Problems
Oct 14, 2009

Problem 1: Determine 2×2 matrices B and C with integral entries such that

$$\begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix} = B^3 + C^3.$$

Problem 2: Let $n \geq 2$ be an integer and T_n be the number of non-empty subsets S of $\{1, 2, 3, \ldots, n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even.

Problem 3: Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability she hits exactly 50 of her first 100 shots?

SOLUTIONS TO SEPT 30 PROBLEMS:

Problem 1: For $n = 1, 2, 3, \ldots$, let $s(n)$ denote the sum of the digits of 2^n. Thus, for example, as $2^8 = 256$, we have $s(8) = 13$. Determine all positive integers n such that

$$s(n) = s(n+1).$$

Solution: The answer is that there are no such integers. Write

$$2^n = a_m 10^m + a_{m-1} 10^{m-1} + \cdots + a_1 10 + a_0,$$

where the a_i's are integers such that $1 \leq a_m \leq 10$, $0 \leq a_k \leq 10$ for $0 \leq k \leq m-1$. Then

$$2^n \equiv a_m + a_{m-1} + \cdots + a_1 + a_0 \equiv s(n) \pmod 3,$$

and so

$$s(n+1) \equiv 2^{n+1} \equiv 2 \cdot 2^n \equiv 2s(n) \pmod 3.$$

Thus, if $s(n) = s(n+1)$, we must have

$$s(n) \equiv 0 \pmod 3$$

and

$$2^n \equiv 0 \pmod 3.$$

. But, this is impossible because 3 does not divide 2^n.

Problem 2:

Right triangle ABC has right angle at C and $\angle BAC = \theta$; the point D is chosen on AB so that $|AC| = |AD| = 1$; the point E is chosen on BC so that $\angle CDE = \theta$. The perpendicular to BC at E meets AB at F. Evaluate $\lim_{\theta\to 0} |EF|$.

Solution: The answer is 1/3. Let G be the point obtained by reflecting C about the line AB. Since $\angle ADC = \frac{\pi-\theta}{2}$, we find that $\angle BDE = \pi - \theta - \angle ADC = \frac{\pi-\theta}{2} = \angle ADC = \pi - \angle BDC = \pi - \angle BDG$, so that E, D, G are collinear. Hence

$$|EF| = \frac{|BE|}{|BC|} = \frac{|BE|}{|BG|} = \frac{\sin(\theta/2)}{\sin(3\theta/2)},$$

where we have used the law of sines in $\triangle BDG$. But by l'Hôpital's Rule,

$$\lim_{\theta\to 0} \frac{\sin(\theta/2)}{\sin(3\theta/2)} = \lim_{\theta\to 0} \frac{\cos(\theta/2)}{3\cos(3\theta/2)} = 1/3.$$

Problem 3:

Evaluate

$$\sqrt[8]{2207 - \frac{1}{2207 - \frac{1}{2207 - \dots}}}.$$

Express your answer in the form $\frac{a+b\sqrt{c}}{d}$, where a, b, c, d are integers.

Solution: The infinite continued fraction is defined as the limit of the sequence $L_0 = 2207, L_{n+1} = 2207 - 1/L_n$. Notice that the sequence is strictly decreasing (by induction) and thus indeed has a limit L, which satisfies $L = 2207 - 1/L$, or rewriting, $L^2 - 2207L + 1 = 0$. Moreover, we want the greater of the two roots. Now how to compute the eighth root of L? Notice that if x satisfies the quadratic $x^2 - ax + 1 = 0$, then we have

$$\begin{aligned} 0 &= (x^2 - ax + 1)(x^2 + ax + 1) \\ &= x^4 - (a^2 - 2)x^2 + 1. \end{aligned}$$

Clearly, then, the positive square roots of the quadratic $x^2 - bx + 1$ satisfy the quadratic $x^2 - (b^2+2)^{1/2}x + 1 = 0$. Thus we compute that $L^{1/2}$ is the greater root of $x^2 - 47x + 1 = 0$, $L^{1/4}$ is the greater root of $x^2 - 7x + 1 = 0$, and $L^{1/8}$ is the greater root of $x^2 - 3x + 1 = 0$, otherwise known as $(3 + \sqrt{5})/2$.