MA 113 3/20/17

$\boxed{1}$ Log in to REEF

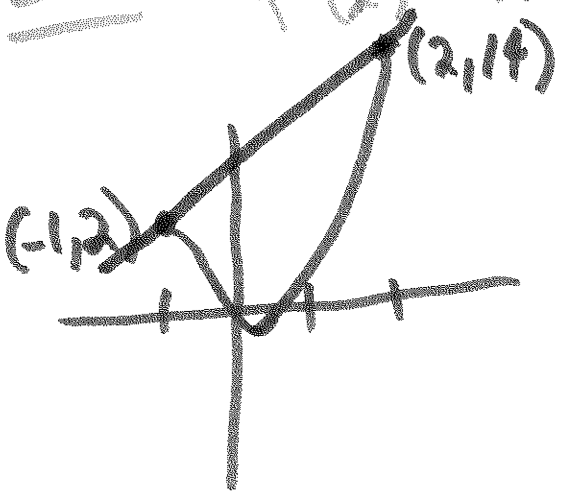
$\boxed{2}$ See Canvas announcement for assignments this week.

$\boxed{3}$ Today: §4.2, Mean Value Theorem

$\boxed{4}$ **Discuss w/ your neighbors:**

If $f'(x)=0$ for all x in (a,b), what can $f(x)$ be? Why?

Ex: $f(x) = x^3 + 2x^2 - x$ on $[-1, 2]$.



To get from $(-1,2)$ to $(2,14)$ on graph of $f(x)$, we have to move generally in the direction given by slope of line between these two points.

Slope is

$$\frac{14-2}{2-(-1)} = \frac{12}{3} = \boxed{4.}$$

NOTE: $f'\left(\frac{-4+\sqrt{76}}{6}\right) = 4$.

So, at point $\left(\frac{-4+\sqrt{76}}{6}, f\left(\frac{-4+\sqrt{76}}{6}\right)\right)$ the graph of f has tangent line w/ slope 4.

__Mean Value Thm:__ Suppose f is a fn that is

- cts on $[a,b]$

and

- diff on (a,b).

Then there is a value c in (a,b) where
$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

i.e. __slope of tangent line__ to $(c, f(c))$ is equal to __slope of secant line__ between $(a, f(a))$, $(b, f(b))$.

__Special Case:__ When $f(a) = f(b)$,

Then $f'(c) = \frac{f(b)-f(a)}{b-a} = 0$.

So, tangent line is horizontal. This case is called Rolle's Thm.

Why is M.V.T. true?

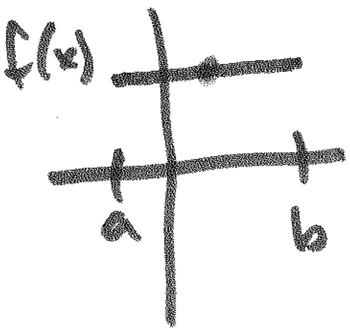
Step 1: Use Fermat's Thm + Extreme Value Thm to prove Rolle's Thm.

Step 2: Use Rolle's Thm to prove M.V.T.

For details, see §4.2. It is worth reading.

Application of M.V.T.:

Theorem: If $f'(x) = 0$ on (a,b), then $f(x)$ must be constant on (a,b).



Q: What does it mean for $f(x)$ to be constant if you don't know the value of the constant?

A: A function f is constant if for any two x_1, x_2 in domain(f), we have $f(x_1) = f(x_2)$.

To prove this implication $f'(x)=0 \Rightarrow f(x)$ constant, pick two points x_1, x_2 in (a,b): let's write

KEEP 2

$a < x_1 < x_2 < b$. Since f is diff on (a,b), it is diff and cts on $[x_1, x_2]$.

By MVT, there is a point c in (x_1, x_2) with

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c) = \underset{\uparrow \text{ by assumption.}}{0}$$

So, $f(x_2)-f(x_1)=0$, thus $f(x_2)=f(x_1)$.

Thus, f is constant.

<u>Thm</u>: If $f'(x)=g'(x)$, on (a,b), then $f = g + \text{constant}$ on (a,b).

<u>why?</u> $f-g$ on (a,b) has derivative $f'-g'=0$.

So, $f-g=$ constant, thus $f=g+\text{const}$.

Ex: Suppose $f(x)$ is cts on $[6,15]$ and diff on $(6,15)$. If $f(6) = -2$ and $f'(x) \leq 10$ for every x in $(6,15)$, what is largest possible value of $f(15)$?

Things you tried
- graph
- $f'(c) = \dfrac{f(b) - f(a)}{b-a}$
- ???

Note: $a = 6$, $b = 15$.
we don't have a "c" given to us.
$f(a) = f(6) = -2$.

MVT says
$$f'(c) = \frac{f(15) - f(6)}{15-6} = \frac{f(15)+2}{9}.$$

So, $f'(c) = \dfrac{f(15)+2}{9}$ for some c in $(6,15)$.

Using $f'(x) \le 10$ for all x in $(6,15)$, we get

$$10 \ge f'(c) = \frac{f(15)+2}{9}.$$

So, $$10 \ge \frac{f(15)+2}{9}$$

$$\Rightarrow \quad 90 \ge f(15)+2$$

$$\Rightarrow \quad 88 \ge f(15).$$

So, $f(15)$ can be no larger than 88.

Ex: Show that $f(x) = 4x^5 + x^3 + 7x - 2$ has exactly one real root.

Review this: Step 1: Use I.V.T. and fact that $f(0) = -2$, $f(1) = 10$ to conclude that a root exists in $(0,1)$.

<u>Step 2:</u> Suppose f has 2 real roots,
i.e. a, b with $f(a) = 0 = f(b)$.

Assume one is root from step 1.

By Rolle's Thm there must be some c in (a,b) with $f'(c) = \dfrac{f(b)-f(a)}{b-a} = 0$.

But, $f'(x) = 20x^4 + 3x^2 + 7 \geq 7$.

So, there is no value c with $f'(c) = 0$,
hence there can only be one real root.

MA113 3/22/17

1. Log in to REEF

2. § 4.3, How derivatives affect shape of a graph, is today

3. See Canvas for assignments this week, announcement from Sunday.
   → Start written assignment early!

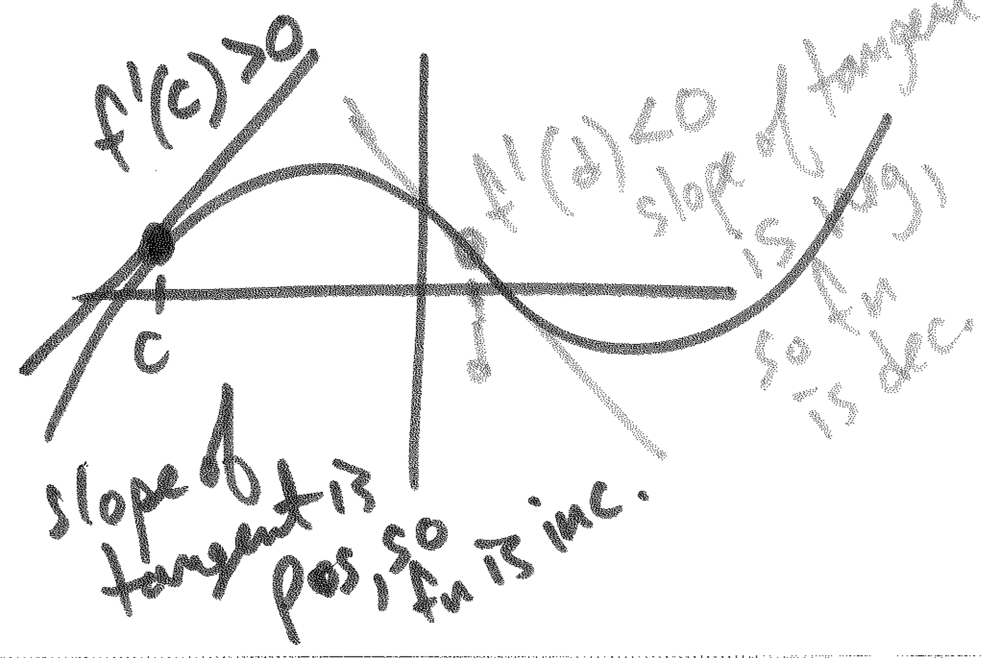
4. Today's big Q: How do we find the local mins and maxs for a differentiable function?

Two answers to this Q:

1) 1^{st} derivative test
2) 2^{nd} derivative test

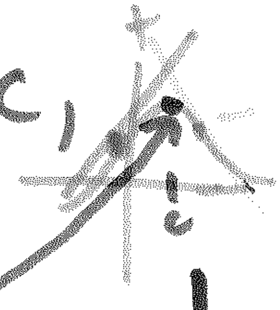
(I) First derivatives

Increasing/Decreasing test: If $f'(x) > 0$, (resp. if $f'(x) < 0$), on (a,b), then $f(x)$ is increasing (resp. dec.) on (a,b).



__1st derivative test__: Suppose c is a critical number of f, + f is cts.

a) If f' changes from pos to neg at c, then f has a local max at c.



b) If f' changes from neg to pos at c, then f has a local min at c.

c) If f' does not change signs at c, then f has no local min or max at c.

__Ex__: $f(x) = \sin x$, see desmos.

II Second derivatives

Q: What does it tell us if $f''(x) > 0$? or $f''(x) < 0$?

Def$^{\underline{n}}$: If the graph of f lies above all of its tangent lines on (a,b), then it is called concave up on (a,b). If graph of f lies below all of its tangent lines (after zooming in) on (a,b), it is called concave down on (a,b).

Concavity Test: (a) If $f''(x) > 0$ for x in (a,b), then f is concave up on (a,b).
(b) If $f''(x) < 0$ for x in (a,b), then f is concave down on (a,b).

Def$^{\underline{n}}$: A point on $y = f(x)$ is an **inflection point** if f is cts there and f changes concavity at that point.

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2nd der. test: Suppose f'' is cts near c.
(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local min at c.
(In words, if f has a hor. tangent line at c and

the graph is above the tangent line near c, then $(c, f(c))$ is a local min pt.)

(b) If $f'(c)=0$ and $f''(c)<0$, then f has a local max at c.

<u>Remark</u>: §4.3 has lots of examples. Read them!

3/24/17

No notes available.

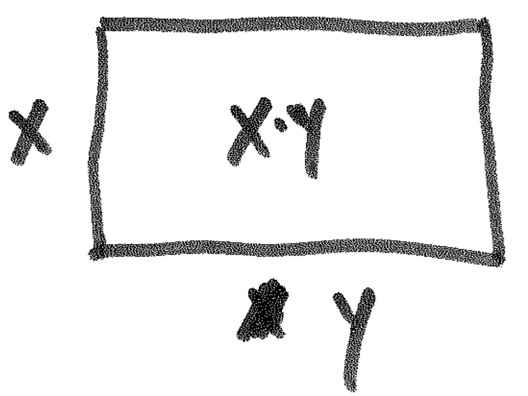
<u>MA113</u> <u>3/27/17</u>

1. REEF today
2. See Canvas announcement for assignments this week.
3. Due to being ill at end of last week, I am way behind in responding to your emails --- give me today + tomorrow + I should be caught up.
4. <u>Today and Wednesday</u>: § 4.7, Optimization

__Ex__: Find two numbers that differ by 110 where the product of those numbers is as small as possible.

NOTE: These are like related rates problems, in that you need to name variables, draw diagrams, set notation, etc.

Set $y - x = 110$, so $y = x + 110$.



drawing represents $x, y > 0$, but problem allows any x, y.

__Goal:__ Minimize

$$P(x) = x \cdot y = x(x+110) = x^2 + 110x.$$

Find critical pts of P:

$P'(x) = 2x + 110 \Rightarrow x = -55$ is a critical pt.

Since $P''(x) = 2 > 0$, $x = -55$ yields an absolute min for P on $(-\infty, \infty)$, since there are no other critical values and graph is concave up.

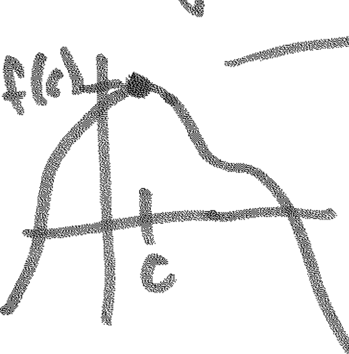
So, $y = -55 + 110 = 55$ gives solution

$x = -55$, $y = 55$ for the problem.

First derivative test for absolute extreme values:

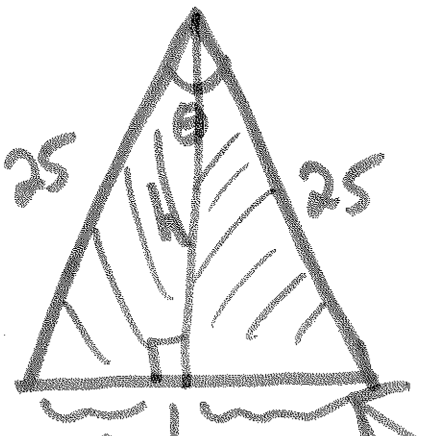
If f is cts on (a,b) and c is a critical number of f in (a,b), then:

a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the abs. max of f on (a,b).



b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the abs. min of f on (a,b).

Ex: Find the angle θ that maximizes area of isoceles triangle w/ legs of length 25.



each length $\frac{b}{2}$

We need to know area of triangle. Drop a height h, could instead maximize half the area, i.e. right-side triangle. <u>Let's stick w/ full Area.</u>

$A = \frac{1}{2} b \cdot h$. is <u>full</u> area.

$$A(b) = \frac{1}{2} \cdot b \cdot \left(\sqrt{25^2 - \left(\frac{b}{2}\right)^2}\right).$$

since $h = \sqrt{25^2 - \left(\frac{b}{2}\right)^2}$ by Pyth. Thm.

<u>Goal</u>: Find b with max area, then compute θ for that b, w/ assumption $0 < b < 50$, since $50 = 25 + 25$.

$(0, 50)$

Maximize $A(b) = \underbrace{\tfrac{1}{2}b}\cdot\underbrace{\sqrt{25^2 - \left(\tfrac{b}{2}\right)^2}}_{\downarrow \text{chain rule}}$ on $(0, 50)$.

By chain rule + product rule,

$$A'(b) = \frac{1}{2}\sqrt{25^2 - \left(\frac{b}{2}\right)^2} + \frac{1}{2}b \cdot \overbrace{\frac{-2b}{4} \cdot \frac{1}{2\sqrt{25^2 - \left(\frac{b}{2}\right)^2}}}$$

$$= \frac{1}{2}\left[\frac{25^2 - \left(\frac{b}{2}\right)^2 + -\frac{b^2}{4}}{\sqrt{25^2 - \left(\frac{b}{2}\right)^2}}\right]$$ ← via algebra, use $\sqrt{25^2 - \left(\frac{b}{2}\right)^2}$ as a common denominator.

$$= \frac{1}{2}\left[\frac{25^2 - \frac{b^2}{2}}{\sqrt{25^2 - \frac{b^2}{4}}}\right].$$

For $0 < b < 50$, denominator is > 0.

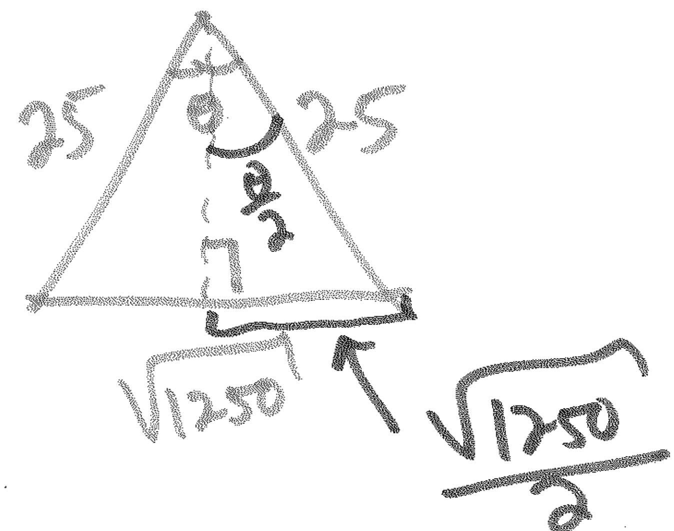
So, $A'(b) = 0$ when $25^2 - \frac{b^2}{2} = 0 \Rightarrow b = \sqrt{1250}$.

is the critical point.

NOTE: Check $A'(b) > 0$ for $b < \sqrt{1250}$,
and $A'(b) < 0$ for $b > \sqrt{1250}$.

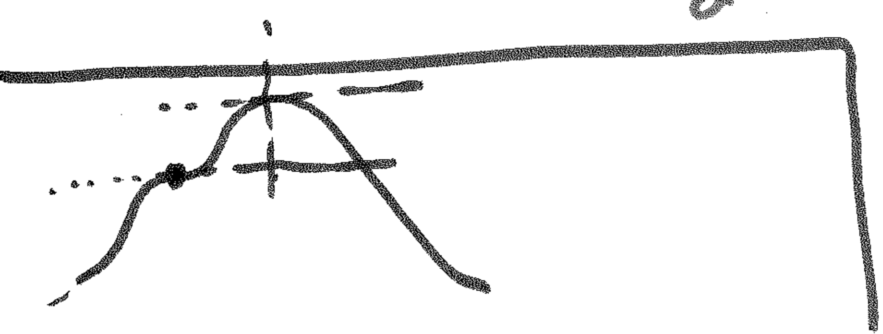
So, by ~~our~~ our 1st der. test for abs. values, $b = \sqrt{1250}$ gives abs. max of A on $(0, 50)$.

To find θ-value corresponding to b,



so $\frac{\theta}{2} = \arcsin\left(\frac{\sqrt{1250}/2}{25}\right)$

$\Rightarrow \theta = 2\arcsin\left(\frac{\sqrt{1250}}{50}\right)$.



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