Making the Discrete Continuous: How Combinatorics Becomes Topology

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At first, topology and combinatorics seem far removed from each other. Combinatorics deals with finite sets and "rigid" structures whereas in topology you often have spaces which you are allowed to deform continuously (hence the classic phrase that a coffee cup and a donut are the same thing to a topologist). Yet the last several decades have seen beautiful applications of topology in combinatorics and discrete geometry. A recent textbook, [10], focuses entirely on combinatorial uses of the Borsuk-Ulam theorem! So how is it that topology finds such a useful place in combinatorics? How do we get from our rigid objects to continuous ones? And most importantly, why should we expect this transition from the discrete to the continuous to be successful? The purpose of this essay is to find some answers to these questions. Our goal is not for the reader to understand lots of broad technical details; in fact, some of the ideas in the final section are very difficult and take years to understand. We will instead highlight some specific situations in combinatorics where topology arises, hopefully enticing the reader into this interesting area of mathematics. This essay therefore has three sections: a gentle introduction to graph coloring, an introduction to simplicial complexes, and a discussion of situations where topology plays a role in graph theory and group theory.

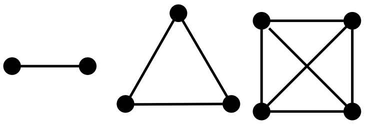

Figure 1: The complete graphs K_2, K_3, and K_4.

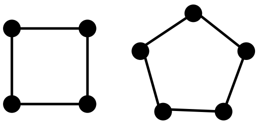

Figure 2: The cycles C_4 and C_5.

Graph coloring

Let's begin with the topic of graph colorings. Recall that a *graph*, denoted $G = (V, E)$, is a finite set V together with some collection E of pairs of elements of V. V is called the *vertex set* and E is called the *edge set.* A graph can be thought of as a set with an adjacency relationship (given by E) between some of its elements. We call two vertices which are endpoints for the same edge *neighbors.* One basic example of a graph is the complete graph K_n. This is the graph on n vertices where every pair of vertices forms an edge. One way to represent a graph is to draw a dot in the plane for each vertex and connect neighbors with a line. We've done this for some complete graphs in Figure 1. Other examples are C_4 and C_5, the cycles of length four and five. These are defined by the drawings we give in Figure 2. Similar diagrams show that for every natural number n, there is a cycle C_n. You can also draw graphs in higher dimensional space. A drawing of a graph G where the edges only intersect at their endpoints, and those only once, is called a *geometric realization* of G. Figure 1 has geometric realizations of K_2 and K_3, but not K_4.

Given a graph G, we say a *coloring* of G is a labeling of the vertices of G with colors (red, green, blue, etc.) so that no two neighbors have the same color. The smallest number of colors with which we can do this is called the *chromatic number* of G, denoted by $\chi(G)$. The study of graph chromatic numbers has a long history. There is a famous theorem called the Four-color theorem that resisted proof for over 100 years and whose proof is still considered imperfect by some mathematicians because it relies heavily on computer calculations. The Four-color problem arose when mapmakers were trying to figure out the smallest number of colors they needed to color their maps without adjacent countries having the same color. (Sound familiar?) It is not too hard to show that you can always do this with five colors; it is really hard to prove that you can always do it with four, which is the statement of the theorem. Graph coloring has many other applications in various fields, see the references for more information.

Let's look at what $\chi(G)$ is for our previous examples. If you label the vertices of K_n with fewer than n colors, two vertices must be colored the same. That is a problem, because those two vertices are neighbors and we want neighbors to have different colors. So, $\chi(K_n) \geq n$. Assigning each vertex of K_n a distinct color results in a good coloring. Hence $\chi(K_n) = n$. You can check for yourself that $\chi(C_4)$ equals two. What about $\chi(C_5)$? Can we produce a coloring using only two colors, like with C_4? (Think about it before you read on...) Hopefully you have discovered that it has chromatic number three. Why? Because if I try to use only two colors and begin by coloring one vertex red, call it v, then the neighbors of v have to both be colored blue. But, their respective neighbors are endpoints for the same edge, so they cannot both be red! We need at least three colors in this case. The same argument shows that any cycle with an odd number of vertices has chromatic number three, while even length cycles only need two colors (think about it).

Right away, you can see that the set of neighbors for each vertex is important in graph coloring; this makes sense, as the condition for a good coloring involves the adjacency relationship. However, if you study more graph theory you will see that considering the neighborhood of a single vertex by itself isn't enough to determine $\chi(G)$ because the chromatic number depends on the overall structure of the graph while the set of neighbors of a single vertex v is a local property, meaning that it only tells you about the structure of G "locally" near v. Cycles of odd length are a good example, because looking at the neighbors of one vertex you might think that two colors will be enough for a good coloring. However, as you keep "spreading out" all over the graph, you eventually see that two won't do, we need three. Believe it or not, odd cycles are a relatively tame example of how weird things can get! To effectively investigate graph colorings, we need to simultaneously handle the neighborhoods of all the vertices in the graph. One way to do this is to turn to topology, so let's set graph colorings aside for awhile and develop some of the topological tools we will need.

From the discrete to the continuous

Typically in combinatorics we investigate finite sets with some kind of structure: the vertex set of a graph has an adjacency structure, the elements of a partially ordered set have a partial order imposed, etc. Without these special structures, finite sets are determined uniquely by their size. So, if we wish to understand how to get to topology from combinatorics, we must begin by understanding how we can associate a topological space to a finite set in a meaningful way. Luckily, to understand the construction of these spaces we don't actually need to know much about point-set topology, as our constructions will take place in Euclidean space.

Let R denote the real numbers, and let R^n denote standard n-dimensional Euclidean space. This is a straightforward generalization of our usual Euclidean spaces, e.g., the real line R, plane R^2, and three-dimensional space R^3. A point in R^n is specified by a list of n real numbers, the coordinates for the point. Thus, $R^n = \{(r_1, \ldots, r_n) : r_i \in R\}$. The points $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, with the 1 in the i^{th} position, are called the *standard basis vectors.* If you haven't had linear algebra or multivariable calculus, draw some pictures depicting R, R^2, and R^3 via coordinate axes and label

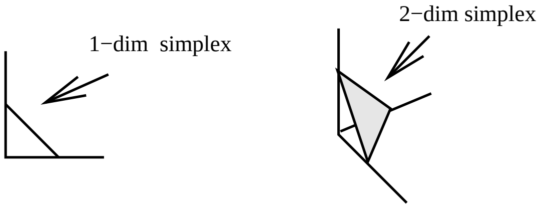

Figure 3: σ^1 and σ^2.

the standard basis vectors to get used to them.

For each set of size n we will construct a space called an $(n-1)$-*simplex*, denoted σ^{n-1}. We define σ^{n-1} to be the convex hull of the standard basis vectors in R^n. Figure 3 shows σ^1 and σ^2. Symbolically, $\sigma^{n-1} := \{a_1e_1 + a_2e_2 + \cdots + a_ne_n : a_1 + \cdots + a_n = 1, a_i \geq 0\}$. This definition, while correct, is unenlightening if you haven't seen convex hulls before. Informally, the convex hull of a set of points is all the space "between and amongst" the points. So, the convex hull of two points in Euclidean space, no matter the dimension, is a line segment. In R^2, if you take the convex hull of the points $(1,0)$, $(0,1)$, $(0,-1)$, $(-1,1)$, and $(-1,-1)$, you get a solid five sided polygon (play around with this example if you are confused!). Convex hulls of points in R^n can get extremely complicated and are studied in convex geometry. Luckily, we only need to concern ourselves with the standard basis vectors and simplices right now.

In R^1 we have only one basis vector, so σ^0 is a single point. Think of this as being a zero dimensional object, hence the 0 superscript, but we need one vector to be the point, hence R^1. In R^2 we have two basis vectors whose convex hull is σ^1, a line segment. Again, a line is one dimensional, hence the one in the superscript, but we used two basis vectors in Euclidean space to produce that line. In R^3 we get a triangle which is a two dimensional object, and in R^4 we get a tetrahedron which is three dimensional. Thus, simplices are the higher-dimensional analogues of triangles and tetrahedra. Notice that if we consider the convex hull of a subset of our standard basis vectors, we get a simplex of smaller dimension contained in the simplex we built using all the basis vectors. So, an n-dimensional simplex has lots of smaller dimensional simplices which make up its boundary. We call these various subsimplices the *faces* of our simplex.

To understand the move from a combinatorial object to a topological space, it is important to understand what we are doing when we pass from an n-set to a simplex: we are saying, "Hey, I have a bunch of points here and there is no structure, so I have total freedom to move among them and I want to capture that idea in a space." Hence, we take our n points, put them in Euclidean space, and then "fill in" all the space between them by taking the convex hull so that we can move amongst the points as we like.

Next we consider sets which have a combinatorial structure. Let $G = (V, E)$ be a graph with n vertices. We want to create a topological space which captures some of the structure of G. Following our construction of a simplex, we enumerate V as $V = \{v_1, \ldots, v_n\}$ and associate to v_i the point in R^n given by the standard basis vector e_i. From here on out, the points e_i and the vertices v_i will be considered the same, so if we say that two points are adjacent, we mean the corresponding vertices are adjacent in our graph. We want to "fill in" the space between these points in a way which reflects the structure in our graph. How should we proceed?

One way forward is to consider subsets $B \subseteq V$ for which every two vertices in B are neighbors. Such a subset is called a *clique* in G, and you should think of this as being a copy of K_m inside G, where $m \leq n$. This leads us to define the *clique complex* (often called the *flag complex*) of G as the space $C(G)$ obtained by taking the union of the convex hulls of the sets $\{e_{i_1}, \ldots, e_{i_k}\}$ such that $K = \{v_{i_1}, \ldots, v_{i_k}\}$ forms a clique in G. This exactly means we fill in the space between sets of vertices which are all connected in G. Note that we don't want to only fill in the space between pairs of adjacent points; that gives us a geometric realization of our graph and uses only 1-dimensional simplices. We want to fill in the entire $(m-1)$-simplex determined by a set of m pairwise adjacent vertices. We want to capture the

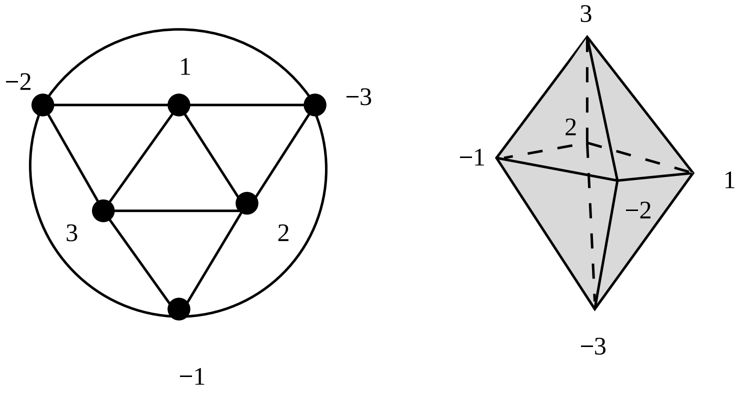

Figure 4: O_3 and $C(O_3)$.

idea that in a complete subgraph of a graph, we can freely move amongst any of those vertices, so we're filling in all the space "between" them.

Thus, for any complete graph K_n we have $C(K_n) \cong \sigma^{n-1}$. This is clear because K_n is a clique, so we take the convex hull of all the points. For a cycle C_n, $n \geq 3$, $C(C_n)$ is a geometric realization of C_n. This is because the only complete subgraphs of cycles are their edges, so we only fill in the space between neighbors. A slightly more complicated example is given by the graph O_3 with vertex set $V(O_3) = \{1, 2, 3, -1, -2, -3\}$ and edge set $E(O_3) = \{\{i, j\} : i \neq -j\}$. Our standard construction of $C(O_3)$ takes place in R^6, but we can actually draw it in R^3. Instead of using $\{e_1, \ldots, e_6\}$, associate $\pm i$ to $\pm e_i$ in R^3. We can still fill in the space we need without running into any trouble. Figure 4 shows that $C(O_3)$ is the boundary of an octahedron. Note that the clique complex of G always contains a geometric realization of G as a 1-dimensional sub-complex, i.e., a space made up of points and line segments.

If you think about it, you will see that the definition of $C(G)$ gives rise to spaces which are built out of simplices. The clique complex of C_m is built from 0- and 1-dimensional simplices, while $C(O_3)$ is built from 0-, 1-, and 2-dimensional simplices. Such spaces are called simplicial complexes. In general, a *simplicial complex* is a space built by "gluing" simplices together by identifying faces which have the same dimension. This description is somewhat vague, particularly about what the "gluing" process is. We ask the reader to allow us this lack of precision to avoid getting bogged down in technicalities and to use the examples as motivation. For a detailed exposition, see [10].

It is exactly the jump from combinatorial objects to simplicial complexes which allows us to make the discrete continuous. The general principle is that whenever we have a set A with combinatorial structure, we want to identify it with a set of points in Euclidean space, also denoted A, which are appropriately spread out. Letting each element in A correspond to a different standard basis vector in $R^{|A|}$ will always work for this, though the example with O_3 shows how different points can sometimes be used. We then take convex hulls to fill in the space between points in any subset B of A that has a certain structure. Because taking the convex hull of B means taking the convex hull of all subsets of B, the structures we can try to study with topology will necessarily be inherited by subsets. The spaces which occur as a result of this process are always simplicial complexes. This process should be understood as taking a discrete object and replacing it with a continuous space capturing some of the combinatorial data. Analyzing the topology of these spaces should then provide information about the combinatorial structure of A.

Applications to graphs and groups

The best way to get used to this principle is to see it in action, so let's return to graph coloring. Consider the family of *Kneser graphs* $\mathrm{KG}_{n,k}$, $n \geq 2k$, with $\binom{n}{k}$ vertices indexed by all k-element subsets of an n-set and edges between vertices labeled by disjoint subsets. It is a good exercise to look at $KG_{5,2}$, more commonly known as the *Petersen graph.* The name for these graphs comes from their connection to a conjecture made in 1955 by M. Kneser. Restated in graph theoretic terms, Kneser's conjecture claimed that $\chi(\mathrm{KG}_{n,k}) = n - 2k + 2$. Kneser himself produced the equivalent of a coloring using $n - 2k + 2$ colors to show that $\chi(\mathrm{KG}_{n,k}) \leq n - 2k + 2$. However, showing the reverse inequality stumped mathematicians for over twenty years. Finally, in 1978 L. Lovász produced a proof using methods from algebraic topology.

What Lovász actually proved was something very general, which implied the Kneser conjecture as a corollary. He defined the neighborhood complex $N(G)$ of an arbitrary graph $G = (V, E)$ as the simplicial complex with points corresponding to V and simplices defined by the sets $A \subset V$ such that the elements of A share a common neighbor in V, i.e., sets A for which there exists some $w \in V$ such that for all $v \in A$, $\{v, w\} \in E$. For example, as shown in Figures 5 and 6, $N(C_4)$ consists of two disjoint lines, $N(C_5)$ is the boundary of a pentagon, $N(K_2)$ is two distinct points, $N(K_3)$ is the boundary of a triangle, and $N(K_4)$ is the boundary of a tetrahedron. The motivation for this construction is that given a good coloring of G and a vertex v of G, none of the neighbors of v can have the same color it has. Subsets of families of vertices that share a common neighbor also share a common neighbor, hence that property is inherited by subsets and leads to the construction of our simplicial complex. Of course, simply defining the complex isn't all we need to do; the topology of the resulting space has to contain some information which is pertinent to the problem at hand. Lovász proved that for all graphs G, if $N(G)$ is k-connected, then $\chi(G) \geq k + 3$, where *k-connectedness* is a notion from algebraic topology which we will discuss after the next example. Though it is a heavy hammer to wield, using this result we can exactly determine the chromatic numbers of C_n and K_n for all n because $N(C_n)$ for even n is (-1)-connected, $N(C_n)$ for odd n is 0-connected, and $N(K_n)$ is $(n - 3)$-connected.

The link between $N(G)$ and $\chi(G)$ sparked serious interest in the study of graph colorings using topological methods. In fact, most of the simplicial complexes which have been developed to investigate $\chi(G)$ have their roots in the neighborhood complex, making it an important construction. Unfor-

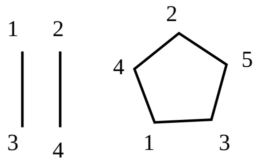

Figure 5: $N(C_4)$ and $N(C_5)$.

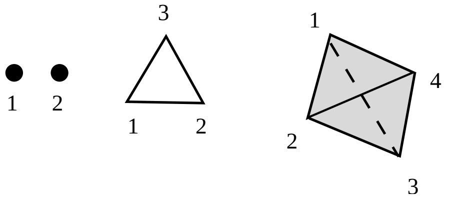

Figure 6: $N(K_2)$, $N(K_3)$, and $N(K_4)$.

tunately the details of Lovász's proof are beyond the scope of this essay. For more information on topological methods for investigating $\chi(G)$, including the proof of the theorem above, see [2], [8], and [10].

Graph theory is one of the cornerstones of combinatorics and has connections to many other areas of mathematics. Our final example shows that the topology of the clique complex of a graph has a decisive role in a question from group theory; this example involves basic concepts from the theory of group presentations, see chapter 26 of [6] for an introduction. Given a graph $G = (V, E)$ on n vertices, we define the *graph group* (also known as the *right-angled Artin group*) of G to be the group F_G with the following presentation:

$$F_G :=< g_1, \ldots, g_n : [g_i g_j] = e \text{ for all edges } \{v_i, v_j\} \in E >,$$

where $[g_i g_j] = g_i g_j g_i^{-1} g_j^{-1}$ is the *commutator* of g_i and g_j. This group is, by definition, the quotient of the free group on n generators by the smallest normal subgroup containing all the elements $[g_i g_j]$ for which $\{v_i, v_j\}$ is an edge of G. So, the adjacency structure of the graph G determines commutativity relations in the group F_G. For example, in F_{K_n} every pair of generators commute and hence F_{K_n} is the free abelian group on n generators. It is important to observe that, for any graph group F_G, there is a homomorphism from F_G to the group of integers under addition which is defined by sending every generator to 1. The kernel of this homomorphism is a subgroup denoted by H_G. Graph groups and these subgroups H_G are related to questions about finiteness properties of groups and have been studied by group theorists.

In 1997, M. Bestvina and N. Brady proved that there is a relationship between higher-order finiteness properties of H_G and higher-order connectivity of $C(G)$. For example, they proved that H_G is finitely generated if and only if $C(G)$ is 0-connected and that H_G is finitely presented if and only if $C(G)$ is 1-connected. These results answered questions in group theory regarding the non-equivalence of various finiteness properties and are difficult to prove. For more information, see [1] and [4].

We have seen how the level of connectivity of simplicial complexes plays a significant role in situations involving graph coloring and finiteness properties for groups. Connectivity is a topic usually not covered until graduate level courses in algebraic topology, but it is worth discussing as it regularly plays a role in combinatorial applications. A simplicial complex Δ (or, more generally, a topological space X) is called *k-connected* if for every r such that $0 \leq r \leq k$, every continuous map from the boundary of B^{r+1}, the unit ball in $(r+1)$-dimensional Euclidean space, into Δ (or X) can be extended to a continuous map from all of B^{r+1} to Δ (or X). For example, Δ being 0-connected means any two points can be connected by a path in Δ, while 1-connected (also called *simply connected*) means that additionally any continuous map from a 1-dimensional loop into Δ is the restriction to the boundary of B^2 of a continuous map from B^2 into Δ. While this definition is short, understanding exactly what k-connectedness captures is not so simple.

When Δ is k-connected, we roughly interpret this to mean that Δ is topologically trivial for dimensions 0 through k. So, by looking at the connectivity of a simplicial complex, you are scanning through increasing dimensions to see where you first encounter complicated aspects of your space. Scanning through stratified levels of information is what we do when we study the chromatic number of a graph or finiteness properties of groups. When we pass from the discrete to the continuous in these situations, some or all of this information is preserved via connectivity. That is why topological methods have been useful in these situations. While there are other topological invariants which play a role in combinatorics, connectivity is certainly a major one and worthy of serious study.

For anyone interested in learning more about the relationship between discreteness and continuity, the textbook "Using the Borsuk-Ulam Theorem," by J. Matoušek, [10], and the more advanced article "Discrete and Continuous: Two sides of the same?" by L. Lovász, [9], are good starting points with extensive bibliographies.

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