Problems
Sept 17, 2009

Problem 1: Prove that there exist infinitely many positive integers which are not representable as sums of fewer than ten squares of odd natural numbers.

Problem 2: Prove that for each positive integer n there exists a circle in the xy-plane which contains exactly n lattice points (i.e. points of the form (a, b) where a and b are integers).

Problem 3: Let

$$a_n = \frac{1}{4n+1} + \frac{1}{4n+3} - \frac{1}{2n+2}, n = 0, 1, 2, \ldots.$$

Does the infinite series $\sum_{n=0}^{\infty} a_n$ converge, and if so, what is its sum?

SOLUTIONS TO SEPT 1 PROBLEMS:

Problem 1: Suppose that one removes two opposite corner squares of a checkerboard. Can the remaining 62 squares be covered with 31 dominos of size 2×1?

Solution: The answer is no. Suppose the squares on a checkerboard are colored black and white. By removing opposite corner squares, you remove two squares of the same color, say black. However, every time a domino is placed on the board, it covers one white and one black square. Thus, if there are 31 dominos covering the board, there must be 31 white and 31 black squares covered, which is impossible since there are 32 white and 30 black squares.

Problem 2: Show that

$$\sum_{k=1}^{n} \frac{1}{k}$$

is never an integer for $n \geq 2$.

Solution: Let N be the largest power of 2 among the integers $1, \ldots, n$. Change the sum $\sum_{k=1}^{n} \frac{1}{k}$ so that each term is over a common denomenator. Then the numerator of each fraction except $1/N$ contains a factor of two. Thus, the numerator is a sum of even numbers plus one odd term (namely, the numerator over N). Thus, we have an odd number divided by an even number, which is not an integer.

Problem 3: Two points A and B lie on one side of a line L. Show how to construct the shortest path, consisting of line segments joined at their ends, from A to B which touches L.

Solution: Let B' be the symmetric point to B on the other side of L. The shortest path from A to B' is the straight line between them, which intersects L at a point P. It is a simple exercise to show that APB is the shortest path from A to B touching L.

Problem 4: If m and n are positive integers with m odd, determine the value of

$$d = GCD(2^m - 1, 2^n + 1).$$

Solution: Define integers k and l by

$$2^m - 1 = kd, \;\; 2^n + 1 = ld,$$

and then we obtain

$$2^m = kd + 1, \;\; 2^n = ld - 1,$$

and so for integers s and t we have

$$\begin{cases} 2^{mn} = (kd+1)^n = sd + 1 \\ 2^{mn} = (ld-1)^m = td - 1, \quad as m is odd. \end{cases}$$

Thus, we have $(s-t)d = -2$, and so d divides 2. But, clearly d is odd, so we must have that $d = 1$.