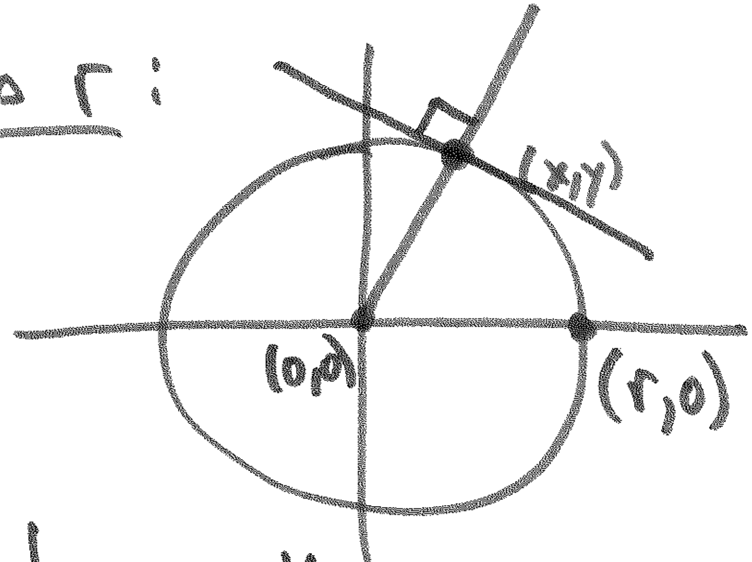
MA 113 2/22/2017

1. Get Logged in to REEF
2. See Canvas announcements for assignments this week.
3. **Today**: §3.5, Implicit Differentiation.
4. With your neighbors: what is the slope of tangent line to the point (x, y) on a circle of radius r centered at the origin?

Circle of radius r:



Before calc was used, assumed that tangent line to circle is $\perp$ to radial line.

Observation 1:
Slope of radial line is $\frac{y}{x}$.

Observation 2: If a line is $\perp$ to radial line, it has slope $-\frac{x}{y}$.

This was the argument before calc was discovered.

Issue: Consider $x^4+y^4=1$.
This is similar to the unit circle, which has equation $x^2+y^2=1$.

To find slope of tangent line for more complicated examples like this, we need to

1. treat y as a fn of x near a fixed y value.
2. Use chain rule when taking derivative of expressions involving y.

Use technique on $x^2+y^2=1$ to demonstrate:

Apply $\frac{d}{dx}$ to both sides of this eqn:

$$\frac{d}{dx}\left(x^2+y^2\right) = \frac{d}{dx}(1)$$

$$\Rightarrow \underbrace{\frac{d}{dx}\left(x^2\right)}_{\text{power rule}} + \underbrace{\frac{d}{dx}\left(y^2\right)}_{\text{chain rule}} = \underbrace{\frac{d}{dx}(1)}_{\text{constant}}$$

$$\Rightarrow 2x + 2y \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \text{solve for } \frac{dy}{dx} = -\frac{x}{y}.$$

This is slope of tangent line to (x,y) on $x^2+y^2=1$.

Q: why chain rule?

$x^2+y^2=1$.

- If $y \geq 0$, write $y=\sqrt{1-x^2}$
- If $y<0$, write $y=-\sqrt{1-x^2}$.

$$x \xmapsto{fn} \sqrt{1-x^2} \xmapsto{fn???} y$$

$y \geq 0$

confusing..... not quite right...

should be

$$x \mapsto 1-x^2 \mapsto \sqrt{1-x^2}.$$

only for $y \geq 0$.... what about $y<0$?

none of this "looks like" $\frac{d}{dx}(y^2)$ hence confusion.

To deal w/ this, do <u>not</u> try to actually write y as a fn of x explicitly. too many cases ($y \geq 0$, $y < 0$, etc) and very messy.

Picture we use: $x \longmapsto y(x) \longmapsto (y(x))^2$.

$\boxed{\text{fn } f(x),\ \frac{df}{dx} \text{ means } f'(x).}$

$\frac{d}{dx}(y^2) = 2 \cdot y \cdot \frac{dy}{dx}$ by chain rule

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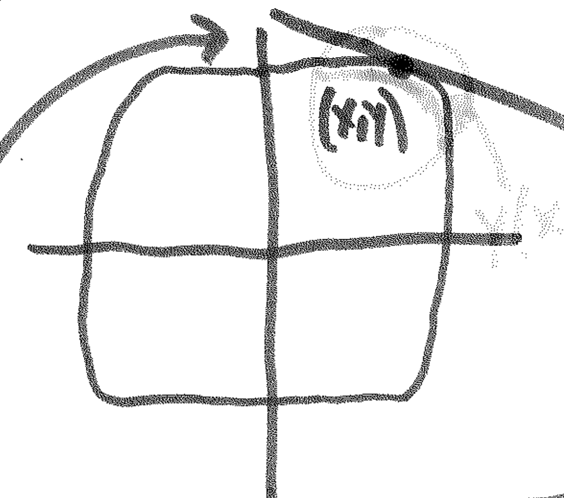
Ex: Find slope of tangent line to (x,y) on curve $x^4 + y^4 = 16$.

Note: Near the point (x,y) for fixed x, y values, y is a fn of x. So, we can apply $\frac{d}{dx}$ to both sides of our eqn.

$$\frac{d}{dx}\left(x^4+y^4\right)=\frac{d}{dx}(16)$$

$$\Rightarrow \quad \frac{d}{dx}\left(x^4\right)+\frac{d}{dx}\left(y^4\right)=\frac{d}{dx}(16)$$

$x^4+y^4=1.$



$$\Rightarrow \quad 4x^3+4y^3\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \quad \text{solve for } \frac{dy}{dx}=\frac{-x^3}{y^3}.$$

Ex: Find y' if $\cos(x+y)=x\cdot\sin(y)$.

Apply $\frac{d}{dx}$ to both sides:

$\frac{d}{dx}\cos(x+y) = -\sin(x+y)\cdot\frac{d}{dx}(x+y)$

chain rule

prod rule

$$\frac{d}{dx}\cos(x+y)=\frac{d}{dx}\left(x\cdot\sin(y)\right)$$

chain rule

$$\Rightarrow \quad -\sin(x+y)\cdot\left(1+\frac{dy}{dx}\right)=\frac{d}{dx}(x)\cdot\sin(y)+x\cdot\frac{d}{dx}\left(\sin(y)\right)$$

$$\Rightarrow -\sin(x+y)\cdot\left(1+\frac{dy}{dx}\right) = \sin(y) + x\cdot\cos(y)\cdot\frac{dy}{dx}$$

$\Rightarrow$ solve for $\frac{dy}{dx}$ $\Rightarrow$ $-\sin(x+y)\cdot\frac{dy}{dx} - x\cos(y)\cdot\frac{dy}{dx}$

$$= \sin(y) + \sin(x+y)$$

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$$\Rightarrow \frac{dy}{dx}\left(-\sin(x+y) - x\cos y\right) = \sin y + \sin(x+y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin y + \sin(x+y)}{-\sin(x+y) - x\cos y}.$$

Application: Thm (Problem 77(a) in §3.5): Suppose f is diff and 1-1, and f^{-1} is also diff.

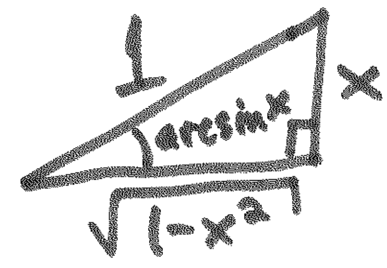
Then $$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

$\underline{\text{Ex}}$: Find $\frac{d}{dx}\arcsin(x)$.

$f(x) = \sin(x)$.
$f^{-1}(x) = \arcsin(x)$.
$f'(x) = \cos(x)$.

By an formula,

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\cos(\arcsin(x))}.$$
$$= \frac{1}{\sqrt{1-x^2}}.$$



$\underline{\text{Ex}}$: $f(x) = \arctan x$, Find $\frac{d}{dx}\arctan x$.

$$\frac{d}{dx}\arctan x = \frac{1}{\sec^2(\arctan x)} \underset{\substack{\uparrow \\ \text{trig}}}{=} \frac{1}{1+\tan^2(\arctan x)}$$
$$= \frac{1}{1+x^2}.$$

MA 113 2/24/17

1. Turn in written assignments
2. Log in to REEF.
3. Webwork B6 due tonight.
4. Exam 2 is Tuesday, March 7.
5. Suppose $y = \ln x$. Then $\boxed{e^y = x}$.
   w/ neighbors, use implicit diff to find $\frac{dy}{dx}$.

Apply $\frac{d}{dx}$ to both sides:

Recall: $y = \ln(x)$

$x \longmapsto y \longmapsto e^y$

$$\frac{d}{dx} e^y = \frac{d}{dx} x$$

$$\Rightarrow e^y \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

Thm: $\frac{d}{dx} \ln x = \frac{1}{x}$.

Recall: If f, f^{-1} are diff, then $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$

We could have used this: $f(x) = e^x$, $f^{-1}(x) = \ln(x)$.

$$(\ln x)' = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

why is $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$?

use eqn $f(f^{-1}(x)) = x$. Set $y = f^{-1}(x)$.

$\Rightarrow f(y) = x.$

Use implicit diff, we get $\dfrac{d}{dx} f(y) = \dfrac{d}{dx} x$

$$\Rightarrow f'(y) \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}.$$

<u>Last time</u>: we used this for inverse trig fns.

<u>Now</u>: use it on $\log_b(x)$ for arbitrary bases.

If $f(x)=b^x$, $f^{-1}(x)=\log_b x$.

$b^x = e^{x\cdot\ln(b)}$

Recall $f'(x)=\ln(b)\cdot b^x$.

Our formula says $(\log_b x)' = \dfrac{1}{\ln(b)\cdot b^{\log_b x}} = \dfrac{1}{\ln(b)\cdot x}$.

Important derivative!

Ex: Find $\dfrac{d}{dx}\underbrace{\log_3}_{\text{outside}}(\underbrace{2x}_{\text{inside}}) = \dfrac{1}{\ln(3)\cdot 2x}\cdot 2 = \dfrac{1}{\ln(3)\cdot x}$.

cancel 2's.

Ex: Find $\dfrac{d}{dx}\underbrace{\ln}_{\text{outside}}(\underbrace{7x}_{\text{inside}}) = \dfrac{1}{7x}\cdot 7 = \dfrac{1}{x}$.

Log laws: $\log_b(a\cdot x) = \underbrace{\log_b(a)}_{\substack{\log_b(a) \\ \text{is also} \\ \text{a constant}}} + \log_b(x)$

If a is a constant $\Rightarrow$ $\log_b(a)$ is also a constant

So, $$\frac{d}{dx}\left(\log_b(a\cdot x)\right) = \frac{d}{dx}\left(\underbrace{\log_b(a)}_{\text{constant}} + \log_b(x)\right) = \frac{d}{dx}\log_b(x).$$

Ex: $$\frac{d}{dx}\ln(x^5) \underset{\text{by chain rule}}{=} \frac{1}{x^5}\cdot 5x^4 = \frac{5}{x}.$$

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Using log laws, $$\frac{d}{dx}\ln(x^5) = \frac{d}{dx}5\ln x = 5\frac{d}{dx}\ln(x) = 5\cdot\frac{1}{x}.$$

Ex: $\frac{d}{dx}\ln\left(x^3+x+1\right) \underset{\substack{\uparrow \\ \text{chain rule}}}{=} \frac{1}{x^3+x+1}\cdot(3x^2+1).$

NOTE: Example 6 in §3.6 has $\frac{d}{dx}\ln|x| = \frac{1}{x}$.
This is worth knowing + understanding.

Ex: $\frac{d}{dx}\ln\left(\frac{x^2+1}{\sqrt{x-1}}\right) = \frac{d}{dx}\left(\ln(x^2+1) - \ln\left(\sqrt{x-1}\right)\right)$

$= \frac{d}{dx}\left(\ln(x^2+1) - \ln\left((x-1)^{1/2}\right)\right)$

$= \frac{d}{dx}\left(\ln(x^2+1) - \frac{1}{2}\ln(x-1)\right)$

$$= \frac{d}{dx}\ln(x^2+1) - \frac{1}{2}\frac{d}{dx}\ln(x-1)$$

$$= \frac{1}{x^2+1}\cdot 2x - \frac{1}{2}\cdot\frac{1}{x-1}\cdot 1 = \frac{2x}{x^2+1} - \frac{1}{2(x-1)}.$$

<u>Thm</u>: $e = \lim_{h\to 0}(1+h)^{1/h}$

Set $f(x)=\ln x$. $f'(x)=\frac{1}{x}$.

$$\frac{1}{1} = f'(1) = \lim_{h\to 0}\frac{\ln(1+h)-\ln(1)}{h} = \lim_{h\to 0}\frac{\ln(1+h)}{h}$$

$$\Rightarrow 1 = \lim_{h\to 0}\frac{\ln(1+h)}{h}. \Rightarrow e = \lim_{h\to 0} e^{\ln(1+h)/h}$$

Take $e^{(\text{both sides})}$ $\quad \underset{\text{algebra}}{\Rightarrow} \quad e = \lim_{h\to 0}(1+h)^{1/h}.$