MA 113 — 2/27/17

1. See Canvas announcement for assignments this week.
2. Log in to REEF.
3. Reminder: Exam 2 on Tues, March 7.
4. Today: §3.7, Rates of Change.
5. Find $f''(t)$ when $f(t) = \frac{t^3}{3} - \frac{5}{2}t^2 + 4t$.

   When is $f''(t) > 0$?

→ Discuss your answers w/ your neighbors!

Q: What is area of a circle of radius r?

$A = \pi r^2$

what is $A'(r) = 2\pi r$.

$A(r)$ grows as r grows.

Q: What is circumference of same circle?

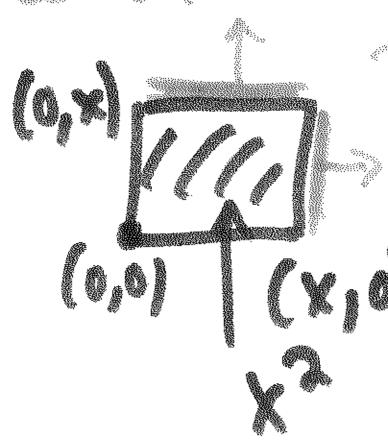
same.

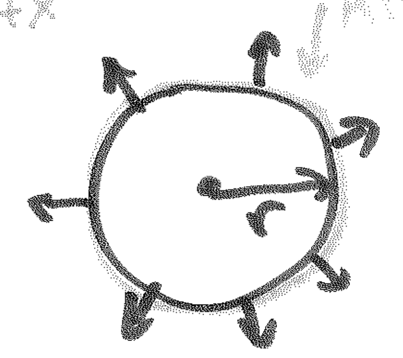
$C = 2\pi r$

what this says is derivative of $A(r)$ is measuring circumference of circle at instant when radius is r.

$2x$ is $x+x$

e.g. $(x^2)' \rightarrow 2x$





Idea: derivatives measure instanteous change.

→ eg. geometric pictures for □, ○

→ eg. average vs inst. velocity.

Suppose $f(x)=y$ is a function relating x, y.

Then if $\Delta x = x_2 - x_1$, where $f(x_1)=y_1$

$\Delta y = y_2 - y_1$, where $f(x_2) = y_2$

Then write $\frac{\Delta f}{\Delta x} = \frac{\Delta y}{\Delta x} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(x_2)-f(x_1)}{x_2 - x_1}$.

average rate of change of f

Key pt: $\underbrace{\lim_{x_2 \to x_1} \frac{f(x_2)-f(x_1)}{x_2-x_1}}_{\text{instantaneous rate of change of } f.} = f'(x_1)$.

Ex: (Physics) If $f(t)$ = position at time t,

$f'(t)$ = inst. velocity

$f''(t)$ = acceleration

$f'''(t)$ = jerk.

Suppose $f(t) = \frac{t^3}{3} - \frac{5}{2}t^2 + 4t$ meters is position.

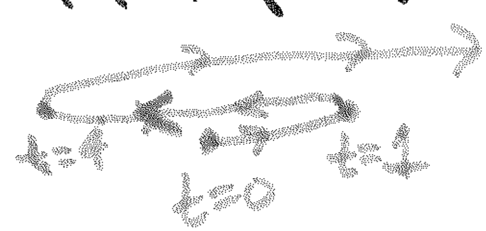
Velocity: $f'(t) = t^2 - 5t + 4$ m/s.

When is particle moving in positive direction?

i.e. Find when $f'(t) > 0$.

Note: $f'(t) = (t-1)(t-4)$, thus $f'(t) > 0$ for t in $(-\infty, 1) \cup (4, \infty)$

Picture of movement:



What is acceleration at time t?

$$f''(t) = 2t - 5 \ m/s^2$$

when $t = \frac{5}{2}$ sec, acceleration is 0.

NOTE: Read Ex #1 in §3.7 for more analysis of a similar example.

REEF: 1

Ex (physics): electrons moving through a wire.

Let ΔQ = net charge through surface during time change Δt.

Average current $= \dfrac{\Delta Q}{\Delta t}$.

Current at time t is

$$I = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}.$$

Most/many problems about current involve

<u>Ohm's Law:</u> Voltage V, current I, resistance R in a circuit, these satisfy

$$V = I \cdot R.$$

On webwork, you will use this to find rates of change for I + R with respect to each other.

<u>Ex:</u> (Biology) If $f(t)$ = population of bacteria at time t,

<u>average growth rate</u> is $\frac{\Delta f}{\Delta t}$.

<u>Inst. growth rate</u> is $\frac{df}{dt}$.

This uses continuous fns to model discrete phenomena.

General phenomena for f in this case:

$$f(t) = A \cdot e^{kt}$$ (small-scale, bacteria growth).

$$\Rightarrow f'(t) = k \cdot A \cdot e^{kt}.$$

<u>Typical real-looking application:</u>

$$f(t) = 203 \cdot 3^t = 203 \cdot e^{\ln(3^t)}$$

$$= 203 \cdot e^{t \cdot \ln(3)}$$

$$= 203 \cdot e^{\ln(3) \cdot t}$$

These are the same structure mathematically... —

Ex: (Economics) $C(x)$ = cost of producing x items.
= cost fn.

$C(x_2) - C(x_1)$ = cost of producing x_2 units instead of x_1 units.

$\frac{\Delta C}{\Delta x}$ = average change in cost per item.

$\frac{dC}{dx}$ = "marginal cost," i.e. $\approx$ cost of producing <u>one</u> more item.

because $C'(x) \approx \frac{C(x+1) - C(x)}{1}$

$= C(x+1) - C(x)$.

MA 113 3/1/2017

$\boxed{1}$ Log in to REEF

$\boxed{2}$ Reminder: Exam 2 on Tues, March 7. See Canvas announcement for info.

3 Today: §3.9, Related Rates

$\boxed{4}$ What is $\frac{d}{d\theta}\tan\theta$? (You need to know this!)

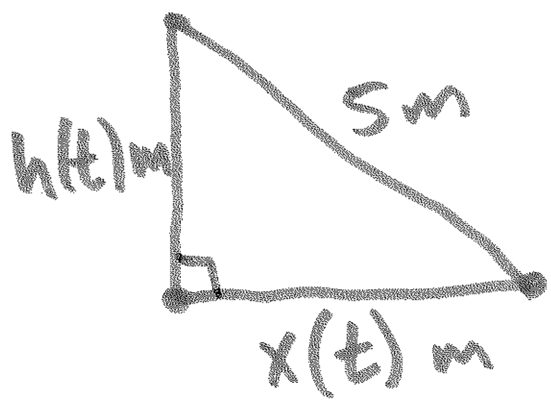
$$\frac{d}{d\theta}\tan\theta = \sec^2\theta$$

§3.9 Related Rates

Idea: Two or more quantities varying with respect to some other quantity (often time). Find an algebraic or other mathematical relationship among these quantities, use implicit diff. to relate their derivatives.

Ex: A 5-meter ladder rests against a wall. Bottom is 1.5 meters away from wall at $t=0$ seconds, bottom slides away from wall at constant rate of 0.8 m/s. What is the velocity of top of ladder at $t=1$ sec?

Step 1: Always try to sketch a picture.



<u>Step 2</u>: relate our quantities.

$\boxed{x(t)^2 + h(t)^2 = 5^2}$ by Pyth. Thm.

<u>Step 3</u>: Consider given info.

$x(0) = 1.5$ m

$\frac{dx}{dt} = 0.8$ m/s

<u>Step 4</u>: what do we want? $\frac{dh}{dt}$ at $t=1$.

We need to relate $\frac{dx}{dt}$ and $\frac{dh}{dt}$. Apply $\frac{d}{dt}$ to using implicit diff.

$$\frac{d}{dt}\left(x(t)^2 + h(t)^2\right) = \frac{d}{dt}\left(5^2\right)$$

$$\Rightarrow \quad 2x(t)\cdot\frac{dx}{dt} + 2h(t)\cdot\frac{dh}{dt} = 0.$$ ← solve for $\frac{dh}{dt}$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{-2x(t)}{2h(t)}\cdot\frac{dx}{dt} = \frac{-x(t)}{h(t)}\cdot\frac{dx}{dt}.$$

I know $\frac{dx}{dt}$, need $x(1)$ & $h(1)$.

$x(1) = x(0) + 0.8 = 1.5 + 0.8 = 2.3$ m.

So, $h(1) = \sqrt{5^2 - 2.3^2} \approx 4.44$ m.

Thus, when $t = 1$, $\frac{dh}{dt} \approx \frac{-2.3}{4.44} \cdot 0.8 \approx -0.41$ m/s

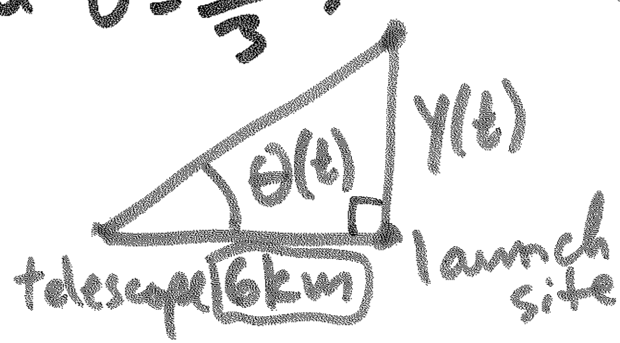
Aside: $x(t) = x(0)_m + t_s \cdot 0.8_{m/s}$ m

Ex: Suppose a telescope is tracking a rocket launch. Scope is 6km from launch site. When Angle θ between telescope and ground is $\frac{\pi}{3}$, θ is changing at 0.9 rad/min. What is velocity of rocket at $\theta = \frac{\pi}{3}$?

t in minutes.

Goal: Find $\frac{dy}{dt}$ when $\theta = \frac{\pi}{3}$.

Picture:



Natural thing to do is write $\tan(\theta(t)) = \frac{y(t)}{6}$.

To relate $\frac{d\theta}{dt}$ + $\frac{dy}{dt}$, use $\frac{d}{dt}$ + implicit diff.

$$\frac{d}{dt}\tan\theta(t) = \frac{d}{dt}\left(\frac{y(t)}{6}\right) \Rightarrow \sec^2\theta(t)\cdot\frac{d\theta}{dt} = \frac{1}{6}\frac{dy}{dt}.$$

$\boxed{\text{REEF: } 6}$

Solve for $\frac{dy}{dt} = 6\sec^2\theta(t)\cdot\frac{d\theta}{dt}$.

At $\theta = \frac{\pi}{3}$, $\frac{dy}{dt} = 6\cdot\sec^2\left(\frac{\pi}{3}\right)\cdot 0.9 \approx 21.6$ km/min.

<u>Ex</u>: A rectangular tank has base 2m by 3m. Water fills at 0.3 m^3/min. How fast does water level rise at time t min?

← $h(t)$ is height of water.

V = Volume of water, given $\frac{dV}{dt} = 0.3\ m^3/\text{min}$.

<u>Goal</u>: Find formula for $\frac{dh}{dt}$.

$V(t) = 2 \cdot 3 \cdot h(t)$ relates volume & height.

Apply $\frac{d}{dt}$, get $\frac{dV}{dt} = 2 \cdot 3 \cdot \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{1}{6} \cdot \frac{dV}{dt}$.

So, $\frac{dh}{dt} = \frac{1}{6} \cdot 0.3 \text{ m/min} = 0.05 \text{ m/min}$.

<u>Read</u> §3.9 because Rel. Rates problems are easiest to do w/ lots of examples.