x^{1/2} + \frac{d}{dx}1$$

$$= 3e^x + 4x^3 - \frac{1}{2}x^{-1/2} + 0$$

$$= 3e^x + 4x^3 - \frac{1}{2\sqrt{x}}.$$

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$\boxed{1}$ Log in to REEF.

$\boxed{2}$ See Canvas Announcement from earlier this week for assignments reminder.

$\boxed{3}$ Discuss w/ your neighbors:
Find derivative of $4x^3 + 3x^{1/3} + e^x$

$\boxed{4}$ What is the derivative of $x \cdot e^x$?
Do you have a guess?

Review: #3 above:

$$\frac{d}{dx}\left(4x^3+3x^{1/3}+e^x\right)=$$

$$\frac{d}{dx}\left(4x^3\right)+\frac{d}{dx}\left(3x^{1/3}\right)+\frac{d}{dx}\left(e^x\right)=$$

$$4\frac{d}{dx}\left(x^3\right)+3\frac{d}{dx}\left(x^{1/3}\right)+\frac{d}{dx}\left(e^x\right)=$$

$$4\cdot 3x^2+3\cdot\frac{1}{3}x^{-2/3}+e^x=$$

$$12x^2+x^{-2/3}+e^x.$$

__Q:__ What about e^{2x}? What is its derivative?

__2 reasonable responses:__

__1:__ $e^{2x} = e^{x+x} = \underline{e^x \cdot e^x}$

__New Q:__ How to take $\frac{d}{dx}$ of a product?

$\Rightarrow$ Product Rule.

__2:__ Composition $f \circ g$: $g(x) = 2x$, $f(x) = e^x$.

$x \longmapsto 2x \longmapsto e^{2x}$

__New Q:__ How to take $\frac{d}{dx}$ of a composition $f \circ g$?

$\Rightarrow$ Chain rule.

Recall: we found if $f(x) = b^x$,

then $\frac{df}{dx}(x) = f'(0) \cdot f(x) = f'(0) \cdot b^x$.

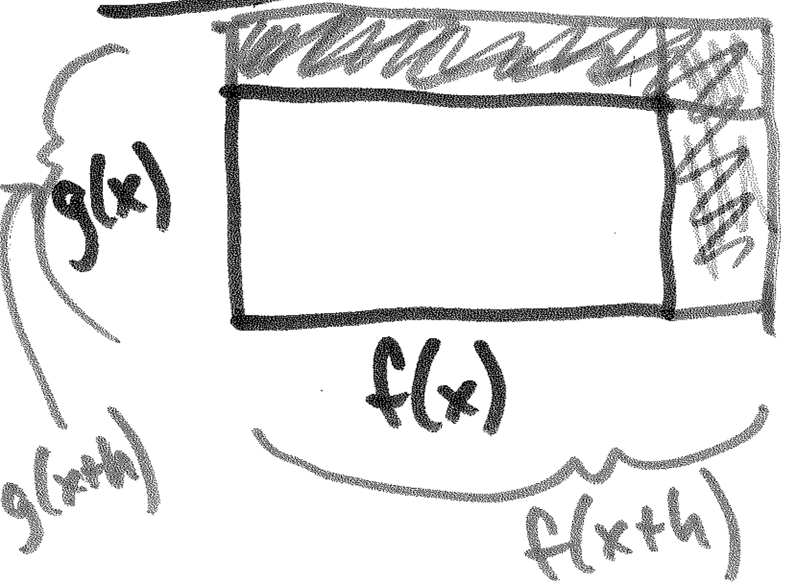
we then **manufactured** a number "e" with $f'(x) = e^x$ where $f(x) = e^x$.

i.e. we **defined** e to be the # w/ this property.

Product Rule: $\frac{d}{dx}(f \cdot g) = \frac{df}{dx} \cdot g + f \cdot \frac{dg}{dx}$

$= f' \cdot g + f \cdot g'$.

Picture:



Imagine f, g are both increasing:

$$\frac{d}{dx}(f \cdot g) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \frac{\text{orange area}}{h}$$

$$= \lim_{h \to 0} \frac{(g(x+h)-g(x)) \cdot f(x) + (f(x+h)-f(x)) \cdot g(x) + (g(x+h)-g(x))(f(x+h)-f(x))}{h} \quad (*)$$

NOTE: two grey numerators are equal if you expand the 2^{nd} one + cancel terms.

$$(*) = \left(\lim_{h \to 0} \frac{(g(x+h)-g(x))}{h}\right) \cdot f(x) + \left(\lim_{h \to 0} \frac{(f(x+h)-f(x))}{h}\right) g(x) + \lim_{h \to 0} \frac{(g(x+h)-g(x)) \cdot (f(x+h)-f(x))}{h}$$

$$= g'(x) \cdot f(x) + f'(x)\, g(x) + 0$$

product Rule.

$$x^4 = \underbrace{x}_{f} \cdot \underbrace{x^3}_{g}$$

<u>Ex</u>: • $\frac{d}{dx} x^n = \frac{d}{dx}\left(x \cdot x^{n-1}\right)$

n a positive integer

$$= \frac{d}{dx}(x) \cdot x^{n-1} + x \cdot \frac{d}{dx}\left(x^{n-1}\right)$$

$$= 1 \cdot x^{n-1} + x \cdot (n-1)x^{n-2}$$

$$= 1 \cdot x^{n-1} + (n-1)x^{n-1}$$

$$= \left(1 + (n-1)\right)x^{n-1} = n x^{n-1}.$$

• $\frac{d}{dx}\left(\underbrace{(x^3+x)}_{f}\underbrace{e^x}_{g}\right) = \left[\frac{d}{dx}(x^3+x)\right]\underline{e^x} + (x^3+x)\underline{\frac{d}{dx}e^x}$

$$= (3x^2+1)e^x + (x^3+x)e^x.$$

<u>Ex</u>: $\frac{d}{dx}\left(\frac{e^x}{\sqrt{x}}\right) = \frac{d}{dx}\left(\frac{e^x}{x^{1/2}}\right) = \frac{d}{dx}\left(e^x \cdot x^{-1/2}\right)$

$= \frac{d}{dx}(e^x) \cdot x^{-1/2} + e^x \cdot \frac{d}{dx}\left(x^{-1/2}\right)$

$= e^x \cdot x^{-1/2} + e^x \cdot -\frac{1}{2} \cdot x^{-3/2}$

$= \frac{e^x}{\sqrt{x}} + \frac{-e^x}{2\sqrt{x^3}}.$

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<u>Quotient Rule:</u> $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g \cdot \frac{d}{dx}f - f\frac{d}{dx}g}{g^2}$

$$= \frac{g \cdot f' - f \cdot g'}{g^2}.$$

How I do $\frac{f}{g}$:

$$\frac{f}{g} = \underset{\text{product}}{f \cdot} \underbrace{(g)^{-1}}_{\text{composition}}$$

So, use product rule
+ chain rule
to do this.