

Problems Sept 30, 2009

Problem 1: For $n = 1, 2, 3, \dots$, let $s(n)$ denote the sum of the digits of 2^n . Thus, for example, as $2^8 = 256$, we have $s(8) = 13$. Determine all positive integers n such that

$$s(n) = s(n + 1).$$

Problem 2:

Right triangle ABC has right angle at C and $\angle BAC = \theta$; the point D is chosen on AB so that $|AC| = |AD| = 1$; the point E is chosen on BC so that $\angle CDE = \theta$. The perpendicular to BC at E meets AB at F . Evaluate $\lim_{\theta \rightarrow 0} |EF|$.

Problem 3:

Evaluate

$$\sqrt[s]{2207 - \frac{1}{2207 - \frac{1}{2207 - \dots}}}$$

Express your answer in the form $\frac{a+b\sqrt{c}}{d}$, where a, b, c, d are integers.

SOLUTIONS TO SEPT 17 PROBLEMS:

Problem 1: Prove that there exist infinitely many positive integers which are not representable as sums of fewer than ten squares of odd natural numbers.

Solution: We show that the positive integers $72k + 42$, $k = 0, 2, 3, \dots$, cannot be expressed as sums of fewer than ten squares of odd natural numbers. Suppose that

$$72k + 42 = x_1^2 + x_2^2 + \dots + x_s^2,$$

for some $k \geq 0$ where x_1, \dots, x_s are odd integers and $1 \leq s < 10$. Now, $x_i^2 \equiv 1 \pmod{8}$ for $i = 1, 2, \dots, s$ and so considering the above displayed equation as an equality modulo 8, we have

$$s \equiv 2 \pmod{8}.$$

Since $1 \leq s < 10$ we must have that $s = 2$ and so

$$72k + 42 = x_1^2 + x_2^2.$$

Treating this equality as a congruence modulo 3, we obtain

$$x_1^2 + x_2^2 \equiv 0 \pmod{3}.$$

Since the square of an integer is congruent to 0 or 1 mod 3, we must have $x_1 \equiv x_2 \equiv 0 \pmod{3}$. Finally, reducing

$$72k + 42 = x_1^2 + x_2^2$$

modulo 9, we obtain the contradiction $6 \equiv 0 \pmod{9}$.

Problem 2: Prove that for each positive integer n there exists a circle in the xy -plane which contains exactly n lattice points (i.e. points of the form (a, b) where a and b are integers).

Partial Solution: For this problem, we will provide the beginning of a solution so that you can try to work out the details on your own. This can be a helpful exercise. Let P be the point $(\sqrt{2}, \frac{1}{3})$. Show that two different lattice points $R = (x_1, y_1)$ and $S = (x_2, y_2)$ must be at different distances from P . Then, start with a very small circle and expand it so that it reaches one lattice point at a time until you have the number you want.

Problem 3: Let

$$a_n = \frac{1}{4n+1} + \frac{1}{4n+3} - \frac{1}{2n+2}, n = 0, 1, 2, \dots$$

Does the infinite series $\sum_{n=0}^{\infty} a_n$ converge, and if so, what is its sum?

Solution: Let $s(N) = \sum_{n=0}^{\infty} a_n$, $N = 0, 1, 2, \dots$. We have that

$$\begin{aligned} s(N) &= \sum_{n=0}^N \left(\frac{1}{4n+1} + \frac{1}{4n+3} - \frac{1}{2n+2} \right) \\ &= \sum_{n=0}^N \left(\frac{1}{4n+1} - \frac{1}{4n+2} + \frac{1}{4n+3} - \frac{1}{4n+4} + \frac{1}{4n+2} - \frac{1}{4n+4} \right) \\ &= \sum_{m=1}^{4N+4} \frac{(-1)^{m-1}}{m} + \frac{1}{2} \sum_{m=1}^{2N+2} \frac{(-1)^{m-1}}{m}. \end{aligned}$$

Letting N go to ∞ , we have

$$\lim_{N \rightarrow \infty} s(N) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} + \frac{1}{2} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m}$$

$$\begin{aligned} &= \frac{3}{2} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \\ &= \frac{3}{2} \ln 2. \end{aligned}$$

Thus, the series converges with sum $\frac{3}{2} \ln 2$.