Bucket handles and Solenoids

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1. Introduction

A **continuum** is a nonempty, compact, connected metric space. A nonempty compact connected subspace of a continuum X is called a **subcontinuum** of X.
A useful theorem about subcontinua, which every first year topology student should prove is:

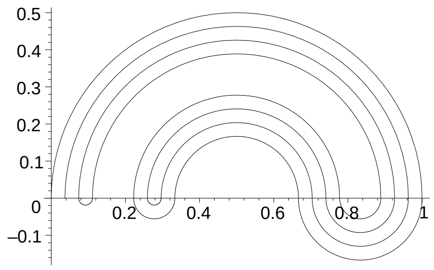
1.1. *Theorem.* *The intersection of a tower of subcontinua of X is a subcontinuum of X.*

A continuum is **indecomposable** if it cannot be written as the union of two proper subcontinua.
The **composant** of a point x in a continuum X is the union of all proper subcontinua of X which contain x. Here is a nice theorem.

1.2. *Theorem.* *The composants of a continuum are dense. The composants of an indecomposable continuum are pairwise disjoint.*

A classical example by contstructed by B. Knaster in the early 1920's is still of interest.

1.1. **The Bucket handle continuum.** Let B_0 be the set of all closed semicircles in the upper half plane centered on $c_0 = .5$ whose diameters have enpoints in the Cantor middle third set. Let $\overline{(B_0)}$ denote the reflection of B_0 about the x-axis. For $1 \leq n$, let
$B_n = c_n - c_0 + \frac{\overline{B_0}}{3^n}$, where $c_n = \frac{2.5}{3^n}$. Then $K_2 = B_0 \cup B_1 \cup B_2 \cup \cdots$ is an indecomposable continuum. The union of the semicircles whose endpoints are endpoints of the Cantor set is called the **visible composant**.
In the figure below, a portion of the visible composant is shown. It is a union of a tower of arcs all with (0,0) as one endpoint. (An arc is a continuum which is homeomorphic with the unit interval I = [0,1].)



It is possible to see the bucket handle as an intersection of a tower of 2-cells, each one obtained from the preceeding one by digging a canal out of it.
Why is K_2 indecomposable? Well, if you can convince yourself that K_2 (1) has only arcs for proper subcontinua, and (2) has no arcs with interior (in K_2), then it is pretty easy to argue that K_2 is indecomposable, for otherwise it would be the union of two arcs A and B and A - B would be a nonempty subset of the interior of K_2.
On the other hand there is another way to construct indecomposable continua which makes it possible to prove indecomposability rather easily.

2. The inverse limit construction

The **inverse limit** $X_\infty = \varprojlim (X_i, f_i)$ of a sequence $(X_i)_1^\infty$ of continua and surjective maps $f_i : X_{i+1} \to X_i$ is defined as the intersection of the subsets
$Q_n = \{(x_i)_1^\infty \,|\text{such that } x_i = f_i(x_{i+1}) \text{ for all } i = 1, \cdots, n\}$ of the product $\Pi_{i=1}^\infty X_i$. The spaces X_i are called the **factor spaces** of X_∞; the maps f_i are called the **bonding maps** of the inverse limit. For each positive integer i, the map $\pi_i : X_\infty \to X_i$ given by
$\pi_i((x_i)_1^\infty) = x_i$ is called the i^{th} **projection map**. It is continuous since it is the restriction of the projection $\rho_i : \Pi_{n=1}^\infty X_n \to X_i$ to X_∞.

2.1. ***Theorem.*** *The inverse limit X_∞ of continua is a continuum. Further if A is a subcontinuum of X_∞ then $A = \varprojlim (A_i, g_i)$, where $A_i = \pi_i(A)$ and $g_i = f_i|_{A_{i+1}}$.*

Proof. $X_\infty = \bigcap_{n=1}^\infty Q_n$ is a continuum by 1.1, since for each n, Q_n is homeomorphic with $\Pi_{i=n}^\infty$. Further, $(a_i)_1^\infty \in A$ if and only for each i, $a_i \in A_i$ and
$g_{i+1}(a_{i+1}) = f_{i+1}(a_{i+1}) = a_i$. □

One easy way to decide if two inverse limits are homeomorphic is described in the next theorem.

2.2. *Theorem. Two inverse limits $X_\infty = \varprojlim (X_i, f_i)$ and $Y_\infty = \varprojlim (Y_i, g_i)$ are homeomorphic if there is a sequence of homeomorphisms $h_i : X_i \to Y_i$ such that $h_i f_i(x) = g_i h_{i+1}(x)$ for each i and all $x \in X_{i+1}$.*

Proof. Define a function $h^* : X_\infty \to Y_\infty$ by $h^*((x_i)_{i=1}^\infty) = (h_i(x_i))_{i=1}^\infty$ for each positive integer i. h^* is into Y_∞ because $g_i h_{i+1}(x_{i+1}) = h_i f_i(x_{i+1}) = h_i(x_i)$, for each i. h^* is continuous since $\pi_i h^* = h_i \pi_i$ is continuous for each i. Call h^* the map **induced by the commutative diagram**.

$$\begin{array}{ccccccc}
X_1 & \xleftarrow{f_1} & X_2 & \xleftarrow{f_2} & \cdots X_i & \xleftarrow{f_i} & \cdots X_\infty \\
{\scriptstyle h_1}\downarrow & & {\scriptstyle h_2}\downarrow & & {\scriptstyle h_i}\downarrow & & {\scriptstyle h^*}\downarrow \\
Y_1 & \xleftarrow{g_1} & Y_2 & \xleftarrow{g_2} & \cdots Y_i & \xleftarrow{g_i} & \cdots Y_\infty
\end{array}$$

In the same way, using the inverse homeomorphisms $h_i^{-1} : Y_i \to X_i$, we can define an induced map from Y_∞ to X_∞ and show that it is the inverse of h^*. Thus a map induced by homeomorphisms is a homeomorphis. □

Here is the theorem describing the indecomposability of a continuum constructed as an inverse limit.
A map $f : X \to Y$ is called **indecomposable** provided whenever $X = A \cup B$, then $f(A) = Y$ or $f(B) = Y$. So a continuum is indecomposable if its identity map is indecomposable.

2.3. *Theorem. X_∞ is indecomposable provided each of its bonding maps is indecomposable.*

Proof. Suppose X_∞ is the union of two proper subcontinua A and B. Then there is an n_1 so that $\pi_{n_1}(A) \neq X_{n_1}$, otherwise by 2.1, $X_\infty = A$. Note also that for all $n \geq n_1$, $\pi_n(A) \neq X_n$. In the same way, there is an n_2 so that $\pi_{n_2}(B) \neq X_{n_2}$. So if we let n be the maximum of n_1 and n_2, then $\pi_n(A)$ and $\pi_n(B)$ are proper subcontinua of X_n. However, $\pi_{n+1}(X_\infty) = X_{n+1} = \pi_{n+1}(A) \cup \pi_{n+1}(B)$, $f_n(\pi_{n+1}(A)) = \pi_n(A)$, and $f_n(\pi_{n+1}(B)) = \pi_n(B)$, a contradiction to the assumed property of the bonding maps f_i. □

Another classical indecomposable continuum, studied somewhat later [2] than the bucket handle is the **dyadic solenoid** $S_2 = \varprojlim (X_i, f_i)$, where each factor space X_i is the unit circle S^1 and each bonding map f_i is the squaring map z^2.

2.4. *Corollary. S_2 is indecomposable.*

Proof. The squaring map is indecomposable, so the theorem follows from 2.3. □

Note that any inverse limit of circles where the bonding maps are power maps z^i, $i \geq 2$ is indecomposable by the same argument. Such continua are referred to as **solenoids**. Solenoids are the only indecomposable continua which admit a group mulitiplication[2]. Another class of indecomposable continua are the **Knaster continua**, which are defined as the continua obtained as inverse limits by using the unit interval I as the factor space and standard maps $w_n, n > 1$, defined by

$$w_n(x) = \begin{cases} nx - i & \text{if } i \text{ is even and } 0 \leq \frac{i}{n} \leq x \leq \dfrac{i+1}{n} \leq 1, \\ i + 1 - nx & \text{if } i \text{ is odd and } 0 < \frac{i}{n} \leq x \leq \dfrac{i+1}{n} \leq 1 \end{cases}$$

Clearly, the map $w_n : I \to I$ is indecomposable when $n > 1$, and so one has the corollary.

2.5. ***Corollary***. *Any Knaster continuum is indecomposable.*

2.6. ***Exercise***. *Show that the dyadic solenoid has the property that each proper subcontinuum is an arc, and has a basis of neighborhoods consisting of sets homeomorphic with the Cantor set crossed with an arc.*

3. An embedding theorem

A homeomorphism of a space X onto a subspace of a space Y is called an **embedding** of X into Y.

3.1. ***Theorem.*** *(A version of the Anderson-Choquet embedding theorem) Suppose the factor spaces X_i of X_∞ are all embedded in a common continuum X (with metric d, say) and the bonding maps f_i satisfy the conditions: (1) There is a positive number K so that* *for each i and each $x \in X_{i+1}$, $d(x, f_i(x)) < \frac{K}{2^i}$, and (2) for each factor space X_i and each positive δ, there is a positive δ' so that if $k > i$ and $p, q \in X_k$ with $d(f_{ik}(p), f_{ik}(q)) > \delta$ then $d(p, q) > \delta'$. Then for each $x = (x_i)_i^\infty \in X_\infty$, the sequnce of coordinates x_i converges to a unique point $h(x) \in X_\infty$ and the function $h : X_\infty \to X$ is an embedding.*

Proof. Condition (1) suffices to guarantee that the sequence of coordinates of a point in X_∞ is Cauchy, and so converges. So the function h is well defined.
To see that h is continuous, let $\epsilon > 0$ be given. Choose N so large that $\sum_{i=N}^{\infty} \frac{K}{2^i} < \epsilon/4$ and and hence by (1) $d(x_N, h(x)) \leq d(x_N, x_{N+1}) + d(x_{N+1}, h(x)) \leq K/2^N + d(x_{N+1}, h(x)) < \epsilon/4$ for all $x \in X_\infty$. Hence if $x \in X_\infty$ and U is an open set in X_N about x_N of diameter less than $\epsilon/4$. Then $\pi_N^{-1}(U)$ is a basic open set in X_∞. Further, if $y = (y_i)_i^\infty \in \pi_N^{-1}(U)$, then $y_N \in U$, and so $d(h(x), h(y) \leq d(h(x), x_N) + d(x_N, y_N) + d(y_N, h(y)) < \epsilon$. This shows h is continuous.
To see that h is 1-1, suppose $x \neq y$ where x and y are in X_∞. Since $x \neq y$, there is an i_0 so that $x_{i_0} \neq y_{i_0}$. Let $\delta = d(x_{i_0}, y_{i_0})/2$, and use (2) to get a $\delta^{\backslash prime} > 0$ so that for $k > i_0$ and $p, q \in X_k$, if $d(f_{i_0 k}(p), f_{i_0 k}(q)) > \delta$, then $d(p, q) > \delta'$. Hence for $k > i_0$, $d(x_k, y_k) > \delta'$, and so $d(h(x), h(y)) \geq \delta'$. This shows h is an embedding. □

3.2. *Theorem*. *K_2 is homeomorphic with an inverse limit of arcs with indecomposable bonding maps.*

Proof. To apply 3.1 to K_2, for each positive integer i, let $a_i = (\frac{1}{3^i}\frac{5}{6}, -\frac{1}{3^i}\frac{1}{6})$, the bottom of the $(i+1)^{st}$ set of semicircles B_i counting from the right, and let A_i be the subarc of the visible composant with endpoints a_0 and a_i. These arcs are the factor spaces. The bonding map $r_i : A_{i+1} \to A_i$ is the retraction which projects a point on the first quarter circle starting at a_{i+1} horizontally to the right onto the matching point on the end quartercircle of A_i, and otherwise projects a point radially onto the nearest point of A_i, except near a_{i+1}. There the last quarter circle of A_{i+1} is mapped homemorphically onto the circular arc from a_0 to $(.5 + \cos((1 - 2(1/2)^i)\pi), .5 + \sin((1 - 2(1/2)^i)\pi))$, and the circular arc from $(1 - (1/2)^i)\pi$ to π on the last semicircle of A_{i+1} is mapped homeomorphically onto the circular arc from $(.5 + \cos((1 - 2(1/2)^i)\pi), .5 + \sin((1 - 2(1/2)^i)\pi))$ to $(.5 + \cos((1 - (1/2)^i)\pi), .5 + \backslash sin((1 - (1/2)^i)\pi))$. It is easy to see that r_i is indecomposable. Note that $d(x, r_1(x)) < 1/3$, $d(x, r_2(x)) < 1/3^2$, and in general $d(x, r_i(x)) < 1/3^i$, so condition (1) of 2.3 is satisfied with $K = 1$. So the function $h : X_\infty \to K_2$ is well-defined and continuous. Since $K_2 = \overline{\bigcup_{i=1}^\infty A_i}$, the continuity of h guarantees that h is onto. We verify that h is 1-1 directly. Suppose $h(x) = h(y)$. Then for some n, x_n and y_n lie in the same circular arc of A_n. Then from then on the coordinates x_{n+i} and y_{n+i} lie along the outer third of the radii drawn from the center of that circular arc. But since $h(x) = h(y)$, for some $m > n$, x_i and y_i lie in the same radius for all $i > m$. But then $x_i = r_i(x_{i+1}) = r_i(y_{i+1}) = y_i$ for all $i > m$. So $x = y$, and h is 1-1. □

So we have realized K_2 as the inverse limit $\varprojlim (A_i, r_i)$ of arcs with indecomposable bonding maps. Now we want to prove the

3.3. *Theorem*. *K_2 is homeomorphic with the Knaster continuum $\varprojlim (I, w_2)$.*

Proof. Let g_1 be any homeomorphism from A_1 to I such that $g_1(a_0) = 0$ and $g_1(a_1) = 1$. Then define $g_2 : A_2 \to I$ by

$$g_2(x) = \begin{cases} \dfrac{g_1(x)}{2} & \text{if } x \in A_1 \\ 1 - \dfrac{g_1(r_1(x))}{2} & \text{if } x \in A_2 - A_1 \end{cases}$$

The function g_2 is a homeomorphism such that $g_1 r_1 = w_2 g_2$. Using the same method, we successively define homeomorphisms $g_i : A_i \to I$ so that $g_{i-1} r_{i-1} = w_2 g_i$. This sequence induces a homeomorphism $g^* : \varprojlim (A_i, r_i) \to \varprojlim (I, w_2)$.

$$\begin{array}{ccccccc} A_1 & \xleftarrow{r_1} & A_2 & \xleftarrow{r_2} & \cdots A_i & \xleftarrow{r_i} & \cdots \varprojlim (A_i, r_i) \\ g_1 \downarrow & & g_2 \downarrow & & g_i \downarrow & & g^* \downarrow \\ I & \xleftarrow{w_2} & I & \xleftarrow{w_2} & \cdots I & \xleftarrow{w_2} & \cdots \varprojlim (I, w_2) \end{array}$$

□

4. Where is $(x_i)_{i=1}^{\infty}$?

It is an interesting exercise to ask where a given point in $\varprojlim (I, w_2)$ goes under the homeomorphism $h(h^*)^{-1} : \varprojlim (I, w_2) \to K_2$ constructed previously.
For example, it is easy to see what point goes to $a_0 = (0, 0)$: the point $(0, 0, \cdots)$.
Where does the point $x = (1, 1/2, 1/4, \cdots) go$? First find its image $y = (h^*)^{-1}(x)$ in $\varprojlim (A_i, r_i)$, then push that on to $h(y)$ in K_2. By the way h^* was constructed, we can compute that $y = (a_1, a_1, a_1, \cdots)$. Then $h(y) = a_1$.
Every inverse limit with one factor space and one bonding map has a **shift homeomorphism**, s, defined by $s((x_1, x_2, x_3, \cdots)) = (x_2, x_3, \cdots)$. So the bucket handle K_2 has a 'shift homeomorphism', $\overline{s} = h(h^*)^{-1} s h^* h^{-1} : K_2 \to K_2$. Where does the shift homeomorphism take the point a_0? It is pretty easy to see that this point is fixed under the shift. What about $\overline{s}(a_1)$? First push a_1 to $(1, 1/2, 1/4, \cdots)$, then shift it to $(1/2, 1/4, \cdots)$, then pull it back to $(h_1^{-1}(1/2), h_1^{-1}(1/2), h\_1^{-1}(1/2), \cdots)$ in $\varprojlim (A_i, r_i)$ then push to $h_1^{-1}(1/2)$ under h. Note that the shift homeomorphism on K_2 depends on the homemorphism h_1 of I to A_1. However, the fixed point a_0 of $\overline{s}$ does not depend on h_1.

4.1. *Exercise:* *There is another fixed point of the shift $\overline{s}$ defined above. Where does it come from in $\varprojlim (I, w_2)$ and where is it at in K_2? Does its location in K_2 depend on h_1?*

5. Mapping the dyadic solenoid lightly onto the bucket handle

David Bellamy [1] was the first to see this, I think.
First you have to think of the group S_1 as the quotient space $I/\{0 = 1\}$ of I with the operation of addition mod 1. In this setting, the squaring map on S^1 becomes the doubling mod 1 map on I. $dbl(t) = \begin{cases} 2t & \text{if } 0 \le t \le 1/2 \\ 2t - 1 & \text{if } 1/2 \le t \le 1 \end{cases}$

Note that this is continuous on $I/\{0=1\}$, and commutes with w_2. Hence, there is an induced map w_2^* of the solenoid onto the bucket handle.

$$\begin{array}{ccccc} I/\{0=1\} & \xleftarrow{dbl} & I/\{0=1\} & \xleftarrow{dbl} & \cdots S_2 \\ {\scriptstyle w_2}\big\downarrow & & {\scriptstyle w_2}\big\downarrow & & {\scriptstyle \cdot\, w_2^*}\big\downarrow \\ I & \xleftarrow{w_2} & I & \xleftarrow{w_2} & \cdots \varprojlim (I, w_2) \end{array}$$

5.1. ***Exercise***. *Use the diagram above to show that* $S_2/\{x=-x\} = K_2$.

6. The hyperspaces of subcontinua of K_2 and S_2

If X is any continuum with metric d say, and A and B are subcontinua of X, the **Hausdorff distance** between A and B, $H_d(A,B)$ is defined by

$H_d(A,B) = \inf\{\backslash epsilon|$each point of either set is within ϵ of some point of the other.$\}$

The set of all subcontinua of X is denoted by $C(X)$.

6.1. ***Theorem***. *H_d is a metric on $C(X)$, and $C(X)$ topologized by this metric is a continuum.*

$C(X)$, topologized with the Hausdorff metric, is called the **hyperspace of subcontinua** of X

6.2. ***Theorem***. *$C(S_2)$ is homeomorphic with the cone over S_2. $C(K_2)$ is homeomorphic with the cone over K_2.*

References

[1] David Bellamy, *A tree-like continuum without the fixed point property*, Houston Math. J. **6** (1979), 1-13.
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[4] K. Kuratowski, **Topology**, vol. 1, Amsterdam, 1968.