MA 113 2/17/17

0. Webwork due tonight.
1. Log in to REEF.
2. Turn in your written assignment at front table.
3. <u>Today</u>: §3.3, Derivatives of Trig functions.
4. <u>Q</u>: Discuss w/ neighbors: What is the angle addition formula for $\sin(x+h)$ (and $\cos(x+h)$)? Do you remember, a few weeks ago, when we discussed the proof of these?

Thm: $\frac{d}{dx}\sin x = \cos x$

$\frac{d}{dx}\cos x = -\sin x$

You need to know derivatives of other trig fns, see §3.3.

Examples of this Thm in action:

Ex: Set $f(x) = \sin(x)$.

$f'(x) = \cos(x)$.

$f''(x) = -\sin(x)$.

$f'''(x) = -\cos(x)$

$f^{(4)}(x) = \sin(x)$

after 4 derivatives, back where we started....

Ex: what is: $f^{(13)}(x) = \cos(x)$, since $f^{(12)}(x) = \sin x$.

Ex: Find $f'(x)$ for $f(x) = \underbrace{e^x}_{g} \cdot \underbrace{(\sin x) \cdot x^2}_{h}$

$$f'(x) = g' \cdot h + h' \cdot g = \frac{d}{dx}(e^x) \cdot (\sin x) \cdot x^2 + \frac{d}{dx}\left(\underbrace{\sin x}_{m} \cdot \underbrace{x^2}_{n}\right) \cdot e^x$$

$\frac{d}{dx} m \cdot n = m' \cdot n + n' m$

$$= \frac{d}{dx}(e^x) \cdot \sin x \cdot x^2 + \left[\frac{d}{dx}(\sin x) \cdot x^2 + \sin x \cdot \frac{d}{dx}(x^2)\right] \cdot e^x$$

$$= e^x \cdot \sin x \cdot x^2 + \left[\cos x \cdot x^2 + \sin x \cdot 2x\right] e^x$$

$$= e^x \cdot \sin x \cdot x^2 + e^x \cdot \cos x \cdot x^2 + e^x \cdot \sin x \cdot 2x$$

$$= x \cdot e^x \left(\sin x \cdot x + \cos x \cdot x + 2\sin x\right).$$

Focus on $\sin x$ (cos x is similar, see textbook):

$$\frac{d}{dx}\sin x = \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h}$$

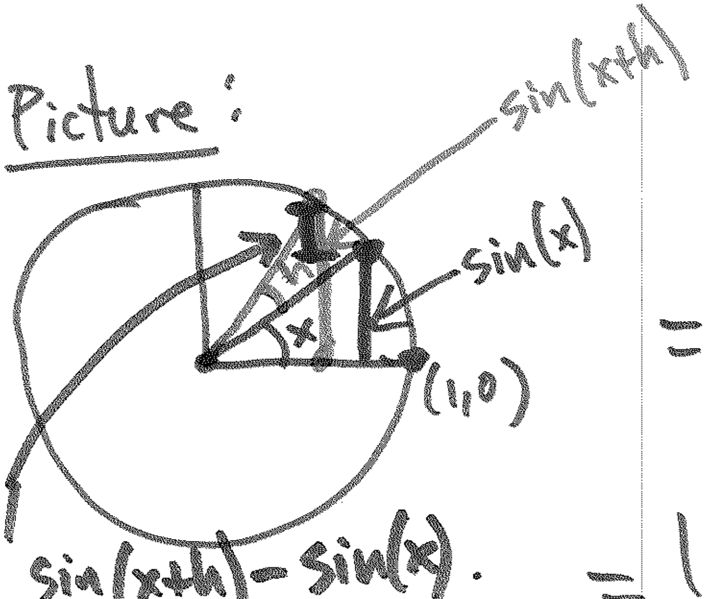
$$= \lim_{h\to 0} \frac{\overbrace{\sin x\cdot\cos h + \cos x\cdot \sin h}^{\text{by angle addition formula.}} - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x\cdot\sin h}{h}$$

$$= \lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0}\frac{\cos x\cdot \sin h}{h}$$

$$= \sin x \cdot \lim_{h\to 0}\frac{\cos h - 1}{h} + \cos x \lim_{h\to 0}\frac{\sin h}{h} = (*)$$

Picture:



$\boxed{\text{Goal is } \frac{d}{dx}\sin x = \cos x}$

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If $\lim_{h\to 0} \frac{\sin h}{h} = 1$

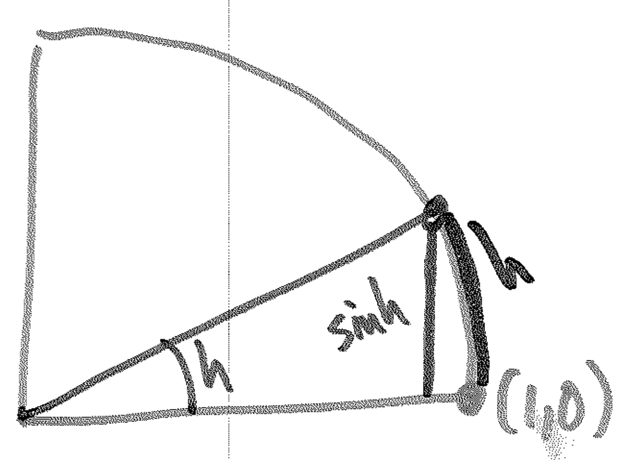
and $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$,

then our proof is complete, and

$(*) = \sin x \cdot 0 + \cos x \cdot 1 = \cos x.$

<u>Thm</u>: $\bullet \underbrace{\lim_{h\to 0} \frac{\sin h}{h} = 1}_{\text{tough! see book, §3.3}}$, $\bullet \underbrace{\lim_{h\to 0} \frac{\cos h - 1}{h} = 0}_{\text{see book, proof uses}}$

<u>Idea</u>:



$\frac{\sin h}{h} = \frac{\text{line length}}{\text{arc length}}$.

As $h \to 0$, these lengths differ by less + less.

Ex: Find $\lim_{x\to 0} \frac{\sin(9x)}{5x} = \lim_{9x\to 0} \frac{\sin(9x)}{5x}$

Use: $\boxed{\lim_{\heartsuit\to 0} \frac{\sin(\heartsuit)}{\heartsuit} = 1}$

$= \lim_{9x\to 0} \frac{9}{9}\cdot\frac{\sin(9x)}{5x}$

$= \frac{9}{5}\cdot \underbrace{\boxed{\lim_{9x\to 0} \frac{\sin(9x)}{9x}}}_{=1}$

$= \frac{9}{5}\cdot 1 = \frac{9}{5}.$

Ex: $\lim_{x\to 0} \frac{\sin^2(3x)}{x^2} = \lim_{x\to 0} \frac{\sin(3x)\cdot\sin(3x)}{x\cdot x} =$

$\lim_{x\to 0}\left(\frac{\sin 3x}{x}\right)\cdot\lim_{x\to 0}\left(\frac{\sin 3x}{x}\right) = 3\cdot 3 = 9$

multiply each of these by $\frac{3}{3}$

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1 Log in to REEF

2 See Canvas for assignments this week:

Announcement → Webwork B4, B5, B6

→ Quiz #5

→ Written Assignment #4 (now on website)

3 Graph $f(x) = e^{x^2}$. What does slope of tangent lines look like? Where positive, where negative?

TODAY: § 3.4, Chain Rule

NOTE: e^{x^2} is a composition: $x \longmapsto x^2 \longmapsto e^{x^2}$

This is what chain rule deals with.

Thm (Chain Rule): If f and g are both differentiable (see §3.4 for a precise statement), then $(f \circ g)' = f'(g(x)) \cdot g'(x)$.

Equivalently, If $y = f(u)$, $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Idea: $f \circ g(x)$ means $x \longmapsto g(x) \longmapsto \underset{\text{"outside"}}{\underbrace{f}}(\underbrace{g(x)}_{\text{"inside" fn}})$.

Ex: $\cancel{\ } \; e^{x^2}$. $\quad x \longmapsto x^2 \longmapsto e^{x^2}$

$g'(x) = 2x$ $\quad \underbrace{g(x) = x^2}_{\text{inside}}$ $\qquad \underbrace{f(x) = e^x}_{\text{outside}}$. $\quad f'(x) = e^x$

Chain rule: $(f \circ g)' = f'(g(x)) \cdot g'(x)$

$$= e^{x^2} \cdot 2x = 2x \cdot e^{x^2}$$

same.

w/ Leibniz notation "$\frac{d}{dx}$" : $y = e^u$, $u = x^2$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot 2x = e^{x^2} \cdot 2x$$

<u>Ex</u>: Let's take $\frac{d}{dx}(x^6) = 6x^5$

chain rule <u>should</u> give us same answer.

$f(x) = x^3$, $g(x) = x^2$.

$f'(x) = 3x^2$

$g'(x) = 2x$

$f \circ g(x) = (x^2)^3 = x^6$.

$$\boxed{\text{NOTE: } x^{2n} = (x^2)^n}$$

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x) = 3 \cdot (x^2)^2 \cdot 2x$$

$$= 3x^4 \cdot 2x = \underline{6x^5} .$$

same answer, which is a good check on method.

__Ex:__ $F(x) = \sqrt[3]{x^4+x}$. Find $F'(x)$.

__Step 1:__ Identify f, g where $F = f \circ g$.

"inside" $\longrightarrow g(x) = x^4 + x$

"outside" $\longrightarrow f(x) = x^{1/3}$

$g'(x) = 4x^3 + 1, \quad f'(x) = \frac{1}{3}x^{-2/3}$

So, $F' = (f \circ g)' = f'(g(x)) \cdot g'(x)$

$= \frac{1}{3} \cdot (x^4+x)^{-2/3} \cdot (4x^3+1)$.

$= \dfrac{4x^3+1}{3 \cdot \sqrt[3]{(x^4+x)^2}}$

__NOTE:__ If you have a composition w/ more than 2 functions, always best to use "$\frac{d}{dx}$" notation.

__Ex:__ Find $\frac{d}{dx}\left(e^{\cos(x^2+1)}\right)$

$$x \longmapsto x^2+1 \longmapsto \cos(x^2+1) \longmapsto e^{\cos(x^2+1)}$$

$y = e^{\cos(x^2+1)} = f \circ g \circ h$ where

$$f(v) = e^v, \quad \underset{v}{\underset{\parallel}{g(u)}} = \cos(u), \quad \underset{u}{\underset{\parallel}{h(x)}} = x^2+1.$$

$$\frac{dy}{dx} = \frac{df}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx}$$

$$= e^v \cdot (-\sin(u)) \cdot 2x = e^v \cdot (-\sin(x^2+1)) \cdot 2x$$

$$= e^{\cos(x^2+1)} \cdot (-\sin(x^2+1)) \cdot 2x$$

$$= -2x \sin(x^2+1) \cdot e^{\cos(x^2+1)}.$$

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__Ex__: __Quotient Rule__:

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{d}{dx}\left(\underline{f} \cdot \underline{(g)^{-1}}\right) = \frac{df}{dx} \cdot (g)^{-1} + f \cdot \frac{d}{dx}\left(\overbrace{(g)^{-1}}^{\text{chain rule!}}\right) =$$

Product rule

outside fn is u^{-1}

$$= \frac{df}{dx} \cdot (g)^{-1} + f \cdot -1(g)^{-2} \cdot \frac{dg}{dx}$$

$$= \frac{\frac{df}{dx} \cdot g}{g \cdot g} + \frac{-f \cdot \frac{dg}{dx}}{g^2} = \frac{\frac{df}{dx} \cdot g - \frac{dg}{dx} \cdot f}{g^2}.$$

Thm: $\frac{d}{dx}\left(b^x\right) = \ln(b)\cdot b^x$.

Why? Write $b^x = e^{\ln(b^x)} = e^{x\ln(b)}$.

$$x \longmapsto x\cdot\ln(b) \longmapsto e^{x\ln(b)}$$

By chain rule, $\frac{d}{dx}\left(b^x\right) = \frac{d}{dx}\left(e^{x\cdot\ln(b)}\right) = e^{x\cdot\ln(b)}\cdot\ln(b)$

$= b^x\cdot\ln(b)$.

Ex: Suppose $F(x) = (x^4+1)^{10}$. Find $F'(x)$.

Choice 1: Expand + use power rule.

Choice 2: use chain rule. $F'(x) = \underbrace{10\left(x^4+1\right)^9}_{\text{outside dn.}}\cdot\overbrace{\left(4x^3\right)}^{\text{inside dn}}$.