Knowing, Feeling, Doing: Assessment in Postsecondary Mathematics

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6 January 2016

What is Mathematical Proficiency?

Formative and Summative Assessment

Examples of Assessments in the Three Domains

Affective (Emotional, Feeling)

Cognitive (Intellectual, Knowing)

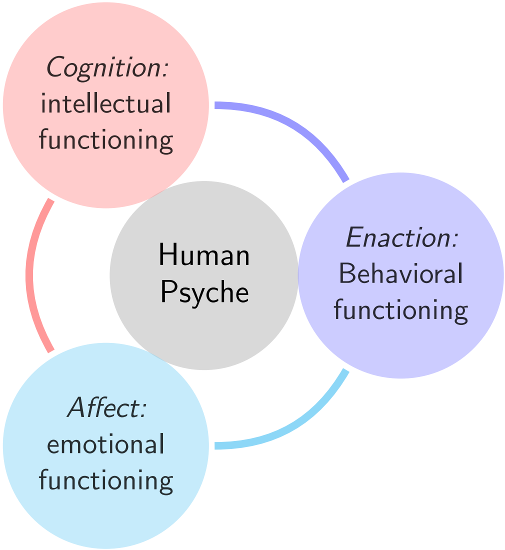
Enactive (Behavioral, Doing)

Things to Consider

Mathematical Proficiency

Assessment begins with a clear identification of our desired outcomes. Thus, we must develop an understanding of the nature of mathematical proficiency.

Modern psychology frames the human psyche as a three-stranded model.



Contemporary definitions of mathematical proficiency reflect this model, for example:

- 2001 NRC report *Adding It Up* introduced a five-strand model of mathematical proficiency, consisting of Conceptual Understanding, Procedural Fluency, Strategic Competence, Adaptive Reasoning, and Productive Disposition.
- Common Core State Standards for K-12 students specify separate sets of standards for "Mathematical Practices" and "Mathematical Content."
- The 2015 MAA CUPM Curriculum Guide specifies four "Cognitive Recommendations" and nine "Content Recommendations" for math major programs.
- The 2015 MAA Common Vision report surveys seven curricular guides from the MAA, AMATYC, ASA, and SIAM, all of which include more than content in their recommendations.

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This isn't sufficient! We need to develop more robust, broadly-implemented methods for giving students feedback and evaluating them based on rich learning outcomes.

Example of explicit learning outcomes

At the University of Kentucky, MA 261 – Number Theory, serves as an "intro to proof" course. Our department voted to adopt the following recommendations to faculty teaching the course.

Recommended content outcomes. Students will deepen their understanding of the following topics:

1. Divisibility, Division Algorithm, Euclidean Algorithm
2. Fundamental Theorem of Arithmetic, Infinitude of Primes
3. Linear Congruences, Chinese Remainder Theorem
4. Fermat's Little Theorem, Wilson's Theorem
5. Direct Proof, Proof by Contradiction, Mathematical Induction

Required mathematical practice outcomes. Students will improve with regard to the following practices:

1. Being persistent, Working through perceived failure, Strategic self-questioning
2. Productive collaboration with others, Asking good questions
3. Constructing examples and non-examples to investigate and understand new definitions and theorems
4. Reading and understanding existing proofs, Recognizing incorrect proofs
5. Developing and communicating original proofs

Formative and Summative Assessment

As described by Carnegie Mellon's Eberly Center for Teaching Excellence:

- *Formative assessment* is intended to monitor student learning, helping students identify their strengths and weaknesses and target areas that need work. These assessments are low- or no-stakes, e.g. incorporated as a required activity during class time or assigning a grade based only on completion of the task.
- *Summative assessment* is intended to evaluate student learning at the end of an instructional unit.

Note that *both* types of assessment can be used to assign a course grade, but formative assessments should generally be graded in a participatory manner rather than in an evaluatory manner.

On Writing Assignments

I am a strong advocate for the use of writing assignments in mathematics courses, especially when addressing affective (emotional, feeling) and enactive (behavioral, doing) aspects of student learning and proficiency. Writing assignments are also powerful additions to mathematical problem sets in the cognitive (intellectual, knowing) domain.

For a more in-depth discussion of this, including grading rubrics for mathematical writing, see my article *Personal, Expository, Critical, and Creative: Using Writing in Mathematics Courses* that was published in PRIMUS in 2014.

While both formative and summative assessments can be applied to all aspects of mathematical proficiency, I believe that *formative assessments based on essay writing* are the best tool for development in the affective domain of the human psyche.

An important point

If you value students doing something, provide a credit-bearing assessment to reinforce it. A few assignments that provide small numbers of “easy points” for students won’t usually have a big impact on their final grades, but will help ensure that students take certain actions or reflect on certain topics that we believe to be important.

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Autobiography

Autobiographical writing serves as excellent formative assessment in the affective domain. All of my undergraduate courses with less than 60 students begin with the following assignment.

- Imagine that you have written a book-length autobiography about your mathematical experiences.
- Write a passage, thought of as a quote from your autobiography, that reveals something significant about you mathematically.
- Be as creative as you like.

I assign a grade based on completion, completely ignoring the quality of the writing, editing, or ideas. If students respond to the prompt in a relevant manner, they get full credit.

Examining challenges

Short reflective essays about challenges in the course promote development in both the affective and enactive domains.

- Write several paragraphs on the following topic: what was the most challenging aspect for you regarding [TOPIC]? What made this difficult for you? Did you overcome the challenge, or are you still struggling with it?
- This will be graded based on completion, i.e. if you write several paragraphs that address these questions then you will get full credit for the problem.

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Levels of cognitive demand

For mathematical problems and exercises, you should develop and assign them based on their "level of cognitive demand." Cognitive demand is a fundamental concept in educational and cognitive psychology and the theory of learning, and all mathematics professors should know about it.

See part II of the series on "Active Learning in Mathematics Courses" at the AMS blog *On Teaching and Learning Mathematics* for an introduction to using levels of cognitive demand in postsecondary math courses: blogs.ams.org/matheducation

Critical reviews of reading

To promote critical analysis skills and develop students' reading abilities, have students write a review of selected readings from your course text.

- Write a three page critical review of [ASSIGNED READING].
- Imagine that you are writing your review for a journal for undergraduates in mathematics and the sciences.
- You must address the mathematical depth and mathematical style of [ASSIGNED READING] in addition to other topics.

Short essays graded using a rubric with five criteria: Writing Style, Arrangement and Development, Editing and Conventions, Mathematical Depth, Mathematical Style.

Extended course projects

Extended course projects are time-consuming to grade, but are the best way I know of to develop students' mathematical written communication skills.

- 10-page written project related to the course
- Suggested topics provided, students may pursue topics from "off-list"
- Directed at both a general university audience and other students in the course
- 7 bibliographic items required, 5 of these must be non-web sources
- A substantial revision process can be required, with instructor and peer-review on first versions

Graded and peer-reviewed using same rubric as described previously, including a sixth criteria for the quality of the revision between first and final versions.

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Reflection on process and practice

Ask students to reflect on their process and practice of doing mathematics, either in small group discussions during class or as part of reflective essays. These can be very effective.

My favorite activity of this type is to link reflective essays about the challenges of homework with assigning unsolved problems.

An unsolved problem I use as homework

For a positive integer n, let $\sigma(n)$ denote the sum of the positive integers that divide n. For example, $\sigma(4) = 1 + 2 + 4 = 7$, and $\sigma(6) = 1 + 2 + 3 + 6 = 12$. Let H_n denote the nth *harmonic number*, i.e.

$$H_n = \sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \,.$$

Let ln denote the natural log function. Does the following inequality hold for all $n \geq 1$?

$$\sigma(n) \leq H_n + \ln(H_n) e^{H_n}$$

NOTE: By work of Lagarias, this problem is equivalent to the *Riemann Hypothesis*.

What I tell students to do with this problem

This is an open, i.e. unsolved, problem given by Jeffrey Lagarias in 2002 in an article in the *American Mathematical Monthly*. Make as much progress as you can on it. Your goal is to do something more than check examples; the examples should lead you to make some interesting observations about the problem, to understand it a bit better. Why do you think it might be true? Why might it be false? Are there any properties of e, H_n, ln, or $\sigma(n)$ that support your comments? Are there special values of n for which this is obviously true? (Seriously, write down everything you're thinking and every idea you try, even if it doesn't go anywhere.)

How do students respond?

A few weeks after giving this assignment, I have students write a reflective essay about what they found most and least challenging in the homework so far, and what their most and least favorite homework problems have been.

What follows are direct quotes from these reflective essays written by undergraduates about their experience working on unsolved problems in my junior-senior-level history of mathematics course.

- I did have a favorite assignment, and that was the unsolved problem. This confuses me a bit because the problem was the essence of theoretical which, as I said before, can give me some trouble. But maybe since there was not really a correct answer I felt like I could attack it from whatever angle I wanted to without consequence. Now that I think about it, this should probably be how I approach all the theoretical problems. Instead of trying to find the correct answer right off the bat, I should write down what I know to be true about the problem and get a better understanding of it first. Anyway, I just really enjoyed how this problem challenged me to come up with my own way of approaching the problem and how I did not feel any pressure to find the correct answer.

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- Working on the unsolved problem involving unit fractions was automatically intimidating to me. Just knowing that this problem has not yet been solved convinced me that I would make little to no progress on the problem. Constantly my mind was reminding me that everything I had found, someone else had found before. There were moments when I was hopeful that my algebra and determination would lead me to some insight. Overall, I was unsatisfied with my work because I left feeling defeated.

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I believe that these passages reveal students identifying and wrestling with issues that are fundamentally related to effectively doing mathematics, and are very difficult to get at with ordinary problems and exercises alone.

Things to consider...

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- Have *modest ambition*[1]. Choose one or two new goals for your teaching each time you teach, and work hard at those. Expertise and deep understanding of teaching is developed slowly over time.
- Don't be afraid of small failures – these are inevitable and common. However, don't put yourself in a position where you take significant risks without consulting with your mentors and department leadership.

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Four Axioms of Teaching

The first two of these axioms come from Federico Ardila, a professor at San Francisco State University.

Axiom #1: *Mathematical talent is uniformly distributed, irrespective of geographic, demographic, and economic boundaries.* Growing and harvesting it is the right/smart thing to do.
Axiom #2: *Everyone can have meaningful and rewarding mathematical experiences.* Mathematics needs users, fans, and ambassadors.
Axiom #3: *Excellence is possible, perfection is not.* Seek and expect excellence from oneself and others, not perfection.
Axiom #4: *All human systems are flawed.* Choose where to invest your energy, accept that flaws will remain.

Thank you for listening!

Questions?

Please contact me with questions or ideas:
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