

## Problems Sept 30, 2009

**Problem 1:** For  $n = 1, 2, 3, \dots$ , let  $s(n)$  denote the sum of the digits of  $2^n$ . Thus, for example, as  $2^8 = 256$ , we have  $s(8) = 13$ . Determine all positive integers  $n$  such that

$$s(n) = s(n + 1).$$

**Problem 2:**

Right triangle  $ABC$  has right angle at  $C$  and  $\angle BAC = \theta$ ; the point  $D$  is chosen on  $AB$  so that  $|AC| = |AD| = 1$ ; the point  $E$  is chosen on  $BC$  so that  $\angle CDE = \theta$ . The perpendicular to  $BC$  at  $E$  meets  $AB$  at  $F$ . Evaluate  $\lim_{\theta \rightarrow 0} |EF|$ .

**Problem 3:**

Evaluate

$$\sqrt[8]{2207 - \frac{1}{2207 - \frac{1}{2207 - \dots}}}$$

Express your answer in the form  $\frac{a+b\sqrt{c}}{d}$ , where  $a, b, c, d$  are integers.

### SOLUTIONS TO SEPT 17 PROBLEMS:

**Problem 1:** Prove that there exist infinitely many positive integers which are not representable as sums of fewer than ten squares of odd natural numbers.

**Solution:** We show that the positive integers  $72k + 42$ ,  $k = 0, 2, 3, \dots$ , cannot be expressed as sums of fewer than ten squares of odd natural numbers. Suppose that

$$72k + 42 = x_1^2 + x_2^2 + \dots + x_s^2,$$

for some  $k \geq 0$  where  $x_1, \dots, x_s$  are odd integers and  $1 \leq s < 10$ . Now,  $x_i^2 \equiv 1 \pmod{8}$  for  $i = 1, 2, \dots, s$  and so considering the above displayed equation as an equality modulo 8, we have

$$s \equiv 2 \pmod{8}.$$

Since  $1 \leq s < 10$  we must have that  $s = 2$  and so

$$72k + 42 = x_1^2 + x_2^2.$$

Treating this equality as a congruence modulo 3, we obtain

$$x_1^2 + x_2^2 \equiv 0 \pmod{3}.$$

Since the square of an integer is congruent to 0 or 1 mod 3, we must have  $x_1 \equiv x_2 \equiv 0 \pmod{3}$ . Finally, reducing

$$72k + 42 = x_1^2 + x_2^2$$

modulo 9, we obtain the contradiction  $6 \equiv 0 \pmod{9}$ .

**Problem 2:** Prove that for each positive integer  $n$  there exists a circle in the  $xy$ -plane which contains exactly  $n$  lattice points (i.e. points of the form  $(a, b)$  where  $a$  and  $b$  are integers).

**Partial Solution:** For this problem, we will provide the beginning of a solution so that you can try to work out the details on your own. This can be a helpful exercise. Let  $P$  be the point  $(\sqrt{2}, \frac{1}{3})$ . Show that two different lattice points  $R = (x_1, y_1)$  and  $S = (x_2, y_2)$  must be at different distances from  $P$ . Then, start with a very small circle and expand it so that it reaches one lattice point at a time until you have the number you want.

**Problem 3:** Let

$$a_n = \frac{1}{4n+1} + \frac{1}{4n+3} - \frac{1}{2n+2}, n = 0, 1, 2, \dots$$

Does the infinite series  $\sum_{n=0}^{\infty} a_n$  converge, and if so, what is its sum?

**Solution:** Let  $s(N) = \sum_{n=0}^{\infty} a_n$ ,  $N = 0, 1, 2, \dots$ . We have that

$$\begin{aligned} s(N) &= \sum_{n=0}^N \left( \frac{1}{4n+1} + \frac{1}{4n+3} - \frac{1}{2n+2} \right) \\ &= \sum_{n=0}^N \left( \frac{1}{4n+1} - \frac{1}{4n+2} + \frac{1}{4n+3} - \frac{1}{4n+4} + \frac{1}{4n+2} - \frac{1}{4n+4} \right) \\ &= \sum_{m=1}^{4N+4} \frac{(-1)^{m-1}}{m} + \frac{1}{2} \sum_{m=1}^{2N+2} \frac{(-1)^{m-1}}{m}. \end{aligned}$$

Letting  $N$  go to  $\infty$ , we have

$$\lim_{N \rightarrow \infty} s(N) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} + \frac{1}{2} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m}$$

$$\begin{aligned} &= \frac{3}{2} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \\ &= \frac{3}{2} \ln 2. \end{aligned}$$

Thus, the series converges with sum  $\frac{3}{2} \ln 2$ .