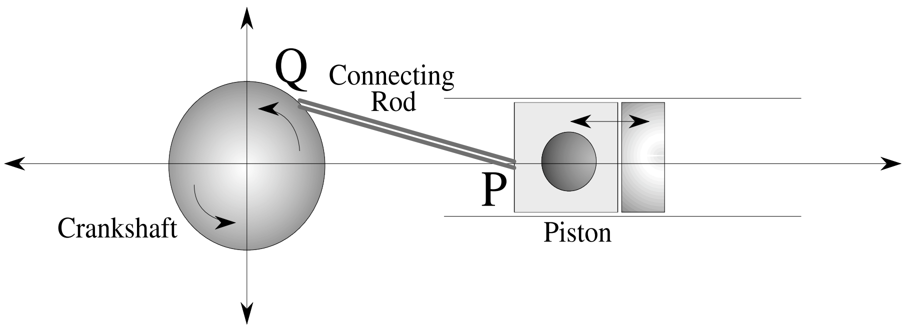
Calculus I

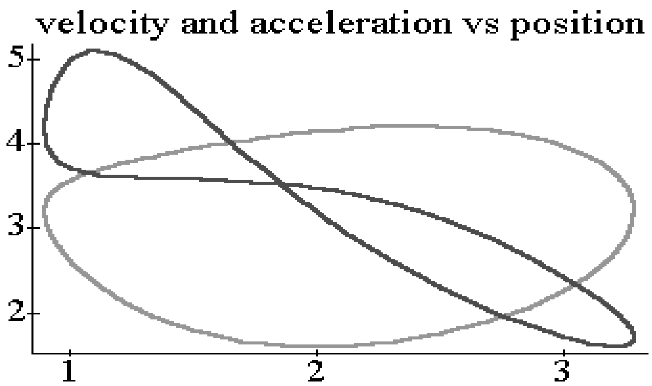
by

Neal Koblitz

A Text for Math 113

(1997–1998)





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Preface

Introduction to the Kentucky Edition:

This text is a minor adaptation of Professor Neal Koblitz's notes from his University of Washington Math 124 course. Professor Koblitz generously permitted their use in this extension to Kentucky of his idea of a complete calculus text for which students pay only the reproduction costs. Errors which have been introduced into the text during the adaptation process are the responsibility of the Affordable Text Project of the University of Kentucky, edited by Paul Eakin, Carl Eberhart, and Avinash Sathaye.

This project has been supported by the National Science Foundation (NSF) **Calculus** and **Advanced Technical Education** (ATE) programs.

Prerequisites: Topics You Should Have Studied Before Taking Ma 113. Basic High School Algebra, Plane Geometry and Trigonometry are prerequisites to MA113. If you have taken a pre-calculus course (perhaps called *advanced math* or *functions* in high school or Ma 110 *Analytic Geometry and Trigonometry* here at UK), you are prepared to take this course.

The following are a few of the ideas from basic algebra and geometry which you will encounter in MA113. The list is far from complete. You should have previously studied and understood all of them. Although you may have forgotten the details of most you should be confident that with a *very* brief review you could re-establish your ability to solve problems with these ideas.

If you have not previously encountered a significant number of the following topics, then you may want to speak to your instructor or teaching assistant. They can discuss your background and preparation with you and help you make sure that you are in the right course.

- Cartesian coordinates, coordinates of points in the plane
- slope of a line,slope of a line perpendicular to a given line
- equation of a line through two points,equation of line given point and slope
- area of circle, rectangle, triangle; surface area of sphere, cylinder, rectangular box
- volume of sphere, rectangular box, cylinder, cone
- circumference of circle, perimeter of rectangle, triangle
- quadratic formula,multiplication and "long division" of polynomials
- addition and multiplication of rational functions
- factorization of $x^2 - y^2$, $x^3 - y^3$, $x^3 + y^3$
- equation of a circle with given center and radius
- points of intersection of a circle and a line
- factor $x^2 - y^2$ and $x^3 - y^3$, expand $(a + b)^2$ and $(a + b)^3$
- absolute value, right triangle, similar triangles
- Pythagorean Theorem, law of sines, law of cosines, radian measure

You should have worked with and previously explored the graphs of functions like the following. You should be able to graph these on a graphics calculator or computer.

- x,x^2,x^3,x^2-3x+2,e^x, e^{-x}
- x^3-x, $1/x$, $1/x^2$, $x+1$, x^2+1, x^3+1
- $1/(x+1)$, $1/(x^2+1)$,$x-1$, $(x-1)^2+2$, $(x^3-1)-3$, $1/(x-1)+1$
- $1/(x^2-1)+1$, $|x|$,$|x-2|$, $\log(x)$, $\ln(x)$,e^{-x^2}
- $\sin(x)$, $\cos(x)$, $\tan(x)$

Advice for Students

The overwhelming emphasis in this course is on applications — story problems. To master story problems one needs a tremendous amount of practice — doing homework, worksheet, and practice exam problems. For this reason, this course will require much more of your time than would a high school or community college calculus course that does not have such a focus on word problems.

Here are some pointers for doing story problems:

1. Carefully read each problem twice before writing anything.
2. Assign letters to quantities that are described only in words; draw a diagram if appropriate.
3. Decide which letters are constants and which are variables. A letter stands for a constant if its value remains the same throughout the problem.
4. Using math notation, write down what you know and then write down what you want to find.
5. Decide what category of problem it is (this might be obvious if the problem comes at the end of a particular chapter, but will not necessarily be so obvious if it comes on an exam covering several chapters).
6. Double check each step as you go along; don't wait until the end to check your work.
7. Use common sense; if an answer is out of the range of practical possibilities, then check your work to see where you went wrong.

Suggestions for Using These Notes

1. Read the example problems carefully, filling in any steps that are left out (ask someone if you can't follow the solution to a worked example).
2. Later use the worked examples to study by covering the solutions, and seeing if you can solve the problems on your own.
3. Give yourself the practice midterms and finals in an exam situation (without access to material other than a single sheet of notes, and with the same time limit as on the exam).
4. The answers to homework and practice exam problems that are in the back of the notes should be used only as a final check on your work, not as a crutch. Keep in mind that sometimes an answer could be expressed in various ways that are algebraically equivalent, so don't assume that your answer is wrong just because it doesn't have exactly the same form as the answer in the back.

Formulas Used in This Course

Algebra

Completing the square: $X^2 + bX + c = (X + \frac{b}{2})^2 \; - \frac{b^2}{4} + c$.

Quadratic formula: roots of $aX^2 + bX + c$ are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Exponents: $a^b \cdot a^c = a^{b+c}$; $\quad \dfrac{a^b}{a^c} = a^{b-c}$; $\quad (a^b)^c = a^{bc}$; $\quad a^{1/b} = \sqrt[b]{a}$.

Geometry

Circle: circumference $= 2\pi r$; area $= \pi r^2$.

Sphere: vol $= \frac{4}{3}\pi r^3$; surface area $= 4\pi r^2$.

Cylinder: vol $= \pi r^2 h$; lateral area $= 2\pi rh$; total surface area $= 2\pi rh + 2\pi r^2$.

Cone: vol $= \frac{1}{3}\pi r^2 h$; lateral area $= \pi r\sqrt{r^2 + h^2}$; total surface area $= \pi r\sqrt{r^2 + h^2} + \pi r^2$.

Analytic geometry

Point-slope formula for straight line: $y = y_0 + m(x - x_0)$.

Circle centered at (h, k): $(x - h)^2 + (y - k)^2 = r^2$.

Ellipse with semimajor axis along x-axis and semiminor axis along y-axis: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Trigonometry

$\sin = \dfrac{\text{opposite}}{\text{hypotenuse}}$; $\cos = \dfrac{\text{adjacent}}{\text{hypotenuse}}$; $\tan = \dfrac{\text{opposite}}{\text{adjacent}}$;

$\sec = \dfrac{1}{\cos}$; $\csc = \dfrac{1}{\sin}$; $\cot = \dfrac{1}{\tan}$; $\tan = \dfrac{\sin}{\cos}$; $\cot = \dfrac{\cos}{\sin}$;

$\sin x = \cos\left(\dfrac{\pi}{2} - x\right)$; $\cos x = \sin\left(\dfrac{\pi}{2} - x\right)$;

$\sin(x + \pi) = -\sin x$; $\cos(x + \pi) = -\cos x$.

Law of cosines: $a^2 = b^2 + c^2 - 2bc\cos A$.

Law of sines: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

Sine of sum of angles: $\sin(x + y) = \sin x \cos y + \cos x \sin y$.

$\sin^2$ and $\cos^2$ formulas: $\sin^2 + \cos^2 = 1$; $\sin^2 x = \dfrac{1 - \cos(2x)}{2}$; $\cos^2 x = \dfrac{1 + \cos(2x)}{2}$.

Contents

CHAPTER 1

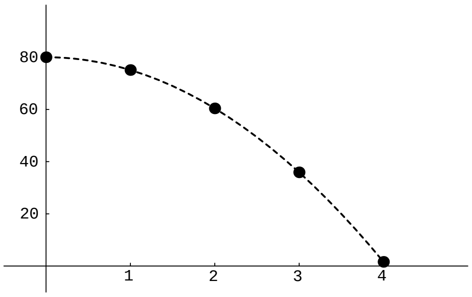
Straight Lines, Polygonal Lines, Slopes, Distance, Circles

Much of the mathematics in this chapter will be a review for you.

In the (x, y) coordinate system we normally write the x-axis horizontally, with positive numbers to the right of the origin, and the y-axis vertically, with positive numbers above the origin. That is, unless stated otherwise, we take "rightward" to be the positive x-direction and "upward" to be the positive y-direction. We normally choose the same scale for the x- and y-axes. For example, the line joining the origin to the point (a, a) makes an angle of $45°$ with the x-axis (and also with the y-axis).

But in applications, more descriptive variable names are used, and often different scales are chosen along the two axes. In fact, the quantities represented by the two variables can be so different that the same scale has no meaning! For example, suppose you drop something from a window, and you want to study how its height above the ground changes from second to second. It is natural to let the letter t denote the time (the number of seconds since the object was released) and to let the letter s denote the height in meters. For each t (say, at one-second intervals) you have a corresponding height s. This information can be tabulated, and then plotted on the (t, s) coordinate plane. Here is an example:

t (sec)	s (meters)
0	80.0
1	75.1
2	60.4
3	35.9
4	1.6



We use the word " quadrant " for each of the four regions the plane is divided into: the 1st quadrant is where points have both coordinates positive, and the 2nd, 3rd, and 4th quadrants are counted off counterclockwise as follows:

2nd	1st
3rd	4th

Suppose we have two points $A(2, 1)$ and $B(3, 3)$ in the (x, y)-plane. We often want to know the change in x-coordinate (also called the "horizontal distance") in going from A to B. This is often written Δx, where the meaning of Δ is "change in" (thus, Δx can be read as "change in x" — it denotes a single number, and should not be read as "delta times x"). In our example, $\Delta x = 3 - 2 = 1$. Similarly, the "change in y" is written Δy. In our example, $\Delta y = 3 - 1 = 2$, the difference between the y-coordinates of the two points. It is the vertical distance you have to move in going from A to B. The general formulas for the change in x and the change in y between a point (x_1, y_1) and a point (x_2, y_2) is:

$$\Delta x = x_2 - x_1, \qquad \Delta y = y_2 - y_1.$$

Note that "the change" is algebraic. For example the values of Δx and Δy in going from B to A will be respectively -1 and -2. You should follow the above formula all the time.

If we have two points $A(x_1, y_1)$ and $B(x_2, y_2)$, then we can draw one and only one line through both points. By the *slope* of this line we mean the ratio of Δy to Δx. The slope is often denoted m: $m = \Delta y/\Delta x = (y_2 - y_1)/(x_2 - x_1)$. For example, the line joining the points A and B in the last paragraph has slope 2.

Note that this slope does not depend on which two points on the line are taken and the order in which they are taken. For instance, if we take the change from B to A, then we will get the ratio of $(-2)/(-1)$ which still evaluates to 2.

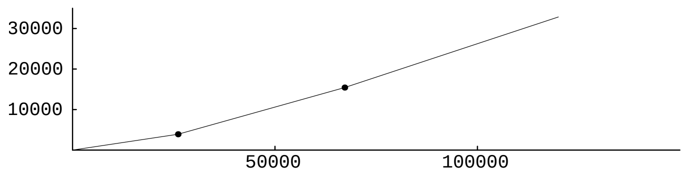
There is a very simple procedure to find all points on the line joining $A(x_1, y_1)$ and $B(x_2, y_2)$. Any point $P(x, y)$ on the line has coordinates

$$x = (1-t)x_1 + tx_2 \ , \ y = (1-t)y_1 + ty_2 \text{ where } t \text{ is any real number.}$$

Convince yourself that $t = 0$ gives the point A, $t = 1$ gives the point B and values of t between $0, 1$ give the line segment from A to B. You should investigate how the different values of t correspond to various subdivisions of the segment AB. Also verify once and for all that the slope is the same if we choose the points A, B or A, P, regardless of the value of t.

Example 1.1. According to the 1990 U.S. federal income tax schedules, a head of household pays 15% on income up to \$26050. If the taxable income is between \$26050 and \$134930, then, in addition, 28% must be paid on the amount between \$26050 and \$67200, and 33% paid on the amount over \$67200 (if any). Interpret the tax bracket information (15%, 28%, or 33%) using mathematical terminology, and graph the tax on the y-axis against the taxable income on the x-axis.

The percentage, when converted to a decimal (0.15, 0.28, and 0.33), is the *slope* of the straight line which is the graph of tax for the corresponding tax bracket. The tax graph is what's called a *polygonal line* , i.e., it's made up of several straight line segments of different slopes. The first line starts at the point (0,0) and heads upward with slope 0.15 (i.e., it goes upward 15 for every increase of 100 in the x-direction), until it reaches the point above $x = 26050$. Then the graph "bends upward," i.e., the slope changes to 0.28. As the horizontal coordinate goes from $x = 26050$ to $x = 67200$, the line goes upward 28 for each 100 in the x-direction. At $x = 67200$ the line turns upward again, and continues with slope 0.33. See the graph below.



The most familiar form of the equation of a straight line is: $y = mx + b$. Here m is the slope of the line: if you increase x by 1, the equation tells you that you have to increase y by m. If you increase x by Δx, then y increases by $\Delta y = m\Delta x$. The number b is called the *y-intercept* , because it is y-coordinate where the line crosses the y-axis.

If you know two points on a line, the formula $m = (y_2 - y_1)/(x_2 - x_1)$ gives you the slope. Once you know a point and the slope, then the y-intercept can be found by substituting the coordinates of the point in the equation: $y_1 = mx_1 + b$, i.e., $b = y_1 - mx_1$. Alternatively, one can use the " point-slope " form of the equation of a straight line, which is: $(y - y_1) = m(x - x_1)$. This relation says that "between the point (x_1, y_1) and any other point (x, y) on the line, the change in y divided by the change in x is the slope m of the line."

In general the equation of a line can be expressed as a linear relation of the form $lx + my + n = 0$ where at least one of the coefficients l, m are nonzero. You should convince yourself that every possible line has such an equation and conversely such an equation always gives a line for its graph.

A convenient way of writing the final equation of a line thru two points (x_1, y_1) and (x_2, y_2) is:

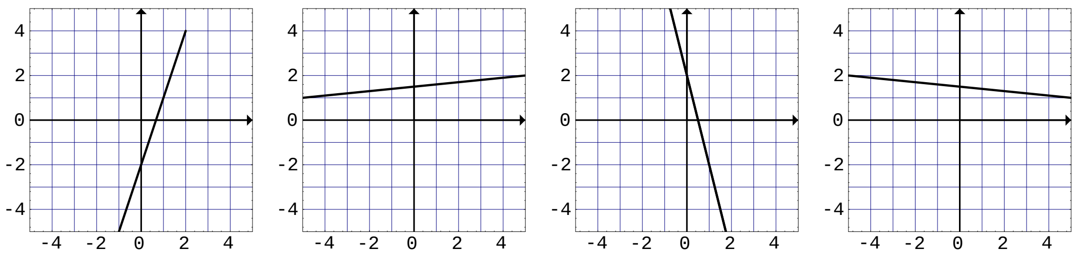
$$(x_2 - x_1)(y - y_1) = (y_2 - y_1)(x - x_1)$$

You should verify that this is correct by checking that the equation is satisfied by both the points and that it has the correct form for the equation of a line. This is enough justification!

For example, if we want to find the equation of the line joining our earlier points $A(2, 1)$ and $B(3, 3)$, we can use this formula:

$$\frac{y - 1}{3 - 2} = \frac{x - 2}{3 - 1} \qquad \text{so that} \qquad y - 1 = 2(x - 2), \qquad \text{i.e.,} \qquad y = 2x - 3.$$

The slope m of a line in the form $y = mx + b$ tells us the direction in which the line is pointing. If m is positive, the line goes into the 1st quadrant as you go from left to right. If m is large and positive, it has a steep incline; while if m is small and positive, then the line has a small angle of inclination. If m is negative, the line goes into the 4th quadrant as you go from left to right. If m is a large negative number (large in absolute value), then the line points steeply downward; while if m is negative but near zero, then it points only a little downward. These four possibilities are illustrated in the graphs below.



$m = 3$ $\qquad$ $m = 0.1$ $\qquad$ $m = -4$ $\qquad$ $m = -0.1$

If $m = 0$, then the line is horizontal: its equation is simply $y = b$.

There is one type of line that cannot be written in the form $y = mx + b$, namely, a vertical line. A vertical line has an equation of the form $x = a$. Sometimes one says that a vertical line has an "infinite" slope.

Sometimes it is useful to find the x-intercept of a line $y = mx + b$. This is the x-value when $y = 0$. Setting $mx + b$ equal to 0 and solving for x gives: $x = -b/m$. For example, the line $y = 2x - 3$ through the points $A(2, 1)$ and $B(3, 3)$ has x-intercept 1.5. Convince yourself that a vertical line like $x = 5$ has x-intercept 5 and has no y-intercept. The special vertical line $x = 0$ coincides with the y-axis and may be said to have infinitely many y-intercepts.

EXAMPLE 1.2. Suppose that you are driving to Seattle at constant speed, and notice that after you have been traveling for 1 hour (i.e., $t = 1$), you pass a sign saying 110 miles to Seattle; and after driving another half-hour you pass a sign saying 85 miles to Seattle. Using the horizontal axis for the time t and the vertical axis for the distance y from Seattle, graph and find the equation $y = mt + b$ for your distance from Seattle. Find the slope, y-intercept, and t-intercept, and describe the practical meaning of each.

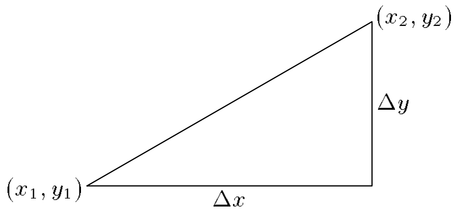
The graph of y versus t is a straight line, because you are traveling at constant speed. The line passes through the two points $(1, 110)$ and $(1.5, 85)$. So its slope is $m = (85 - 110)/(1.5 - 1) = -50$. The meaning of the slope is that you are traveling at $|-50| = 50$ mph. The slope m is negative because you are traveling *toward* Seattle, i.e., your distance y is *decreasing*. The word "velocity" is often used for $m = -50$, when we want to indicate direction, while the word "speed" refers to the magnitude (absolute value) of velocity, which is 50 mph. To find the equation of the line, we use the point-slope formula:

$$\frac{y - 110}{t - 1} = -50, \qquad \text{so that} \qquad y - 110 = -50(t - 1) \qquad \text{so that } y - 50t + 160.$$

The meaning of the y-intercept 160 is that when $t = 0$ (when you started the trip) you were 160 miles from Seattle. To find the t-intercept, set $0 = -50t + 160$, so that $t = 160/50 = 3.2$. The meaning of the t-intercept is: the time when you'll be in Seattle. After traveling 3 hrs 12 min, your distance y from Seattle will be 0.

1. Distance Between Two Points; Circles

Given two points (x_1, y_1) and (x_2, y_2), recall that their horizontal distance from one another is $\Delta x = x_2 - x_1$, and their vertical distance from one another is $\Delta y = y_2 - y_1$. The actual distance from one point to the other is the hypotenuse of a right triangle with legs Δx and Δy:



The Pythagorean theorem then says that the distance between the two points is the square root of the sum of the squares of the horizontal and vertical sides:

$$\text{distance} = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

For example, the distance between our two points $A(2, 1)$ and $B(3, 3)$ is equal to $\sqrt{(3-2)^2 + (3-1)^2} = \sqrt{5} = 2.236\cdots$.

As a special case of the distance formula, suppose we want to know the distance of a point (x, y) to the origin. According to the distance formula, this is equal to $\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$.

A point (x, y) is at a distance r from the origin if and only if $\sqrt{x^2 + y^2} = r$, or, if we square both sides: $x^2 + y^2 = r^2$. This is the **equation of the circle of radius r centered at the origin**. The special case $r = 1$ is called the **unit circle** ; its equation is $x^2 + y^2 = 1$.

Similarly, if $C(h,k)$ is any fixed point, then a point (x,y) is at a distance r from the point C if and only if $\sqrt{(x-h)^2+(y-k)^2}=r$, i.e., if and only if

$$(x-h)^2+(y-k)^2=r^2.$$

This is the **equation of the circle of radius** r **centered at the point** (h,k). For example, the circle of radius 5 centered at the point $(0,5)$ on the y-axis has equation $(x-0)^2+(y-5)^2=25$. If we expand $(y-5)^2=y^2-10y+25$ and cancel the 25 on both sides, we can rewrite this as: $x^2+y^2-10y=0$.

It is easy to recognize the equation of a circle. Its terms of degree 2 can be always arranged to be x^2+y^2 after dividing the whole equation by a suitable number if needed. For example you should convince yourself that the equation $2x^2+2y^2+4x+8y=5$ describes a circle with center $(-1,-2)$ and radius $\sqrt{15/2}$. Hint: expand the model equation given above and compare coefficients after dividing the given equation by 2.

HOMEWORK 1.

1. For each pair of points $A(x_1, y_1)$ and $B(x_2, y_2)$ find (i) Δx and Δy in going from A to B, (ii) the slope of the line joining A and B, (iii) the equation of the line joining A and B in the form $y = mx + b$, and (iv) the distance from A to B:

(a) $A(2, 0)$, $B(4, 3)$; (b) $A(1, -1)$, $B(0, 2)$;
(c) $A(0, 0)$, $B(-2, -2)$; (d) $A(-2, 3)$, $B(4, 3)$;
(e) $A(-3, -2)$, $B(2, 3)$; (f) $A(0.01, -0.01)$, $B(-0.01, 0.05)$.

2. Graph each of the following lines, after changing to the form $y = mx + b$, and also find the y-intercept and x-intercept:

(a) $y - 2x = 2$; (b) $x + y = 6$; (c) $x = 2y - 1$; (d) $3 = 2y$; (e) $2x + 3y + 6 = 0$.

3. Find the equation of the circle of radius 3 centered at: (a) $(0, 0)$; (b) $(5, 6)$; (c) $(-5, -6)$; (d) $(0, 3)$; (e) $(0, -3)$; (f) $(3, 0)$.

4. Graph the circles

(a) $x^2 + y^2 + 10y = 0$; (b) $x^2 - 10x + y^2 = 24$; (c) $x^2 - 6x + y^2 - 8y = 0$.

5. Let x stand for temperature in degrees Celsius (centigrade), and let y stand for temperature in degrees Fahrenheit. A temperature of 0°C (freezing of water) corresponds to 32°F, and a temperature of 100°C (boiling of water) corresponds to 212°F. Find the equation of the line that relates temperature Fahrenheit y to temperature Celsius x. Graph the line, and find the y- and x-intercepts. What is the practical meaning of the intercepts?

6. A car rental firm has the following charges for a certain type of car: \$25 per day with 100 free miles included, \$0.15 per mile for more than 100 miles. Suppose you want to rent a car for one day, and you know you'll use it for more than 100 miles. What is the equation relating the cost y to the number of miles x that you drive the car?

7. An instructor gives a 100-point final exam, and decides that a score 90 or above will be a grade of 4.0, a score of 40 or below will be a grade of 0.0, and between 40 and 90 the grading will be linear. Let x be the exam score, and let y be the corresponding grade. Find a formula of the form $y = mx + b$ which applies to scores x between 40 and 90.

8. A photocopy store advertises the following prices: 5¢/per copy for the first 20 copies, 4¢/per copy for the 21st through 100th copy, and 3¢/per copy after the 100th copy. Let x be the number of copies, and let y be the total cost of photocopying. (a) Graph the cost as x goes from 0 to 200 copies. (b) Find the equation in the form $y = mx + b$ that tells you the cost of making x copies when x is more than 100.

9. In the Kingdom of Zog the tax system works as follows. Someone who earns less than 100 gold coins per month pays no tax. Someone who earns between 100 and 1000 golds coins pays tax equal to 10% of the amount over 100 gold coins that he or she earns. Someone who earns over 1000 gold coins must hand over to the King all of the money earned over 1000 in addition to the tax on the first 1000. (a) Draw a graph of the tax paid y versus the money earned x, and give formulas for y in terms of x in each of the regions $0 \le x \le 100$, $100 \le x \le 1000$, and $x \ge 1000$. (b) Suppose that the King of Zog decides to use the second of these line segments (for $100 \le x \le 1000$) for $x \le 100$ as well. Explain in practical terms what the King is doing, and what the meaning is of the y-intercept.

10. Market research tells you that if you set the price of an item at \$1.50, you will be able to sell 5000 items; and for every 10 cents you lower the price below \$1.50 you will be able to sell another 1000 items. Let x be the number of items you can sell, and let P be the price of an item. (a) Express P linearly in terms of x, in other words, express P in the form $P = mx + b$. (b) Express x linearly in terms of P.

11. The tax for a single taxpayer is described in the figure below. Use this information to graph tax versus taxable income (i.e., x is the amount on Form 1040, line 37, and y is the amount on Form 1040, line 38). Find the slope and y-intercept of each line that makes up the polygonal graph.

1990 Tax Rate Schedules

Caution: *use ONLY if your taxable income (form 1040, line 37) is $50,000 or more. If less, use the* **Tax Table**. *(Even though you cannot use the tax rate schedules below if your taxable income is less than $50,000, we show all levels of taxable income so that taxpayers can see the tax rate that applies to each level.)*

Schedule X—Use if your filing status is **Single**

If the amount on Form 1040, line 37, is: Over—	But not over—	Enter on Form 1040, line 38	of the amount over—
$0	$19,450	.... **15%**	**$0**
19,450	47,050	**$2,917.50+28%**	**19,450**
47,050	97,620	**$10,645.50+33%**	**47,050**
97,620	....	Use **Worksheet** below to figure your tax.	

Schedule Z—Use if your filing status is **Head of household**

If the amount on Form 1040, line 37, is: Over—	But not over—	Enter on Form 1040, line 38	of the amount over—
$0	$26,050	.... **15%**	**$0**
26,050	67,200	**$3,907.50+28%**	**26,050**
67,200	134,930	**$15,429.50+33%**	**67,200**
134,930	....	Use **Worksheet** below to figure your tax.	

12. Do the points shown in the following table below lie on a line? If so, find the equation for this line.

x	5.2	5.3	5.4	5.6	5.9	6.5
y	27.8	29.2	30.6	33.4	37.6	46.0

13. If the point $(3, k)$ lies on the line with slope $m = -2$ passing through the point $(2, 5)$, find k.

14. Does the point $(3, -2)$ lie on the line through the points $(8, 0)$ and $(-7, -6)$?

15. Use the slopes to determine whether the points $(7, -1)$, $(10, 1)$, and $(6, 7)$ are the vertices of a right triangle.

16. Is there a value of k so that the graphs of $x^3 - x$ and $kx^2 - 1$ meet in three distinct points? In four points? In 2 points? In one point? Do not meet? If so give specific values of k; if not, explain why.

CHAPTER 2

Functions

We say that a variable y is a function of another variable x, if given any value of x we have a well defined procedure to compute the value of y. The simplest example of a function is given by the equation $y = 2x+3$, which associates the value $2x+3$ to y when any value of x is chosen. Thus, if $x = 1$, then $y = 2(1) + 1 = 3$. We will find it convenient to use names to denote such rules. Thus we may prefer to say $y = f(x)$, where the function $f(x) = 2x + 3$. This notation has the advantage that we can describe the assignment of values to x in a concise fashion. Thus the sentence that "if $x = 1$, then $y = 2(1) + 1 = 3$ " can be simply stated as $y = f(1) = 3$.

If we consider a new function $g(x) = 1/x$, then we see that $g(0)$ does not make sense and hence we should allow restrictions on the values allowed for the variable x. The set of such allowed values is called the domain of the function. Sometimes, we may want to restrict the values of the variable for practical reasons which may have little to do with evaluation problems.

For example, if x denotes the length of the side of a square and the function $y = A(x)$ denotes the area of such a square, then it makes sense to state tha domain of the function $A(x)$ to be the set of nonnegative real numbers, even though the formula $A(x) = x^2$ makes perfect sense for negative x also.

This $A(x)$ is an example of a function which is defined by a geometric procedure first, but then we make an algebraic formula. Sometimes we may need several algebraic formulas and sometimes the function may not have any known formula. For instance the daily average temperature at a place is a function of the date, but usually all we expect to know completely is a few values of this function! Sometimes, such functions are interpolated by ordinary mathematical functions for estimation and prediction, but in reality, we are only stipulating the existence of such functions!

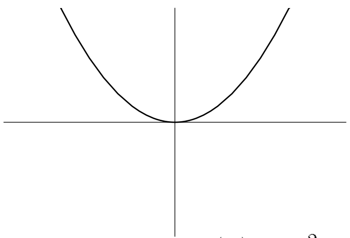
Another example of a function without a formula is the one where the domain is a set of people and the function associates the chosen first name to each person. Here neither the domain nor the values of the function are real numbers and we don't expect any formulas. Fortunately, our functions would usually be about real numbers.

Given a domain D and a function $f(x)$ we say that the set of all possible values obtained by evaluating the function $f(x)$ is said to be the image or the range of the function f. For instance, the range of the area function $A(x)$ described above is the set of all nonnegative real numbers.

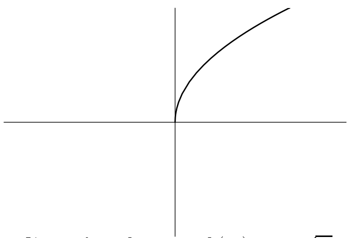
Mathematical functions can be usually described by a graph which consists of plotting all possible points $(x, f(x)$ as x varies in the domain. Such graphs will usually look like curves. We now think about which curves can be graphs of functions. Given a graph and a value $x = t$, to get the value of $f(t)$, we simply have to draw the vertical line $x = t$ and see where it hits the graph.

The condition that the graph comes from a function states that every such vertical line (as x varies in the domain) must hit the graph is **exactly one** point. This is the so-called " vertical line test ".

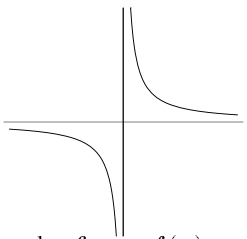
Given a value of x, a function has to give you only one value of y. Thus, vertical lines are not graphs of functions. For example, the line $x = 1$ fails the test in two ways. If we take $x = 1$, then both $(1, 2)$ and $(1, 3)$ are on the graph, so the vertical line hits in more than one point. (Actually it hits infinitely often.) Also, if we take any other value, say $x = 2$, then the vertical line does not hit the graph at all! So, we have a failure of the condition, if our domain contains at least one more point besides $x = 1$.



Graph of $y = f(x) = x^2$



Graph of $y = f(x) = \sqrt{x}$



Graph of $y = f(x) = 1/x$

We have seen that the domain of a function is a matter of choice, as long we stick to points for which the function can be uniquely evaluated. Given a mathematical formula, it makes sense to think about a " natural domain ", that is the largest set of real numbers which can be in the domain without violating the definition.

For example, the square-root function $y = f(x) = \sqrt{x}$ is the rule which says, given an x-value, take the nonnegative number whose square is x. This rule only makes sense if x is positive or zero. We say that the *natural* domain of this function is $x \geq 0$.

We often use a more precise notation which is:

$$\text{The domain of the function } \sqrt{x} \text{ is} \{x | x \geq 0\}$$

Alternately, we can use interval notation, and write that the domain is $[0, \infty)$. (In interval notation, square brackets mean that the endpoint is included, and a parenthesis means that the endpoint is not included.) The fact that the domain of $y = \sqrt{x}$ is $[0, \infty)$ means that in the graph of this function (see above) we have points (x, y) only above x-values on the right side of the x-axis.

We may often drop the word "natural" and say that it is "the domain" of the given function.

Another basic example of a function whose domain is not the entire x-axis is: $y = f(x) = 1/x$, the reciprocal function. We know that we cannot substitute $x = 0$ in this formula. The function makes sense, however, for any nonzero x, so we get that the (natural) domain is: $x \neq 0$. The graph of this function does not have any point (x, y) with $x = 0$. As x gets close to 0 from either side, the graph goes off toward infinity. We call the vertical line $x = 0$ an **asymptote** of the given graph.

To summarize, usually there are two reasons why certain x-values are excluded from the domain of a function (i) we cannot divide by zero, and (ii) we cannot take the square root of a negative number. A possible third reason is that the variable x is associated with quantities for which only certain values are allowed. We have already illustrated this in the are function above.

In a story problem, often letters different from x and y are used. In using a computer it is not uncommon to denote variables by entire words. For example, the volume V of a sphere is a function of the radius r, given by the formula $V = f(r) = \frac{4}{3}\pi r^3$. Also, letters different from f may be used. For example, if y is the velocity of something at time t, we write $y = v(t)$ with the letter v (instead of f) standing for the velocity function (and t playing the role of x).

In using a computer system, it might make more sense to write $V = Vol(radius) = \frac{4}{3}\pi r^3$. The computer can do the calculations in either symbolism with equal ease, but it is easy to recall your own reasoning after many days, of you use the long notations.

The letter playing the role of x is called the **independent variable** , and the letter playing the role of y is called the **dependent variable** (because its value "depends on" the value of the independent variable). In story problems, when one has to translate from English into mathematics, a crucial step is to determine what letters stand for variables. If only words and no letters are given, then you have to decide which letters to use. Some letters are traditional. For example, almost always, t stands for time.

EXAMPLE 2.1. An open-top box is made from an $a \times b$ rectangular piece of cardboard by cutting out a square of side x from each of the four corners, and then folding the sides up. Find a formula for the volume V of the box as a function of x, and find the domain of this function.

Here the box we get will have height x, and rectangular base of dimensions $a - 2x$ by $b - 2x$. Thus,

$$V = f(x) = x(a - 2x)(b - 2x).$$

Here a and b are constants, and V is the variable that depends on x, i.e., V is playing the role of y. Note that for the problem to make sense, the constants a, b have to be positive!

This formula makes mathematical sense for any x, but in the story problem the domain is much less. In the first place, x must be positive. In the second place, it must be less than half the length of either of the sides of the cardboard. Thus, the domain is

$$\{x|0 < x < \frac{a}{2} \text{ and } 0 < x < \frac{b}{2}\}$$

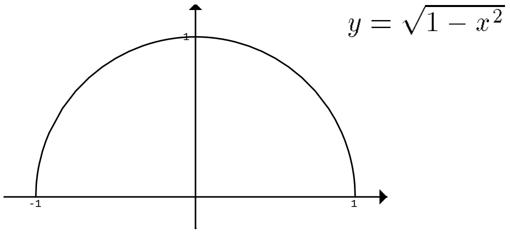
In interval notation, we write: the domain is the interval $(0, \frac{1}{2}\min(a, b))$.

We now give more examples of the domain of a purely mathematical function.

EXAMPLE 2.2. (Circle of radius r centered at the origin)

The equation for this circle is usually given in the form $x^2 + y^2 = r^2$. To write the equation in the form $y = f(x)$ we solve for y, obtaining $y = \pm\sqrt{r^2 - x^2}$. But *this is not a function*, because when we have an x it does not give us a *single* value of y but rather *two values* (provided that x is between r and $-r$). To get a function, we must choose one of the two signs in front of the square root. If we choose the positive sign, for example, we get the upper semicircle $y = f(x) = \sqrt{r^2 - x^2}$. The domain of this function is the interval $[-r, r]$, i.e., x must be between $-r$ and r (including the endpoints). If x is outside of that interval, then $r^2 - x^2$ is negative, and we cannot take the square root. In terms of the graph, this just means that there are no points on the curve whose x-coordinate is greater than r or less than $-r$.

Note that it is equally possible to take the negative square root and then the graph is that of the lower semicircle. This is a different function with domain $[-r, r]$.



EXAMPLE 2.3. Find the domain of

$$y = f(x) = \frac{1}{\sqrt{4x - x^2}}.$$

To answer this question, we must rule out the x-values that make $4x - x^2$ negative (because we cannot take the square root of a negative) and also the x-values that make $4x - x^2$ zero (because if $4x - x^2 = 0$, then when we take the square root we get 0, and we cannot divide by 0). In other words, the domain consists of all x for which $4x - x^2$ is strictly positive. We give three different methods to find out when $4x - x^2 > 0$.

First method. Factor $4x - x^2$ as $x(4 - x)$. The product of two numbers is positive when either both are positive or both are negative, i.e., if either $x > 0$ and $4 - x > 0$, or else $x < 0$ and $4 - x < 0$. The latter alternative is impossible, since if x is negative, then $4 - x$ is greater than 4, and so cannot be negative. As for the first alternative, the condition $4 - x > 0$ can be rewritten (adding x to both sides) as $4 > x$, so we need: $x > 0$ and $4 > x$ (this is sometimes combined in the form $4 > x > 0$, or, equivalently, $0 < x < 4$). In interval notation, this says that the domain is the interval $(0, 4)$.

Second method. Write $4x - x^2$ as $-(x^2 - 4x)$, and then complete the square, obtaining $-\Big((x-2)^2 - 4\Big) = 4 - (x-2)^2$. For this to be positive we need $(x-2)^2 < 4$, which means that $x - 2$ must be less than 2 and greater than -2: $-2 < x - 2 < 2$. Adding 2 to everything gives $0 < x < 4$.

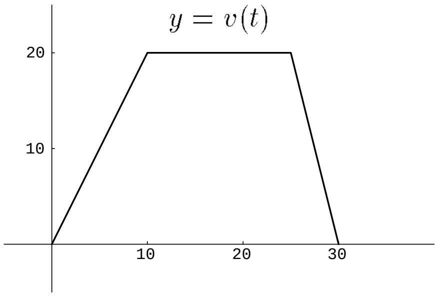
Third method. Again factor $4x - x^2$ as $x(4 - x)$. Thus the product is zero when $x = 0$ or $x = 4$. Mark these two points on a usual number line. This yields three pieces of the number line which, in interval notation, are:

$$I_1 = (-\infty, 0)\ ,\ I_2 = (0, 4)\ ,\ I_3 = (4, \infty)$$

Our function $4x - x^2$ must keep a constant sign on each of these intervals. We can test the signs easily by testing any convenient point in each interval. For example, in I_1, try $x = -1$ and consider $x(4 - x) = (-1)(4 - (-1)) = -5 < 0$. In I_2, try $x = 1$ to get $1(4 - 1) = 3 > 0$ and in I_3, try $x = 5$ to get $5(4 - 5) = -5 < 0$. So the needed interval is I_2. Since we need to avoid the zeros of our function, the answer is $I_2 = (0, 4)$. In general, you will have to investigate the endpoints separately.

A function does not always have to be given by a single formula. For example, suppose that $y = v(t)$ is the velocity function for a car which starts out from rest (zero velocity) at time $t = 0$; then increases its speed steadily to 20 m/sec, taking 10 seconds to do this; then travels at constant speed 20 m/sec for 15 seconds; and finally applies the brakes to decrease speed steadily to 0, taking 5 seconds to do this. The formula for $y = v(t)$ is different in each of the three time intervals. The graph of this function is shown below, along with the three formulas:

$$y = v(t) = \begin{cases} 2t, & \text{if } 0 \le t \le 10; \\ 20, & \text{if } 10 \le t \le 25; \\ 120 - 4t, & \text{if } 25 \le t \le 30. \end{cases}$$



Not all functions are given by formulas. A function can be given by an experimentally determined table of values. For example, the population y of the U.S. is a function of the time t: we can write $y = f(t)$. This is a perfectly good function — you could graph it if you had data for various t — but you couldn't find an algebraic formula for it.

Another inportant point is that this is a fuzzy function! Its mathematical form, once written down, gives a good function, but whether it is "really equal" to the actual function being discussed is always in doubt and subject to disputes and opinions. Even though the function may exist and be well defined, it is impossible for anyone to claim to know it completely or to be able to prove the claim!

This raises an interesting philosophical question. Does a function exist if its description seems to make sense, but nobody can ever determine all its function values - or even a formula for it! The uncertainty principle of Physics makes this problem even more intriguing, since it states that it is impossible to precisely determine both the time and the position of a moving particle. In Mathematics, we don't have to worry about such uncertainties!

1. Shifts and Expansions

Many functions in applications are built up from simple functions by inserting constants in various places. It is important to understand the effect such constants have on the appearance of the graph.

Horizontal shifts. *If you replace x by $x - C$ everywhere it occurs in the formula for $f(x)$, then the graph shifts over C to the right.* (If C is negative, then this means that the graph shifts over $|C|$ to the left.) For example, the graph of $y = (x-2)^2$ is the parabola $y = x^2$ shifted over to have its vertex at the point 2 on the x-axis. The graph of $y = (x+1)^2$ is the same parabola shifted over to the left so as to have its vertex at -1 on the x-axis.

If you have to do such sketches by hand, often the trick is to draw the original parabola $y = x^2$ and then just shift the y-axis to the desired new location. Think why you must shift the y-axis for an apparent shift in x values. Also, you should remember that to shift a curve to the left is the same as shifting the y-axis to the right by the same amount.

Vertical shifts. *If you replace y by $y - D$, then the graph moves up D units.* (If D is negative, then this means that the graph moves down $|D|$ units.) If the formula is written in the form $y = f(x)$ and if y is replaced by $y - D$ to get $y - D = f(x)$, we can equivalently move D to the other side of the equation and write $y = f(x) + D$. Thus, this principal can be stated: *to get the graph of $y = f(x) + D$, take the graph of $y = f(x)$ and move it D units up.* For example, the function $y = x^2 - 4x = (x-2)^2 - 4$ can be obtained from $y = (x-2)^2$

(see the last paragraph) by moving the graph 4 units down. The result is the x^2-parabola shifted 2 units to the right and 4 units down so as to have its vertex at the point $(2, -4)$.

Just as explained above, this can also be accomplished by shifting the x-axis accordingly. It is of course, much easier to redraw the axes in a different location than to redraw the whole curves.

Warning. Do not confuse $f(x) + D$ and $f(x + D)$. For example, if $f(x)$ is the function x^2, then $f(x) + 2$ is the function $x^2 + 2$, while $f(x + 2)$ is the function $(x + 2)^2 = x^2 + 4x + 4$. To sketch $f(x) + 2$, the graph of $f(x)$ has to be pushed up by 2 units; but to sketch the graph of $f(x + 2)$, the same graph has to be pushed to the left by two units.

EXAMPLE 2.4. (Circles) An important example of the above two principles is the circle $x^2 + y^2 = r^2$. This is the circle of radius r centered at the origin. (As we saw, this is not a single function $y = f(x)$, but rather two functions $y = \pm\sqrt{r^2 - x^2}$ put together; in any case, the two shifting principles apply to equations like this one which are not in the function form $y = f(x)$.) If we replace x by $x - C$ and replace y by $y - D$ — getting the equation $(x - C)^2 + (y - D)^2 = r^2$ — the effect on the circle is to move it C to the right and D up, thereby obtaining the circle of radius r centered at the point (C, D). This tells us how to write the equation of any circle, not necessarily centered at the origin.

We will later want to use two more principles concerning the effects of constants on the appearance of the graph of a function.

Horizontal expansion. *If A is positive and x is replaced by x/A in a formula, then the effect on the graph is to expand it by a factor of A in the x-direction (away from the y-axis).* This wording supposes that $A > 1$. If A is between 0 and 1, then "expand by a factor of A" means "contract by a factor of $1/A$." For example, replacing x by $x/0.5 = 2x$ has the effect of contracting toward the y-axis by a factor of 2.

If A is negative, then we expand by a factor of $|A|$ and then flip about the y-axis. Thus, replacing x by $-x$ has the effect of taking the mirror image of the graph with respect to the y-axis. For example, the function $y = \sqrt{-x}$, which has domain $x \leq 0$, is obtained by taking the graph of $\sqrt{x}$ and flipping it around the y-axis into the second quadrant.

Vertical expansion. *If B is positive and y is replaced by y/B in a formula, then the effect on the graph is to expand it by a factor of B in the vertical direction, away from the x-axis.* Note that if we have a function $y = f(x)$, replacing y by y/B is equivalent to multiplying the function on the right by B: $y = Bf(x)$. The effect on the graph is to expand the picture away from the x-axis by a factor of B if $B > 1$, to contract it toward the x-axis by a factor of $1/B$ if $0 < B < 1$. If $B < 0$, then as before the operation is to expand by $|B|$ and then flip about the x-axis.

As above, we can avoid redrawing the graph in the case of vertical or horizontal expansions. We simply choose an appropriate new scale on the axes, keeping our original curve in tact. While this may be convenient, it may not be the best visual illustration. As we will illustrate below, we can declare a circle to be an ellipse or an ellipse to be a circle by a simple change of scales on the axes. Thus, if we want comparable scales on the two axes, then this trick is not available. See the illustrations below.

EXAMPLE 2.5. (Ellipses) A basic example of the two expansion principles is given by an **ellipse of semimajor axis a and semiminor axis b**. We get such an ellipse by starting with the unit circle — the circle of radius 1 centered at the origin, the equation of which is $x^2 + y^2 = 1$ — and stretching it by a factor of a horizontally and by a factor of b vertically. To get the equation of the resulting ellipse, which crosses the x-axis at $\pm a$ and crosses the

y-axis at $\pm b$, we replace x by x/a and y by y/b in the equation for the unit circle. This gives

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Finally, if you want to analyze a function that involves *both* shifts and expansions, it is usually necessary to deal with the expansions first, and then the shifts. For instance, if you want to expand a function by a factor of A in the x-direction and then shift C to the right, you do this by replacing x first by x/A and then by $(x - C)/A$ in the formula. As an example, suppose that, after expanding our unit circle by a in the x-direction and by b in the y-direction to get the ellipse in the last paragraph, we then wanted to shift it a distance h to the right and a distance k upward, so as to be centered at the point (h, k). The new ellipse would have equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

A concrete example might be the function $y = (x - 2)^2/4 + 1$. Naturally, you want to start with the model parabola $y = x^2$. Then you should first do the vertical expansion $y = x^2/4$, either by just renaming the scale or preferably by sketching the curve again in the new formula. The resulting curve can then be shifted horizontally and vertically as needed. (See illustrations below.) You should experiment with changing the order of these operations and see what trouble you get into.

HOMEWORK 2.

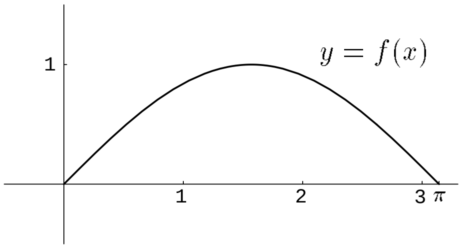
1. Find the domain of each of the following functions:

1. $y = f(x) = \sqrt{2x - 3}$.
2. $y = f(x) = \dfrac{1}{x+1}$.
3. $y = f(x) = \dfrac{1}{x^2 - 1}$.
4. $y = f(x) = \sqrt{-1/x}$.
5. $y = f(x) = \sqrt[3]{x}$.
6. $y = f(x) = \sqrt[4]{x}$.
7. $y = f(x) = \sqrt{r^2 - (x-h)^2}$ (r, h are positive constants)
8. $y = f(x) = \sqrt{1 - (1/x)}$.
9. $y = f(x) = \dfrac{1}{\sqrt{1 - (3x)^2}}$.
10. $y = f(x) = \sqrt{x} + \dfrac{1}{x-1}$.
11. $y = f(x) = \dfrac{1}{\sqrt{x} - 1}$.

Starting with the graph of $y = \sqrt{x}$, the graph of $y = 1/x$, and the graph of $y = \sqrt{1 - x^2}$ (the upper unit semicircle), sketch the graph of each of the following functions:

12. $y = f(x) = \sqrt{x - 2}$.
13. $y = f(x) = 4 + \sqrt{x + 2}$.
14. $y = f(x) = -\sqrt{-x}$.
15. $y = f(x) = -4 + \sqrt{-(x - 2)}$.
16. $y = f(x) = \dfrac{1}{x+1}$.
17. $y = f(x) = 1 + \dfrac{1}{x-1}$.
18. $y = f(x) = -1 - \dfrac{1}{x+2}$.
19. $y = f(x) = \dfrac{x}{1-x}$.
20. $y = f(x) = 2 + \sqrt{1 - (x-1)^2}$.
21. $y = f(x) = 2\sqrt{1 - (x/3)^2}$.
22. $y = f(x) = 4 + 2\sqrt{1 - (x-5)^2/9}$.
23. $y = f(x) = \sqrt{100 - 25(x-1)^2} + 2$.

In questions 24–30, $y = f(x)$ is given by the following graph:



Using that graph, sketch:

24. $y = f(x - 1)$.
25. $y = 1 + f(x + 2)$.
26. $y = 1 + 2f(x)$.
27. $y = 2f(3x)$.
28. $y = 2f(3(x - 2)) + 1$.
29. $y = 0.5f(3x - 3)$.
30. $y = f(\frac{x}{3} + 1) + 2$.

2.

Graphs of standard elementary functions The following is a list of functions whose graphs you will need to become familiar with. For each, plot the graph for a suitable range of values of the independent variable, x. If there are additional parameters make plots for several values of the parameters. Label printed copies of the graphs with the following information:
i) the function which the graph illustrates
ii) the axes (x and y)
iii) the location of roots
iv) asymptotes

1. $f(x) = \sin(x)$.
2. $f(x) = a\sin(bx + c)$
3. $f(x) = a\cos(bx) + c$.
4. $f(x) = \tan(x)$.
5. $f(x) = \ln(x)$.
6. $f(x) = e^x$.
7. $f(x) = e^{-x}$.
8. $f(x) = x$.
9. $f(x) = ax + b$.
10. $f(x) = x^2$.
11. $f(x) = ax^2 + bx + c$.
12. $f(x) = x^3$.
13. $f(x) = ax^3 + bx^2 + cx + d$.
14. $f(x) = \frac{1}{x}$.
15. $f(x) = \frac{1}{x^2}$.
16. $f(x) = \frac{1}{x^3}$.
17. $f(x) = |x|$.
18. $f(x) = |x - 2|$.

CHAPTER 3

Instantaneous Rate Of Change: The Derivative

Suppose that y is a function of x, say $y = f(x)$. It is often necessary to know how sensitive the value of y is to small changes in x about a fixed value. We can measure this change as $\frac{\Delta y}{\Delta x}$, where as usual Δx denotes a change in x and Δy the corresponding change in y. This ratio is called a **Difference Quotient** for the obvious reason that it is a quotient of differences.

If we think of a linear function like $y = mx + b$, then it is easy to see that the ratio is really independent of our choice of the value of x as well as the change Δx; it is simply the slope m of the line. For other functions, life is not so easy!

EXAMPLE 3.1. Take, for example, $y = f(x) = \sqrt{625 - x^2}$ (the upper semicircle of radius 25 centered at the origin). When $x = 7$, we find that $y = \sqrt{625 - 49} = 24$. Suppose we want to know how much y changes when x increases a little, say to 7.1 or 7.01.

Let us look at the ratio $\frac{\Delta y}{\Delta x}$ for our function $y = f(x) = \sqrt{625 - x^2}$ when x changes from 7 to 7.1. Here $\Delta x = 7.1 - 7 = 0.1$ is the change in x, and

$$\begin{aligned}\Delta y &= f(x + \Delta x) - f(x) = f(7.1) - f(7)\\ &= \sqrt{625 - 7.1^2} - \sqrt{625 - 7^2} = 23.9706 - 24 = -0.0294.\end{aligned}$$

Thus, $\Delta y/\Delta x = -0.0294/0.1 = -0.294$. Geometrically, this means that the **chord** of the circle drawn from the point $(7, 24)$ to the point $(7.1, 23.9706)$ has **slope** equal to -0.294.

In general, if we draw the chord from the point $(7, f(7))$ to a nearby point on the semicircle $(7 + \Delta x,\ f(7 + \Delta x))$, the slope of this chord is the **difference quotient**

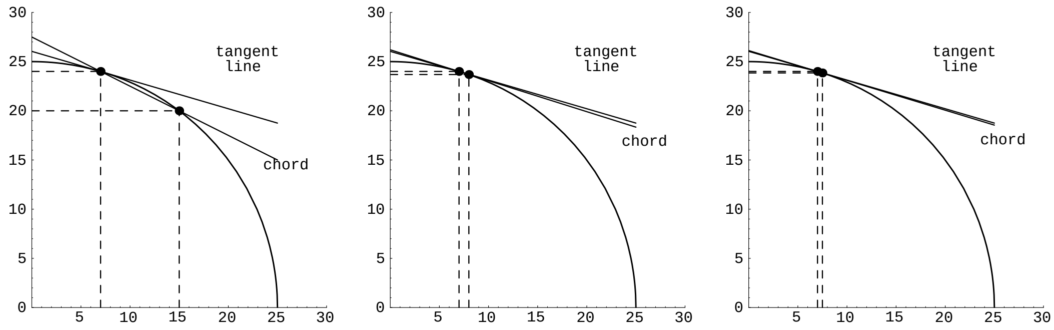
$$\text{slope of chord } = \frac{f(7 + \Delta x) - f(7)}{\Delta x} = \frac{\sqrt{625 - (7 + \Delta x)^2} - 24}{\Delta x}.$$

For example, if x changes only from 7 to 7.01, then the difference quotient (slope of the chord) is equal to $(23.997081 - 24)/0.01 = -0.2919$.

Note that unlike the line, the difference quotient depends on x as well as Δx.

As our second x value $7 + \Delta x$ moves in towards 7, the chord joining $(7, f(7))$ to $(7 + \Delta x, f(7 + \Delta x))$ shifts slightly. As can be seen in the picture below, as Δx gets smaller and smaller, the chord joining $(7, 24)$ to $(7 + \Delta x, f(7 + \Delta x))$x gets closer and closer to the **tangent line** to the circle at the point $(7, 24)$. Recall that the tangent line is the line that just grazes the circle at that point. In case of the circle, it doesn't meet the circle at any other point, but this is not likely to happen at more complicated functions.

Thus, as Δx gets smaller and smaller, the slope $\Delta y/\Delta x$ of the chord gets closer and closer to the slope of the tangent line.



$\Delta x = 8$ $\Delta x = 1$ $\Delta x = 0.5$

The slope of the tangent line to the circle at $(7, 24)$ is called the **derivative** of our function $f(x) = \sqrt{625 - x^2}$ at 7. It is denoted as $f'(7)$ (we say "f prime of 7," or equivalently, "the derivative of f at 7"). The slope of the chord joining $(7, f(7))$ to $(7+\Delta x, f(7+\Delta x))$, namely, $\Delta y/\Delta x = (f(7+\Delta x) - f(7))/\Delta x$, gets closer and closer to this value $f'(7)$. We write

$$f'(7) = \lim_{\Delta x \longrightarrow 0} \frac{f(7+\Delta x) - f(7)}{\Delta x}$$

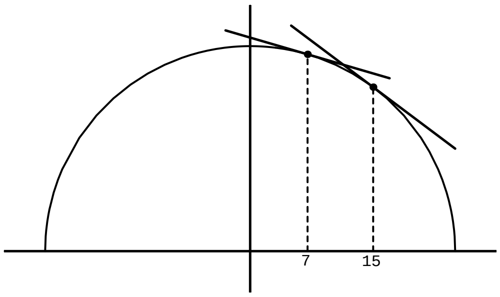
and we say that $f'(7)$ is the **limiting value** or simply the **limit** as Δx approaches zero of the difference quotient $(f(7+\Delta x) - f(7))/\Delta x$.

In the particular case of a circle, there's a simple way to find the derivative. Namely, the tangent to a circle at a point is perpendicular to the radius drawn to the point of contact, and so its slope is the negative reciprocal of the slope of the radius. The radius joining $(0, 0)$ to $(7, 24)$ has slope $24/7$. Hence, the tangent line has slope $-7/24 = -0.29166\cdots$. We write: $f'(7) = -0.29166\cdots$. Notice that when Δx is small, such as 0.01, the slope of the chord joining $(7, f(7))$ to $(7+\Delta x, f(7+\Delta x))$ is a good approximation to the value $f'(7)$. For example, when $\Delta x = 0.01$, we saw that this difference quotient is -0.2919, which is close to $f'(7)$.

Now suppose that we choose a different x, say $x = 15$, and we want to know how fast y is changing as x increases a little from 15. As an approximation we could take the chord joining $(15, f(15))$ to $(15+\Delta x, f(15+\Delta x))$ when $\Delta x = 0.01$:

$$\frac{\Delta y}{\Delta x} = \frac{f(15.01) - f(15)}{0.01} = \frac{19.992476 - 20}{0.01} = -0.7504.$$

The limiting value of this difference quotient as Δx becomes smaller and smaller is what we mean by $f'(15)$. The value $f'(15)$ is the slope of the tangent to the circle at the point $(15, 20)$. As before, we can find this slope exactly, since the tangent to a circle is perpendicular to the radius. The answer is $f'(15) = -15/20 = -0.75$. Again we see that, if Δx is small (like 0.01), then the slope of the chord joining $(x, f(x))$ to $(x+\Delta x, f(x+\Delta x))$ is very close to this value $f'(x)$ (here $x = 15$).



For different x we generally get different values of the derivative $f'(x)$. In our example of the circle, we had $f'(7) = -7/24$ and $f'(15) = -3/4$.

Warning: A geometrical method used in this example for finding the slope of the tangent line is usually not available for general functions. In fact, historically, clever methods were developed for the conic sections; but the concept of defining the tangent line with an abstract limiting slope took hundreds of years to develop!

The only time $f(x)$ has the same derivative for all different x is when we have a straight line $y = f(x) = mx + b$. Then the difference quotient $\Delta y/\Delta x$ will always be m, and the derivative at any point on the line will also be m. The **tangent line** to the line $y = mx + b$ is simply the line $y = mx + b$ itself, and this has slope m. But except for straight lines, a function $y = f(x)$ has different "slopes" (of the tangent line) for different x.

We wish to calculate the slopes of tangents for several examples. For ease of notation and calculation, let us replace the change Δx by a single letter h. For a function $y = f(x)$, we take two points (x, y) and $(x + h, f(x + h))$ and our difference quotient and the slope of the tangent take the form:

Difference quotient is $\dfrac{f(x+h) - f(x)}{h}$, and the tangent has slope $lim_{h\longrightarrow 0}\dfrac{f(x+h) - f(x)}{h}$.

Of course, the name of the independent variable x can also change by context, as in the next example.

Example 3.2. Suppose we are on an airless planet where the gravitational constant is 2 m/sec^2 (the moon has gravitational constant close to this). Then it turns out that an object dropped from rest falls through a distance of exactly t^2 meters after t seconds have elapsed. (Falling body problems will be taken up more systematically later.) That is, if y denotes the distance fallen in meters, we have $y = f(t) = t^2$. Suppose that we are interested in the moment $t = 1$ sec, and we want to know how much the object falls as t increases from 1 to $1 + h$ seconds. The distance fallen in that short time interval is $\Delta y = (1 + h)^2 - 1^2$, which simplifies to $2h + h^2$. For example, in the time from 1 sec to 1.01 sec the object falls a distance $\Delta y = 0.0201$ m. The difference quotient for this 0.01-sec time interval is $\Delta y/\Delta t = 0.0201/0.01 = 2.01$ m/sec. It is customary to calle this difference quotient the *average velocity during the time interval* $[1, 1.01]$. In other words, it is the distance traveled divided by time in this interval. More generally, the **average velocity** between 1 and $1+h$ sec. is

$$\frac{(1+h)^2 - 1^2}{h} = \frac{1 + 2h + h^2 - 1}{h} = \frac{2h + h^2}{h} = 2 + h.$$

Now the derivative of $y = f(t) = t^2$ at the point $t = 1$ is the limiting value of this difference quotient as h approaches zero:

$$f'(1) = \lim_{h\longrightarrow 0} 2 + h = 2,$$

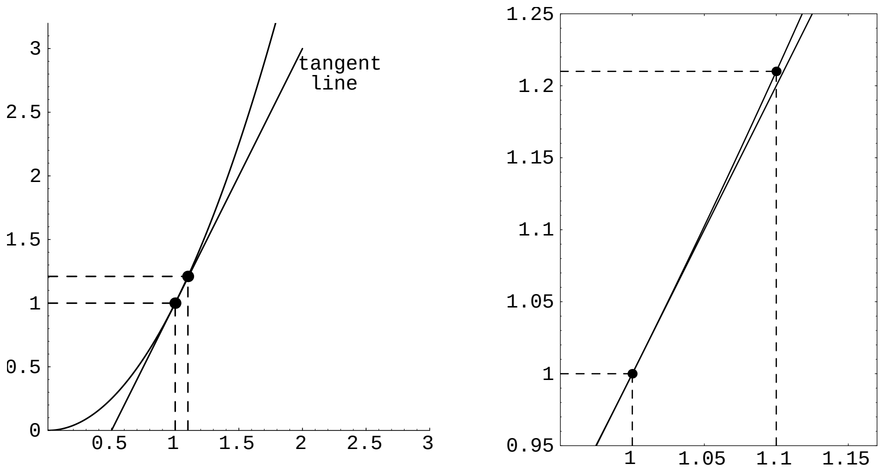
because as h gets closer and closer to 0, the number $2+h$ gets closer and closer to 2. In the case of a distance function like our $f(t)$ (distance fallen), the derivative $f'(1)$ has the practical meaning of the *instantaneous velocity* at time $t=1$. The **instantaneous velocity** at time t is the limiting value of the average velocity during a shorter and shorter time interval from t to $t+h$.

Note that our calculation was independent of the sign of $\Delta t = h$, even though our wording seems to suggest that h is positive. In reality, we need to get the same answer regardless of the sign of h. Leter, we will discuss what happens when it does depend on the sign of h.

To understand the distinction between average and instantaneous velocity, think of a car: the average velocity during a time interval means the distance traveled divided by time, while the instantaneous velocity at a particular instant means the speed indicated on the speedometer at that moment.

The graph below shows distance fallen as a function of time in this example: $y=t^2$. The diagram next to it is an enlargement near the point $t=1$ sec, where $y=1$ meter (i.e., the point (1,1) on the curve). The slope of the line joining $(1,1)$ to $(1.1, 1.21)$ on the curve $y=t^2$ is the average velocity of the falling object during the time interval from 1 to 1.1 sec. The slope of the line which just grazes the parabola at the point $(1,1)$ is the instantaneous velocity of the object at time 1 sec. Thus, we have two interpretations of the derivative: as the slope of the tangent line in the graph, or as the instantaneous velocity.

The curve is a parabola and it also has a geometric construction for the tangent. Let O be the vertex and l the tangent at O. At any point P of the parabola drop a perpendicular from P to L and let Q be the foot of the perpendicular. Then the tangent at P is the line PM where M is the midpoint of OQ. (See picture). Of course, this is far more complicated than the circle. You should try to prove it by using our calculations above.



EXAMPLE 3.3. As a final example, we look at the reciprocal function $y=f(x)=1/x$. Suppose we want to find the derivative $f'(x)$ when $x=2$.

One way is to approximate. We can join the point $(2, 0.5)$ to the point $(2+h, 1/(2+h))$ for small values of h, say 0.1, 0.01, 0.0001, or 0.0000001, and then compute the slope of the chord. We get the following table:

x	2	2	2	2
$x+h$	2.1	2.01	2.0001	2.0000001
Difference quotient	-0.238095	-0.248756	-0.249988	-0.24999999875

At this point we might guess that the slope of these chords is approaching -0.25, so that the derivative is $-1/4$. But to be more precise, we need an algebraic procedure to determine for sure what the limit is.

That is, as in the example $y = t^2$, we write out the difference quotient, which represents the slope of the chord joining the points $(2, f(2))$ and $(2+h, f(2+h))$, and then we try to simplify:

$$\begin{aligned}\frac{f(2+h)-f(2)}{h} &= \frac{1/(2+h)-1/2}{h} = \frac{2(2+h)}{2(2+h)} \cdot \frac{1/(2+h)-1/2}{h} \\ &= \frac{2-(2+h)}{2(2+h)h} = \frac{-h}{2(2+h)h} = -\frac{1}{2(2+h)}.\end{aligned}$$

As h approaches zero, the denominator $2(2+h)$ approaches $2 \cdot 2 = 4$, and so we get

$$f'(2) = \lim_{h \longrightarrow 0} \frac{f(2+h)-f(2)}{h} = \lim_{h \longrightarrow 0} -\frac{1}{2(2+h)} = -\frac{1}{4}.$$

HOMEWORK 3.

1. Draw the graph of the function $y = f(x) = \sqrt{169 - x^2}$ between $x = 0$ and $x = 13$. Find the slope $\Delta y/\Delta x$ of the chord between the points of the circle lying over (a) $x = 12$ and $x = 13$, (b) $x = 12$ and $x = 12.1$, (c) $x = 12$ and $x = 12.01$, (d) $x = 12$ and $x = 12.001$. Now use the geometry of tangent lines on a circle to find (e) the exact value of the derivative $f'(12)$. Your answers to (a)–(d) should be getting closer and closer to your answer to (e).

2. Use geometry to find the derivative $f'(x)$ of the function $f(x) = \sqrt{625 - x^2}$ in the text for each of the following x: (a) 20, (b) 24, (c) -7, (d) -15. Draw a graph of the upper semicircle, and draw the tangent line at each of these four points.

3. Let $y = f(t) = t^2$, where t is the time in seconds and y is the distance in meters that an object falls on a certain airless planet. Draw a graph of this function between $t = 0$ and $t = 3$. Make a table of the average velocity of the falling object between (a) 2 sec and 3 sec, (b) 2 sec and 2.1 sec, (c) 2 sec and 2.01 sec, (d) 2 sec and 2.001 sec. Then use algebra to find a simple formula for the average velocity between time 2 and time $2 + \Delta t$. (If you substitute $\Delta t = 1, 0.1, 0.01, 0.001$ in this formula you should again get the answers to parts (a)–(d).) Next, in your formula for average velocity (which should be in simplified form) determine what happens as Δt approaches zero. This is the instantaneous velocity. Finally, in your graph of $y = t^2$ draw the straight line through the point $(2, 4)$ whose slope is the instantaneous velocity you just computed.

4. If an object is dropped from an 80-meter high window, its height y above the ground at time t sec is given by the formula $y = f(t) = 80 - 4.9t^2$. (Here we are neglecting air resistance; the graph of this function was shown at the beginning of the first section.) Find the average velocity of the falling object between (a) 1 sec and 1.1 sec, (b) 1 sec and 1.01 sec, (c) 1 sec and 1.001 sec. Now use algebra to find a simple formula for the average velocity of the falling object between 1 sec and $1 + \Delta t$ sec. Determine what happens to this average velocity as Δt approaches 0. That is the instantaneous velocity at time $t = 1$ sec (it'll be negative, because the object is falling).

5. Draw the graph of the function $y = f(x) = 1/x$ between $x = 1/2$ and $x = 4$. Find the slope of the chord between (a) $x = 3$ and $x = 3.1$, (b) $x = 3$ and $x = 3.01$, (c) $x = 3$ and $x = 3.001$. Now use algebra to find a simple formula for the slope of the chord between $(3, f(3))$ and $(3 + \Delta x, f(3 + \Delta x))$. Determine what happens when Δx approaches 0. In your graph of $y = 1/x$, draw the straight line through the point $(3, 1/3)$ whose slope is this limiting value of the difference quotient as Δx approaches 0.

6. Find an algebraic expression for the difference quotient $\big(f(1 + \Delta x) - f(1)\big)/\Delta x$ when $f(x) = x^2 - (1/x)$. Simplify the expression as much as possible. Then determine what happens as Δx approaches 0. That value is $f'(1)$.

7. Draw the graph of $y = f(x) = x^3$ between $x = 0$ and $x = 1.5$. Find the slope of the chord between (a) $x = 1$ and $x = 1.1$, (b) $x = 1$ and $x = 1.001$, (c) $x = 1$ and $x = 1.00001$. Then use algebra to find a simple formula for the slope of the chord between 1 and $1 + \Delta x$. (Use the expansion $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$.) Determine the limit as Δx approaches 0, and in your graph of $y = x^3$ draw the straight line through the point $(1, 1)$ whose slope is equal to the limit you just found.

CHAPTER 4

The Derivative Function

In the last section we saw how, given a function $y = f(x)$ and some particular value of x, we can speak of the derivative of the function at x, denoted by $f'(x)$. In geometrical terms, $f'(x)$ is the slope of the tangent line which grazes the curve at the point $(x, f(x))$. In this section, instead of taking just one particular value of x (or a few particular values, as in Problem 2 of the last homework), we let x vary. That is, we are interested in the rule which for every x in the domain of our function $f(x)$ determines the value of the derivative $f'(x)$. That gives us a new function f', called "f prime" or "the derivative of f."

EXAMPLE 4.1. Find the derivative function $f'(x)$ for the first function discussed in the last section: $f(x) = \sqrt{625 - x^2}$.

The point on the upper semicircle above x has coordinates $(x, \sqrt{625 - x^2})$. The radius out to that point has slope $\Delta y/\Delta x = (\sqrt{625 - x^2} - 0)/(x - 0) = (\sqrt{625 - x^2})/x$. Since the tangent to the circle at that point is perpendicular to this radius, its slope is the negative reciprocal, i.e., $-x/\sqrt{625 - x^2}$. Thus, $f'(x) = -x/\sqrt{625 - x^2}$. For example, if you plug $x = 20$, 24, -7, or -15 into this formula, you get the answers to Problem 2 of the last homework.

Of course, the above calculation depends on the clever geometrical method for finding the tangent. We need a more reliable method!

Notation. There are various other ways to denote the derivative $f'(x)$. If $y = f(x)$, we can also write y', or else dy/dx, or $\frac{d}{dx} f(x)$. For example, using the different types of notation in Example 1, where $f(x) = \sqrt{625 - x^2}$, we can write:

$$y' = f'(x) = \frac{dy}{dx} = \frac{d}{dx}\Big(\sqrt{625 - x^2}\Big) = -\frac{x}{\sqrt{625 - x^2}}.$$

EXAMPLE 4.2. Find the derivative function for the second function in the last section: $y = f(t) = t^2$.

The derivative $f'(t)$ is the **instantaneous velocity** , which we find by taking the limit as $h = \Delta t$ approaches zero of the average velocity during the time interval between t and $t + h$. That average velocity is the difference quotient

$$\begin{aligned} \frac{\text{distance traveled}}{\text{time}} &= \frac{(t+h)^2 - t^2}{h} \\ &= \frac{t^2 + 2t(h) + (h)^2 - t^2}{h} = \frac{2t(h) + (h)^2}{h} = 2t + h. \end{aligned}$$

The limit of this as h approaches zero is $2t$. Thus, $f'(t) = \dfrac{d}{dt}\left(t^2\right) = 2t$. In the case of a distance function, the derivative (the instantaneous velocity) is often denoted with a dot rather than a prime; so we can write: $\dot{y} = \dfrac{d}{dt}\left(t^2\right) = 2t$.

EXAMPLE 4.3. Find the derivative function for the third function in the last section: $y = f(x) = 1/x$.

We go through the procedure at the end of the last section, but with any possible x rather than with 2. The difference quotient — the slope of the chord joining the point $(x, 1/x)$ on the reciprocal curve to the nearby point $(x+h,\ 1/(x+h))$ — is:

$$\frac{f(x+h)-f(x)}{h} = \frac{1/(x+h)-1/x}{h} = \frac{x(x+h)}{x(x+h)} \cdot \frac{1/(x+h)-1/x}{h} =$$
$$= \frac{x-(x+h)}{x(x+h)h} = \frac{-h}{x(x+h)h} = -\frac{1}{x(x+h)}.$$

As h approaches zero, the denominator approaches $x(x+0) = x^2$, and so we get:

$$y' = f'(x) = \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}.$$

In Example 1 we had a special geometrical situation (the tangent being perpendicular to the radius) which does not apply for any function except circles. But in Examples 2 and 3 we had a more general procedure for finding the derivative function: (1) form the difference quotient; (2) simplify algebraically; and (3) take the limit as h becomes smaller and smaller. In other words, we define the derivative function $f'(x)$ as follows:

$$f'(x) = \lim_{h \longrightarrow 0} \frac{f(x+h)-f(x)}{h}.$$

Warning. If you happen to know a bunch of derivative formulas from an earlier course, for the time being you should know that they are true but unproved! In examples like the ones above and the homework below, you are required to know how to find the derivative formula starting from our current definition of the derivative.

1. Corners and Cusps

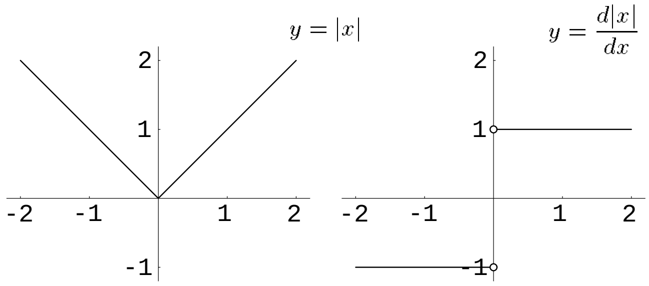
Sometimes one encounters a point in the domain of a function $y = f(x)$ where there is **no derivative**, because there is no well defined tangent line. At other times, our limiting process can fail if the tangent is vertical. First we consider the lack of a well defined tangent.

In order for the notion of the tangent line at a point to make sense, the curve must be " smooth " at that point. This means that if you imagine a particle traveling at some steady speed along the curve, then the particle does not experience an abrupt change of direction. There are two types of situations you should be aware of — corners and cusps — where there's a sudden change of direction and hence no derivative.

EXAMPLE 4.4. Discuss the derivative of the absolute value function $y = f(x) = |x|$. (This is the function that for negative x drops the minus sign and for nonnegative x leaves it unchanged.)

If x is positive, then this is the function $y = x$, whose derivative is the constant 1. (Recall that when $y = f(x) = mx + b$, the derivative is the slope m.) If x is negative, then we're dealing with the function $y = -x$, whose derivative is the constant -1. If $x = 0$, then the function has a corner, i.e., there is no tangent line. A tangent line would have to point in the direction of the curve — but there are *two* directions of the curve that come together at the origin. Thus,

$$y' = \frac{d}{dx}|x| = \begin{cases} 1 & \text{if } x > 0; \\ -1 & \text{if } x < 0; \\ \text{undefined} & \text{if } x = 0. \end{cases}$$

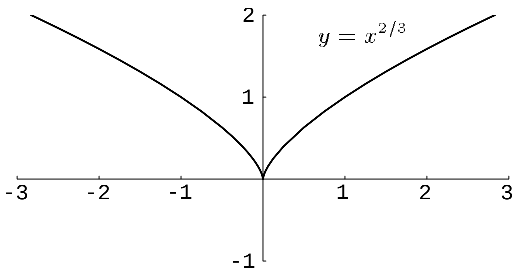


It is worthwhile discussing our difference quotients at $x = 0$. We are interested in $\frac{|0+h|}{h} = \frac{|h|}{h}$. Clearly, this quotient is 1 if $h > 0$ and -1 if $h < 0$. This explains why there is no possible limit as $h \longrightarrow 0$. We can also imagine limits as h goes to 0 thru positive values and deduce that it is $+1$. Similarly, the limit as h goes to 0 thru negative values is -1. In this case, it is possible to justify the statement that the function has two tangents with slopes 1 and -1. More on this later.

Example 4.5. The function

$$y = f(x) = x^{2/3}$$

pictured at the right has a derivative for every nonzero value of x (which we will find a formula for later), but it does not have a derivative at the origin, which is a cusp.

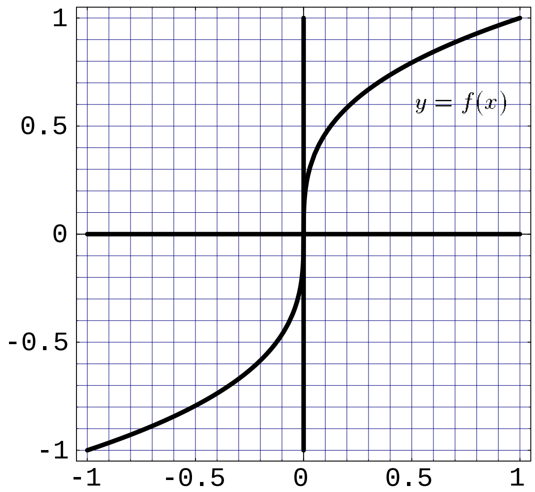


From the picture, it appears that this has a vertical tangent given by the equation $x = 0$. The algebraic procedures for establishing this are too complicated to introduce right now, but we will take it up later.

2. Graphical Differentiation

Often in applications one has a function $y = f(x)$ which is not given by a formula at all, but rather by drawing a smooth curve through points corresponding to a table of experimental data. Whenever the function $f(x)$ is given to you as a curve without any formula, you can find the derivative function $f'(x)$ using a graphical method. Namely, (1) divide up the domain of your function into regular intervals; (2) for each x which is an endpoint of an interval, look at the point over x on the curve; (3) put a small ruler along the curve at that point $(x, f(x))$ so that it just grazes the curve there (i.e., it is tangent to the curve there); (4) find the slope of the ruler, by measuring distance up divided by distance across (for example, if you are doing this against a piece of graph paper, and if you go over to the right 5 little boxes and up 8 little boxes, then the slope is 8/5=1.6); (5) tabulate the slopes obtained for all of the x values; (6) graph these values, drawing a smooth curve between them. The curve you draw is the derivative function $f'(x)$.

Example 4.6. Use graphical differentiation to find the graph of $f'(x)$ for the curve $y = f(x)$ pictured below.

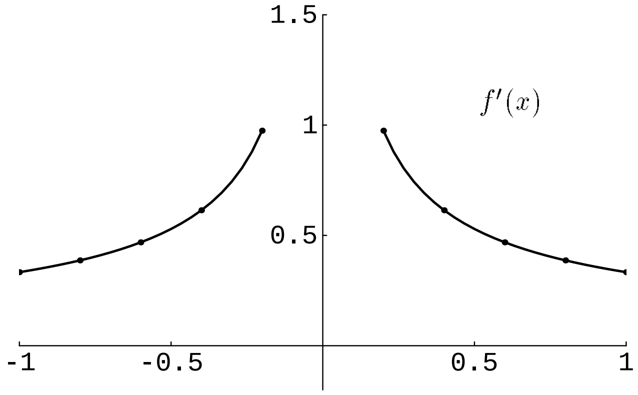


Following the 6-step procedure, we make a table of values of $f'(x)$ for x at intervals of 0.2 from -1 to 1:

x	-1.0	-0.8	-0.6	-0.4	-0.2	0.0	0.2	0.4	0.6	0.8	1.0
$f'(x)$	0.3	0.4	0.5	0.6	1.0	infinite	1.0	0.6	0.5	0.4	0.3

Note that this is a very approximate procedure, so our values of $f'(x)$ can be relied upon only to 1 or 2 significant figures. The estimates of the slopes are also subjective and a function of the accuracy of measurement.

Based on this table of values, we obtain an approximate graph of the derivative function $f'(x)$. Note that the derivative becomes infinite as we get closer and closer to $x = 0$, because the tangent to the curve there is vertical (has infinite slope). We say that $f'(0)$ "does not exist" (or "is infinite").

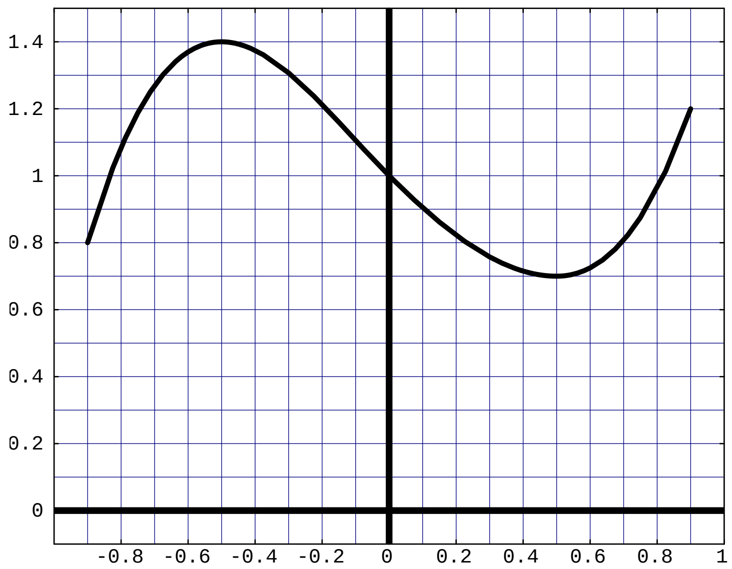


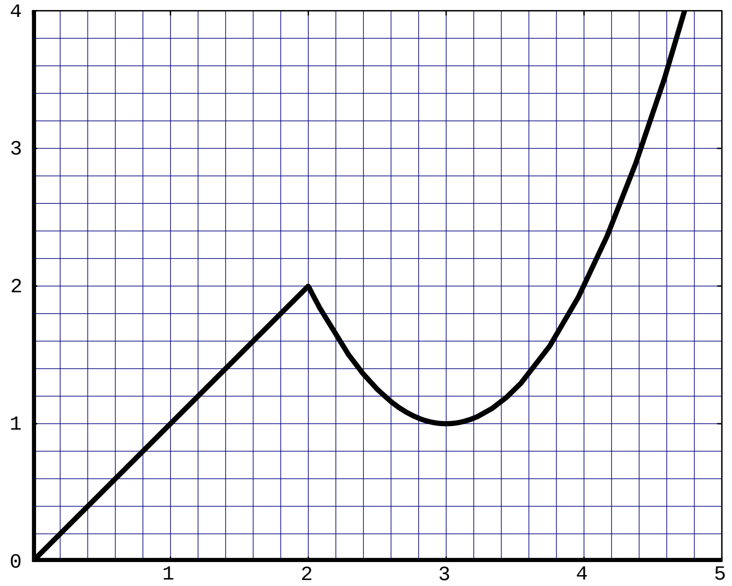
HOMEWORK 4.

1. Using the procedure used above in Example 4.1, find the derivative function for $y = f(x) = \sqrt{169 - x^2}$.

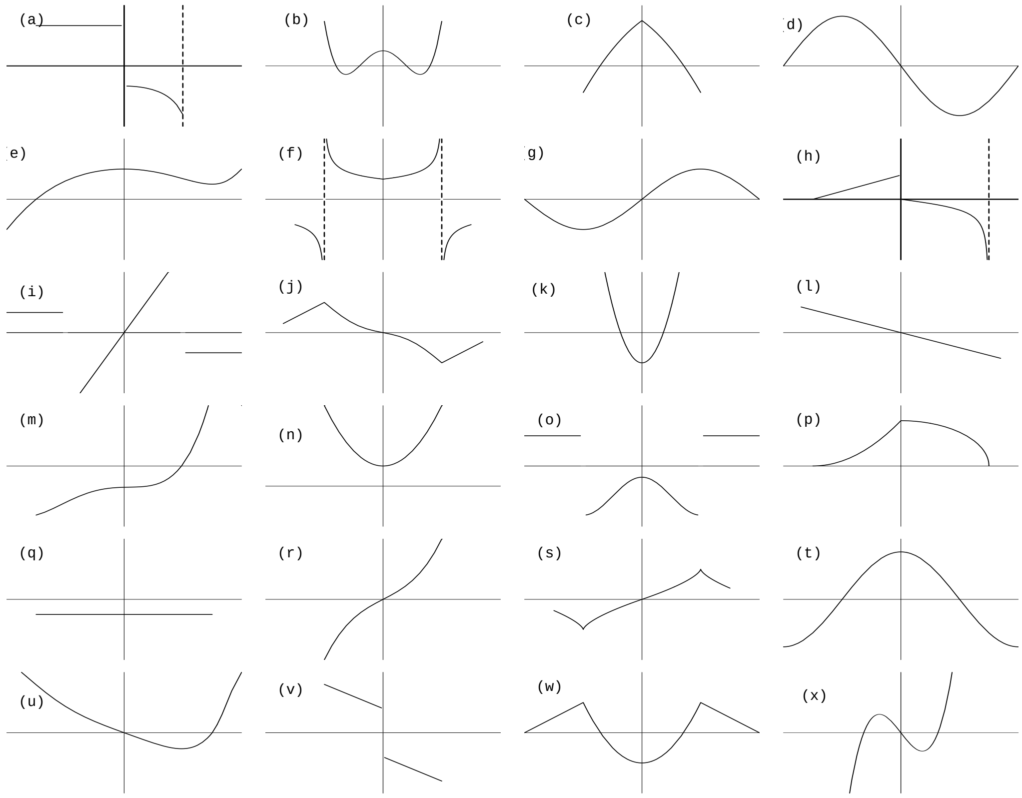
2. Using the procedure that we used in Examples 4.2 and 4.3, find the derivative function for: (a) $y = f(t) = 80 - 4.9t^2$, (b) $y = f(x) = x^2 - (1/x)$, (c) $y = f(x) = ax^2 + bx + c$ (where a, b, and c are constants), (d) $y = f(x) = x^3$. Please show every step clearly.

3. For each of the following two graphs $y = f(x)$, use graphical differentiation to sketch the graph of the derivative function. For x at regular intervals from one end of the domain to the other, estimate the derivative $f'(x)$ using a ruler, as explained in the text. Show clearly the points where the derivative is zero and where the derivative does not exist (or is infinite).





4. The 24 graphs below are labeled by letters from (a) to (x). For each of the following graphs of $f(x)$, give the letter of the graph that looks most like it could be the graph of the derivative function $f'(x)$:
(1) b, (2) c, (3) e, (4) g, (5) h, (6) j, (7) l, (8) p, (9) r, (10) s, (11) t, (12) u, (13) w, (14) x.



CHAPTER 5

The Tangent Line Approximation

1. The Power Rule

Before discussing the main theme of this section, we introduce the formula for the derivative of a power function, i.e., a function of the form $y = f(x) = x^n$. Here n can be anything — positive or negative, integer or fraction (or even irrational, like π). The formula is: $f'(x) = n\,x^{n-1}$. That is,

(Power Rule) $$\frac{d}{dx}\,x^n = n\,x^{n-1}.$$

We shall use this derivative formula in this section, even though we have not yet explained why it is valid — this will be done later. For some special values of n we did derive this formula in the last section, namely:

$$\frac{d}{dx}x^2 = 2x, \qquad \frac{d}{dx}x^3 = 3x^2,$$
$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}x^{-1} = (-1)x^{-2} = -\frac{1}{x^2}, \qquad \frac{d}{dx}\,x^1 = 1\cdot x^0 = 1.$$

Here are some examples of the $n\,x^{n-1}$ formula for other n:

$$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{(1/2)-1} = \frac{1}{2\sqrt{x}}, \qquad \frac{d}{dx}\sqrt[3]{x} = \frac{d}{dx}x^{1/3} = \frac{1}{3}x^{-2/3},$$
$$\frac{d}{dx}\left(x^{2/3}\right) = \frac{2}{3}x^{-1/3}.$$

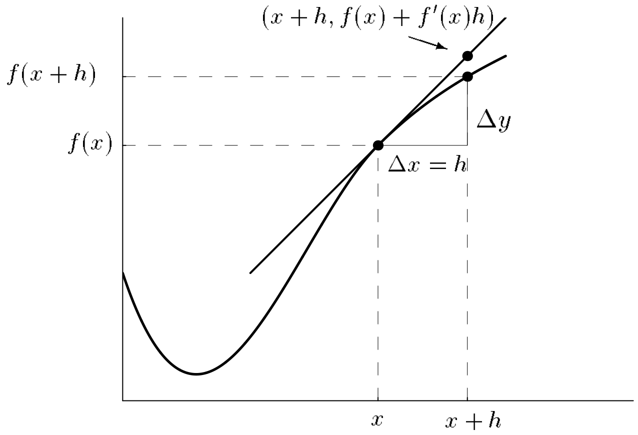
The last of these formulas tells us how to find the slopes of the tangent lines to the graph of $y = x^{2/3}$ in Example 4.5 of the last section. As expected, as x gets closer and closer to 0, the slopes become infinite (since 0 to a negative power is infinite).

2. The Tangent Line Approximation

The tangent line approximation is one of the most fundamental uses of the derivative. Suppose we have a function $f(x)$ which we can easily evaluate at some particular x. For example, the function $f(x) = 1/x$ has an obvious value when $x = 1$ or when $x = 100$; the function $f(x) = \sqrt{x}$ has an obvious value when $x = 4$ or when $x = 0.01$. Now suppose that we want to evaluate our function not at the easy x but rather at some nearby $x+\Delta x = x+h$. For example, we want to find the reciprocal not of 1 but rather of 1.01; or we want to find the square root not of 4 but rather of 3.99. The tangent line approximation is a way of doing this quickly but not with perfect precision — the result will be a little off (the accuracy depends on the particular function and on the size of Δx — the smaller the Δx the better the accuracy).

In a tangent line approximation problem we will know $f(x)$ and want to find $f(x+h)$. The derivative f' will also be relatively easy to compute at the point x. The formula derived below expresses $f(x+h)$ in terms of $f(x)$,$f'(x)$ and of course h.

To derive the formula, we draw a picture of our situation:



We want to find the y-coordinate of the point on the curve above $x+h$. Physically, we can get to that point by starting at our known point $(x, f(x))$ and following the curve until we go a horizontal distance $h = \Delta x$ to the right (we go to the left if h is negative). In doing this, we move upward a distance $\Delta y = f(x+h) - f(x)$. If we can determine Δy, then we know what to add to our known value $f(x)$ to get the value we're interested in: $f(x+h)$. Now $\Delta y/h$ is the slope of the chord joining the two points on the curve. The idea of the tangent line approximation is to use instead the slope of the tangent line to the curve at the point $(x, f(x))$. (This will give us a slightly different value for the change in y — namely, it gives us the change in y if we were to follow the tangent line rather than the actual curve when we move to the right a horizontal distance of h.) That is,

$$\text{(slope of chord)} \approx \text{(slope of tangent line)}$$

or

$$\frac{\Delta y}{h} \approx f'(x)\,.$$

And so

$$\Delta y \approx f'(x)\,h\,.$$

This gives us the **(VERY IMPORTANT!) tangent line approximation** formula

$$\boxed{f(x+h) = f(x) + \Delta y \approx f(x) + (h)f'(x).}$$

In using this formula, the basic steps are: (1) decide what your function f is in the given problem, and what you're taking for x and $x+h$; (2) find the derivative $f'(x)$; and (3) evaluate the formula.

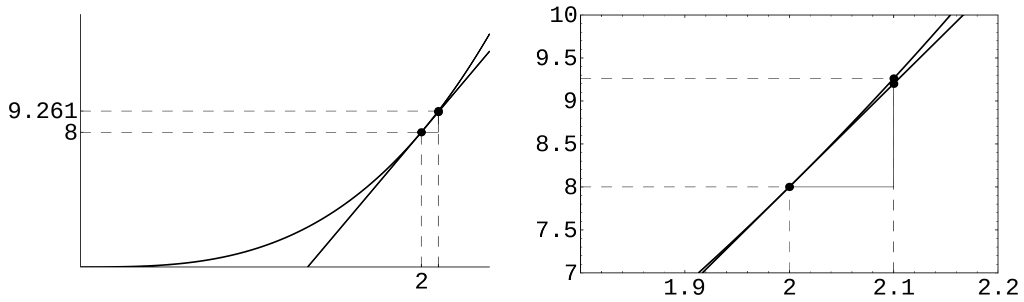
Example 5.1. Find 2.1^3.

Here our function is $f(x) = x^3$, which we can evaluate in our heads at the point $x = 2$, namely $f(2) = 2^3 = 8$. We want to know its value at the nearby point $x + h = 2.1$. Thus, $h = 0.1$. In the tangent line approximation formula, we need to know $f'(x)$. Now

$\frac{d}{dx}x^3 = 3x^2$ (Use the Power Rule above). Hence $f'(x) = f'(2) = 3 \cdot 2^2 = 12$. We are now ready to use the tangent line approximation formula:

$$f(2 + 0.1) \approx f(2) + (0.1)f'(2) = 8 + 0.1 \cdot 12 = 9.2.$$

Of course, we could easily find $(2.1)^3$ more exactly by calculator (or precisely by multiplying by hand): the result is 9.261.



Since the tangent line approximation does not give the exact answer, and since we could have found 2.1^3 easily enough without it, you might wonder why one bothers with the tangent line approximation. There are several answers: (1) In easy examples, with experience you can perform the tangent line approximation in your head, thereby getting a better "feel" for how the function changes when you make a small change from x to $x+h$. For instance, in Example 5.1 we can say that "the cube of a number near 2 is going to be near 8, and the difference between the cube and 8 is going to be about 12 times the difference between our number and 2." (2) In many practical applications, the tangent line approximation gives simple answers to questions about experimental error, especially *percent error.* (3) Later on, in the section on implicit functions, we will encounter examples where y cannot be written in terms of x, and so there is no way to evaluate the y-coordinate of the point on the curve above $x + h$ except through an approximation method, such as the tangent line approximation.

A more important answer will come much later. A generalization of the tangent line approximation will develop into a systematic method for calculation with any desired precision and this will explain what your calculators or computers are doing. This is the topic concerning Taylor polynomials. In fact, the method can finally be pushed into making precise power series expansions of most functions. This will come much later!

In other words, do not look down upon the imprecise answers, we are just warming up!

EXAMPLE 5.2. Find 1/4.99.

Here we're interested in evaluating the reciprocal function very near to an x-value whose reciprocal we can evaluate in our heads, namely: $f(5) = 1/5 = 0.2$. That is, our function is $f(x) = 1/x$, our x is 5, and $x + h = 4.99$, i.e., $h = -0.01$. Since $f'(x) = -1/x^2 = -1/25 =$

-0.04, we have:

$$\begin{aligned} 1/4.99 &= f(4.99) = f(5 + (-0.01)) \approx f(5) + (-0.01) f'(5) \\ &= 0.2 - 0.01 \cdot (-0.04) = 0.2004. \end{aligned}$$

Using a calculator, we can find a better value: $1/4.99 = 0.200400801 \cdots$. A more sophisticated calculation can do better: 0.20040080160320641282565130. Thus, in this case the tangent line approximation is accurate to 5 decimal places.

You should consider the question of what decimal expansion, if any, would be a completely precise answer? (**Hint:** In this case, it is a recurring decimal with a pattern of 498 digits!)

EXAMPLE 5.3. On the circle of radius 10 centered at the origin find a point in the first quadrant whose x-coordinate is 6.01. Also, do the same for $x = 5.99$.

First the point $x = 6.01$. Here we know the "easy point" $(6, 8)$ on the circle, and we want to know the y-coordinate of the nearby point $(6.01, ?)$. The function is $y = f(x) = \sqrt{100 - x^2}$, our x is 6, and our $x + h = 6.01$, i.e., $h = 0.01$. Using the technique in the last section (e.g., Problem 1 on the homework), we can find the derivative: $f'(x) = -x/\sqrt{100 - x^2} = -6/\sqrt{100 - 6^2} = -0.75$. Thus, the tangent line approximation tells us that $f(6.01) \approx f(6) + 0.01 \cdot f'(6) = 8 + 0.01(-0.75) = 7.9925$. (A more precise value, computed by calculator, is $7.992490225 \cdots$.)

The point $x = 5.99$ can be done similarly by taking $h = -0.01$ instead. So, we get $f(5.99) \approx 8 + (-0.01)(-0.75) = 8.0075$. (A more precise value, computed by calculator, is $8.0074902435 \cdots$.)

EXAMPLE 5.4. If 1 quart equals 0.9464 liter, and 1 liter of water occupies exactly 1000 cc (cubic centimeters), i.e., a cubic container 10 cm on a side, then what are the dimensions (in cm) of a cubic container that holds 1 quart of water?

In a story problem like this, we must first analyze the operation that is being performed. The question can be reworded as follows: if a cube holding 1000 cc has side 10 cm, then what is the side of a cube holding 0.9464 as much? The process of going from the volume to the side is the cube-root function. Since "0.9464 as much as 1000 cc" obviously means 946.4 cc, we can rephrase the question once again: if the cube-root of 1000 is 10, then what is the cube-root of 946.4?

Let s be the length of the side of the cube and V its volume, then

$$s = f(V) = \sqrt[3]{V}\,.$$

In the tangent line approximation we take $V = 1000$ and $V + \Delta V = 946.4$, i.e., $\Delta V = h = -53.6$. By the Power Rule, we have $f'(V) = \frac{1}{3} V^{-2/3}$. When $V = 1000$, we have $1000^{-2/3} = 1/100$, and so $f'(1000) = 1/(3 \cdot 100) = 1/300$. So the tangent line approximation says that $\sqrt[3]{946.4} = f(946.4) \approx f(1000) + (-53.6) \cdot f'(1000) = 10 - 53.6/300 = 9.821$ cm. (Note: a more exact value obtained by calculator is $9.81804256 \cdots$.)

Alternate Method. Write $V = s^3$. Then by the tangent line approximation $\Delta V \approx 3s^2 \Delta s$. Since $s = 10$ and $\Delta V = -53.6$, we have $-53.6 \approx 300 \Delta s$. Solving for Δs gives $\Delta s \approx -0.179$, so (new s) $= 10 + \Delta s \approx 10 - 0.179 = 9.821$ cm.

This method illustrates an important concept. It is often easier to use a relation between quantities, rather than one explicit functional evaluation. Solving explicitly for s was not necessary and indeed led to more complicated calculations!

The next example involves *percent error* . If you measure a value x, then the error (meaning the maximum possible error, or perhaps the maximum error that is likely) is the amount by which the true value could possibly differ from your measured value. We shall denote the error Δx. In most cases the true value could be on either side of the measured value, so the error is actually $\pm\Delta x$. By the **percent error** we mean the percent that this error is of the measured value, namely:

$$\text{percent error} = 100\frac{\Delta x}{x}.$$

For example, if you measure the speed of a car to be 50 mph to within 1 mph on either side, that means you have a percent error of $\pm 100(1)/50 = \pm 2\%$. If you know the percent error, say $\pm p\%$, then to convert this to absolute error h, use the formula: $\Delta x = \frac{p}{100}x$.

The reason for using the percent error is that it is a more realistic estimate of the error size. For instance an error of 10 feet in measuring a mile is not as serious as an error of 10 feet in measuring off a football field of 100 yards! The former percent error is only $100(10)/5280 = 0.189$, but the latter is $100(10)/300 = 3.333$.

Example 5.5. If you determine the length of a side of a cubic container by measuring the volume of liquid it holds, then an error of $\pm 1\%$ in measuring the volume will lead to what percent error in your value for the length of a side? An error of $\pm p\%$ will lead to what percent error for the side?

This is a lot like Example 5.4. Again our process is: starting with the volume of a cube, determine its side. That is, our function is again the cube-root function. But we're not given any concrete values — and to answer the question we don't need any.

So let V stand for the measured value of the volume, and let $s = f(V) = \sqrt[3]{V}$ be the length of a side. We want to know the error Δs that is caused if V is replaced by $V \pm \Delta V$, where ΔV is a 1% error, i.e., $\Delta V = 0.01V$. Actually, what we want to know is the *percent* error in the side s, i.e., $100(\Delta s)/s$. According to the tangent line approximation,

$$\Delta s \approx f'(V)\Delta V = \frac{1}{3}V^{-2/3}\cdot 0.01V = \frac{1}{3}\cdot 0.01\sqrt[3]{V},$$

where the last equality comes from the rule of exponents: $V^{-2/3}\cdot V = V^{-2/3}\cdot V^1 = V^{-2/3+1} = V^{1/3} = \sqrt[3]{V}$. But $\sqrt[3]{V}$ is precisely the value $s = f(V)$. So our percent error in s is

$$100\frac{\Delta s}{s} \approx 100\left(\frac{1}{3}\right)\frac{0.01\sqrt[3]{V}}{s} = \frac{1}{3}\%.$$

If we take $-\Delta V$ in place of ΔV, we get $-1/3\,\%$. Thus, the percent error in s is $\pm\frac{1}{3}\%$. This answers our first question.

If the error in V is $\pm p\%$ instead of $\pm 1\%$, everything would go through the same way, except with a factor of p everywhere:

$$100\frac{\Delta s}{s} \approx 100\frac{f'(V)\Delta V}{s} = 100\frac{(1/3)V^{-2/3}\cdot 0.01p\,V}{s} = \frac{1}{3}p,$$

so the error in s is $\pm\frac{1}{3}p\%$. That is, *a certain percent error in measuring the volume of a cube leads to a percent error one-third as great in the value computed for the side.*

Note. Notice where the 1/3 came from in the last example: the exponent in $x^{1/3}$ that comes down in the rule for the derivative of x^n. In the same way, it turns out that whenever your function is of the form $y = f(x) = Cx^n$ (where C is any constant), there is a simple relationship between the percent error in your measurement of x and the percent error in y obtained using the tangent line approximation. More examples of this will show up in the homework.

HOMEWORK 5.

1. Compute the following values using the tangent line approximation with an appropriate $f(x)$, and in each case see how close you are by computing the true value by calculator or computer:
(a) $(2.1)^2$, (b) $1/1.1$, (c) $1/0.99$, (d) $1/101$, (e) $\sqrt{4.01}$, (f) $\sqrt{26}$, (g) $\sqrt{24}$, (h) $\sqrt[3]{8.012}$.

2. What is the y-coordinate of the point on the circle $x^2 + y^2 = 25$ whose x-coordinate is 3.08? Get an approximate answer using the tangent line approximation.

3. One liter is 1.0567 quarts. Use the tangent line approximation to answer the question: One quart is how many liters?

4. You want to tile a square floor, which you guess to be 15 feet by 15 feet. Based on your guess, you buy 225 square feet of tile. But after you've used up all of the tile, you still have 5 square feet not covered with tile. What are the actual dimensions of the square floor? Use the tangent line approximation.

5. Roughly speaking, an acre is about 200 feet by 200 feet. More precisely, an acre is 43560 square feet. What are the dimensions of an acre of land in the shape of a square? Use the tangent line approximation.

6. If a right triangle has legs 6 and 8, then the smallest circle in which one can put this right triangle has area 25π (its diameter is the hypotenuse of the triangle, which is 10). Suppose that one leg of the right triangle is known to be exactly 6, but the other leg is known to be 8 with an error of $\pm h$. What is the area of the smallest circle containing the triangle? Give the error estimate for this area in terms of h.

7. You measure a cubic container, and find it to be about 10 cm on a side. From this you conclude that it holds 1000 cc (cubic centimeters). However, your measurement is accurate only to within ± 0.1 cm. In other words, you can only be sure that a side of the cube is between 9.9 and 10.1 cm. Use the tangent line approximation to estimate the error in your value of 1000 for the volume. Then give the *percent error* in your measurement of the length of a side and the *percent error* in your value for the volume.

8. The distance in feet that an object falls in t seconds is given by the formula $y = 16t^2$. (a) If an object is dropped from a height of 400 ft, how long will it take to fall to the ground? (b) If your value of 400 ft for the initial height of the object is accurate only to $\pm 1\%$, how accurate is your answer to part (a)? Determine both the absolute error and the percent error. (c) If your value of 400 ft is accurate to $\pm p\%$, then how accurate (as a percent error, expressed in terms of p) is your answer to part (a).

9. Suppose that you know that a circle has area A with an error of $\pm 1\%$. (a) Find a formula for the radius r in terms of the area A. (b) If your value for A is off by $\pm 1\%$, how far off (as a percent error) is your value for r obtained using the formula in part (a)? (c) What if your value for A is off by $\pm p\%$?

10. If you know that the hypotenuse of a right triangle is exactly 10, and one of the legs is 6 ± 0.20, then what is the other leg? Include an error estimate in your answer.

11. The diameter of a sphere is measured with an error of $\pm p\%$, and this measurement is used to calculate the volume of the sphere. What is the percent error in the value computed for the volume?

12. What is the permissible percent error in measuring the side of a cube if you want at most a $\pm p\%$ error in your value for (a) the surface area of the cube, (b) the volume of the cube.

13. The distance around a sphere's equator is measured with an accuracy of $\pm p\%$. This value is used to compute the sphere's (a) radius, (b) volume, (c) surface area. In each case, determine the percent error.

14. A spherical container is filled with water, and the volume of water the container holds is measured to within $\pm p\%$. This information is then used to compute the surface area of the spherical inside surface of the container. What is the percent error in the value obtained for this surface area?

15. The height of a cylinder is known exactly, but its diameter is measured only to within $\pm p\%$. This information is then used to compute (a) the volume of the cylinder, (b) the circumference of the top of the cylinder, (c) the lateral surface area of the cylinder (top and bottom not included). In each case determine the percent error.

16. Use the tangent line approximation to find an x near 2 such that $x^5 + x^2 = 35$. (Hint: Replace the left hand side by its tangent line approximation at $x = 2$ and solve for h.)

17. A company is making solid cube paperweights. The material used to make the cubes costs 2¢/per cubic cm, while the paint used to paint the 6 square faces costs 0.5¢/per square cm. If the paperweight is $3\,\mathrm{cm} \times 3\,\mathrm{cm} \times 3\,\mathrm{cm}$, then we calculate that the total cost of the material is 54¢ + 27¢ = 81¢. If you want the total cost to be 75¢ rather than 81¢, by how much do you have to decrease the side of the paperweight? Use the tangent line approximation.

CHAPTER 6

Rules For Finding Derivatives

1. Linearity of the Derivative

This means two things:

1. If you know the derivative $f'(x)$ of a function $f(x)$, then the derivative of a constant multiple $cf(x)$ is simply $cf'(x)$. *The derivative of a constant c times a function is equal to c times the derivative of the function*:
$$\frac{d}{dx}\big(cf(x)\big) = c\frac{d}{dx}f(x).$$
This principle is obvious if you think of the derivative as a rate of change. For example, it says that if $f(t)$ is the distance function for a bicycle, and if the distance function for a car is always 5 times the distance for the bicycle, then the car must be going at 5 times the velocity of the bicycle:
$$\frac{d}{dt}\big(5f(t)\big) = 5\frac{d}{dt}f(t)\,.$$
2. If you know the derivatives $f'(x)$ and $g'(x)$ of two functions $f(x)$ and $g(x)$, then the derivative of the sum $f(x) + g(x)$ is simply $f'(x) + g'(x)$. *The derivative of the sum of two functions is equal to the sum of the derivatives of the functions*:
$$\frac{d}{dx}\big(f(x) + g(x)\big) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$$
This principle is also obvious when one thinks of rates. It says that the rate at which the sum of two distances is changing is equal to the sum of the rates at which each distance is changing.

If you have already caught on to the idea of the derivative as a limit of the difference quotient $\frac{\Delta y}{\Delta x}$, then these are easy to understand. If y is replaced by cy then clearly the quotient always gets multiplied by the same constant c and so does the derivative which is its limit! Similarly, if $y = w + z$ is a sum of two functions, then their difference quotients clearly add up and give the sum formula.

Principle (2) can be used repeatedly, if we have a sum of more than two functions. It can also be used in combination with the first principle. When combined in this way, the two principles together tell us that it is easy to find the derivative of any **linear combination** of functions whose derivatives we know. By a "linear combination" we mean any function that is constructed using the process of addition and multiplying by constants. For example, a polynomial is a linear combination of power functions: $c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$.

EXAMPLE 6.1. In the section on the derivative function, we found the derivative of t^2 (or x^2), of $mx + b$, and of the reciprocal function. Use that, together with linearity of the derivative, to do Problem 2(a)-(c) of that section much more quickly.

We have:

$$\frac{d}{dt}\left(80-4.9t^2\right)=\frac{d}{dt}80-4.9\frac{d}{dt}t^2=0-4.9(2t)=-9.8t.$$
$$\frac{d}{dx}\left(x^2-\frac{1}{x}\right)=\frac{d}{dx}x^2-\frac{d}{dx}x^{-1}=2x-\left(-x^{-2}\right)=2x+\frac{1}{x^2}.$$
$$\frac{d}{dx}\left(ax^2+bx+c\right)=a\frac{d}{dx}x^2+\frac{d}{dx}(bx+c)=2ax+b.$$

Example 6.2. Find the derivative of $y=\left(1-5x^3\right)^2$.

Since $\left(1-5x^3\right)^2=1-10x^3+25x^6$, the derivative is

$$\begin{aligned}\frac{d}{dx}\left(1-10x^3+25x^6\right)=&\frac{d}{dx}1-10\frac{d}{dx}x^3+25\frac{d}{dx}x^6\\ =&0-10\cdot 3x^2+25\cdot 6x^5=-30x^2+150x^5.\end{aligned}$$

2. The Product Rule

Suppose that we want to find the derivative of a function that is written as a product $f(x)=u(x)v(x)$, where each of the two factors $u(x)$ and $v(x)$ is a function whose derivative we know. For example, $f(x)=x^3\sqrt{25-x^2}$ is the product of a function $u(x)=x^3$ whose derivative was already computed (namely, $u'(x)=3x^2$) and a function $v(x)=\sqrt{25-x^2}$ whose derivative we also know (namely, $v'(x)=-x/\sqrt{25-x^2}$).

We derive the product rule using the tangent line approximation formula for $u(x)$ and $v(x)$:

$$\begin{aligned}u(x+h)&\approx u(x)+(h)u'(x);\\ v(x+h)&\approx v(x)+(h)v'(x).\end{aligned}$$

We now look at the difference quotient for $f(x)=u(x)v(x)$, whose limit as h approaches zero is the derivative $f'(x)$:

$$\begin{aligned}\frac{f(x+h)-f(x)}{h}&=\frac{u(x+h)v(x+h)-u(x)v(x)}{h}\\ &\approx\frac{\Big(u(x)+(h)u'(x)\Big)\Big(v(x)+(h)v'(x)\Big)-u(x)v(x)}{h}\\ &=\frac{u(x)v(x)+(h)\Big(u(x)v'(x)+u'(x)v(x)\Big)+(h)^2u'(x)v'(x)-u(x)v(x)}{h}\\ &=\Big(u(x)v'(x)+u'(x)v(x)\Big)+(h)u'(x)v'(x).\end{aligned}$$

As h approaches zero, the last term $(h)u'(x)v'(x)$ drops out, and we are left with $u(x)v'(x)+u'(x)v(x)$. Thus,

$$\text{if } f(x)=u(x)v(x),\qquad \text{then}\qquad f'(x)=u(x)v'(x)+u'(x)v(x).$$

This rule can be written in shorthand as follows:

(Product rule) $$\frac{d}{dx}(uv)=uv'+u'v.$$

In words, *the derivative of a product is equal to the first function times the derivative of the second function plus the derivative of the first function times the second function.*

The above calculations certainly motivate the formula and do give an idea of the proof. They still lack many details for a complete proof. You should think about what is possibly wrong with them. The details will be discussed in class.

WARNING: Never take the derivative of a product by multiplying the derivatives of the terms. With addition you have $(u+v)' = u' + v'$, but with multiplication $(uv)' \neq u'v'$.

EXAMPLE 6.3. Find the derivative of $f(x) = x^3\sqrt{25-x^2}$, and determine the points where the curve has a horizontal tangent line.

By the product rule, $f'(x) = x^3\left(-x/\sqrt{25-x^2}\right) + 3x^2\sqrt{25-x^2}$. To say that a point $(x, f(x))$ has a horizontal tangent line means that the slope of the tangent line is 0, i.e., $f'(x) = 0$. So we must solve the equation $-x^4/\sqrt{25-x^2} + 3x^2\sqrt{25-x^2} = 0$. We do this by first clearing denominators (multiplying everything through by $\sqrt{25-x^2}$). The result is: $-x^4 + 3x^2(25-x^2) = 0$. When you have an equation like this, it is a good idea to try to factor the expression on the left. Right away we can factor out x^2; then we gather terms in what remains: $x^2(-x^2+3(25-x^2)) = x^2(-4x^2+75) = 0$. This has roots $x = 0$ or $4x^2 = 75$, i.e., $x = \pm\sqrt{75/4} = \pm 2.5\sqrt{3} = \pm 4.33\cdots$. Thus, there are three points with horizontal tangent lines: $(0,0)$, $(4.33, 202.97)$, and $(-4.33, -202.97)$.

EXAMPLE 6.4. Use the product rule to show how the nx^{n-1} formula can be obtained for $n = 4, 5, 6, \ldots$ (i.e., the successive values of n after the values $n = 0, 1, 2, 3$ for which we already derived the nx^{n-1} formula earlier).

So suppose we don't yet know a formula for the derivative of x^4, but we do know the derivative of x^3 (namely, $3x^2$). Then we can write $x^4 = x^3 \cdot x$, so that, by the product rule,

$$\frac{d}{dx}x^4 = \frac{d}{dx}\left(x^3 \cdot x\right) = x^3\left(\frac{d}{dx}x\right) + \left(\frac{d}{dx}x^3\right)x = x^3 \cdot 1 + 3x^2 \cdot x = 4x^3.$$

This confirms the nx^{n-1} rule when $n = 4$. Now that we have established the rule for the derivative of x^4, we can use the same method to treat x^5:

$$\frac{d}{dx}x^5 = \frac{d}{dx}\left(x^4 \cdot x\right) = x^4\left(\frac{d}{dx}x\right) + \left(\frac{d}{dx}x^4\right)x = x^4 \cdot 1 + 4x^3 \cdot x = 5x^4.$$

And similarly for $n = 6$:

$$\frac{d}{dx}x^6 = \frac{d}{dx}\left(x^5 \cdot x\right) = x^5\left(\frac{d}{dx}x\right) + \left(\frac{d}{dx}x^5\right)x = x^5 \cdot 1 + 5x^4 \cdot x = 6x^5.$$

By now it should be clear that, continuing in this way, we can establish the validity of the nx^{n-1} rule for all positive integers n.

3. The Quotient Rule

Suppose that we want to find the derivative of a function that is written as the quotient of two functions whose derivatives we know: $f(x) = u(x)/v(x)$. The rule is:

(Quotient Rule) $$\frac{d}{dx}\frac{u(x)}{v(x)} = \frac{v(x)u'(x) - u(x)v'(x)}{v(x)^2},$$

or, in shorthand notation, $\left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}$. This rule is a little more complicated than the product rule: *the derivative of a quotient is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all over the square of the denominator.* We shall not prove the quotient rule here. We will derive the quotient rule later in this section, when we cover the chain rule.

EXAMPLE 6.5. Use the quotient rule to find the derivative of the derivative of the upper semicircle function $y = \sqrt{25 - x^2}$. (The derivative of the derivative is called the " second derivative.")

Since the derivative is $-x/\sqrt{25-x^2}$, we want the derivative of an expression of the form $u(x)/v(x)$, where $u(x) = -x$ and $v(x) = \sqrt{25-x^2}$. Since $u' = -1$ and $v' = -x/\sqrt{25-x^2}$, the quotient rule gives us:

$$\frac{d}{dx}\frac{-x}{\sqrt{25-x^2}} = \frac{\sqrt{25-x^2}(-1) - (-x)\left(-x/\sqrt{25-x^2}\right)}{(\sqrt{25-x^2})^2} = \frac{-\sqrt{25-x^2} - \frac{x^2}{\sqrt{25-x^2}}}{25-x^2}.$$

The last expression can be simplified by multiplying the top and bottom by $\sqrt{25-x^2}$. After a little cancelation, we arrive at $-25/(25-x^2)^{3/2}$.

Alternative method: The above calculations can be simplified by using proper notations and some clever simplification. First set $y = \sqrt{25-x^2}$ and note that $y' = -x/y$. Applying the quotient rule and linearity we get

$$(y')' = -\frac{yx' - xy'}{y^2} = (-1/y^2)y - x(-x/y) = (-1/y^2)\frac{(y^2+x^2)}{y}.$$

Noticing that $x^2 + y^2 = 25$, we get that

$$y'' = -\frac{25}{y^3} = -\frac{25}{(25-x^2)^{3/2}}.$$

This idea will be developed into implicit differentiation method later.

EXAMPLE 6.6. Find the derivative of $\sqrt{100-x^2}/\sqrt{x}$ in two ways: using the quotient rule, and using the product rule.

First, using the quotient rule and the fact that $\sqrt{100-x^2}$ has derivative $\frac{-x}{\sqrt{100-x^2}}$ and $\sqrt{x}$ has derivative $\frac{1}{2\sqrt{x}}$, we obtain:

$$\begin{aligned}\frac{d}{dx}\frac{\sqrt{100-x^2}}{\sqrt{x}} &= \frac{\sqrt{x}(-x/\sqrt{100-x^2}) - (1/2\sqrt{x})\sqrt{100-x^2}}{x} \cdot \frac{2\sqrt{x}\cdot\sqrt{100-x^2}}{2\sqrt{x}\cdot\sqrt{100-x^2}} \\ &= \frac{-x^2-100}{2x^{3/2}\sqrt{100-x^2}}.\end{aligned}$$

Alternately, we could use the product rule if we first write our function in the form $x^{-1/2}\sqrt{100-x^2}$:

$$\begin{aligned}\frac{d}{dx}\left(x^{-1/2}\sqrt{100-x^2}\right) &= x^{-1/2}\frac{-x}{\sqrt{100-x^2}} - \frac{1}{2}x^{-3/2}\sqrt{100-x^2} \\ &= -\frac{x^{1/2}}{\sqrt{100-x^2}} - \frac{1}{2}x^{-3/2}\sqrt{100-x^2},\end{aligned}$$

which is algebraically equivalent to the answer we got using the quotient rule.

Alternative method: As above, the work can be cleaned up by using a few names. Let $y = \sqrt{100-x^2}$ and $z = x^{1/2}$. Then $y' = -\frac{x}{\sqrt{100-x^2}} = -\frac{z^2}{y}$. Also, $z' = \frac{1}{2z}$. The derivative of the function becomes:

$$\frac{y}{z} = \frac{zy' - yz'}{z^2} = \frac{1}{z^2}(-\frac{z^3}{y} - \frac{y}{2z}) = -\frac{1}{2yz^3}(2z^4 + y^2).$$

Getting the final answer is now the matter of plugging in the values of y, z and simplifying.

A bit of advice on avoiding the quotient rule. Do **not** use the quotient rule when the numerator is a constant, because it is unnecessarily cumbersome in that case. For example, to find the derivative of $10/x^2$, write this as $10x^{-2}$ and use the power rule: the derivative is $10 \cdot (-2)x^{-3} = -20/x^3$.

On the other hand it is a good idea to try your quotient rule on $1/x$, whose derivative is known to be $-1/x^2$. It helps you to remember that the negative term in the numerator is $-uv'$.

4. The Second Derivative and Acceleration

As mentioned before, the second derivative is simply the derivative of the derivative. If $y = f(x)$ is our function, then there are various ways to denote the second derivative $\dfrac{d}{dx}\left(\dfrac{d}{dx} f(x)\right)$:

$$y'' = f''(x) = \frac{d^2}{dx^2} f(x) = \frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{d}{dx} f(x)\right).$$

Example 6.7. Find the second derivative of (a) $y = x^n$, (b) $y = 1/x$, (c) $ax^2 + bx + c$ (where a, b, c are constants).

(a) We have

$$\frac{d}{dx}\left(\frac{d}{dx} x^n\right) = \frac{d}{dx}\left(nx^{n-1}\right) = n(n-1)x^{n-2}.$$

Then (b) is the special case $n = -1$, i.e., the second derivative of x^{-1} is $2x^{-3}$.

In part (c) we have $\dfrac{d}{dx}\big(\dfrac{d}{dx}(ax^2 + bx + c)\big) = \dfrac{d}{dx}(2ax + b) = 2a$.

Dot Notation. If the time t is playing the role of the x-variable, i.e., if $y = f(t)$, then we often use dots rather than primes for the derivatives:

$$\dot{y} = \frac{d}{dt} f(t), \quad \ddot{y} = \frac{d^2}{dt^2} f(t).$$

If the y-variable is denoted by a letter other than y, we still use the dot notation. For example, if the distance function is denoted $s = s(t)$, then we write $\ddot{s}(t)$ for the second derivative of the distance function.

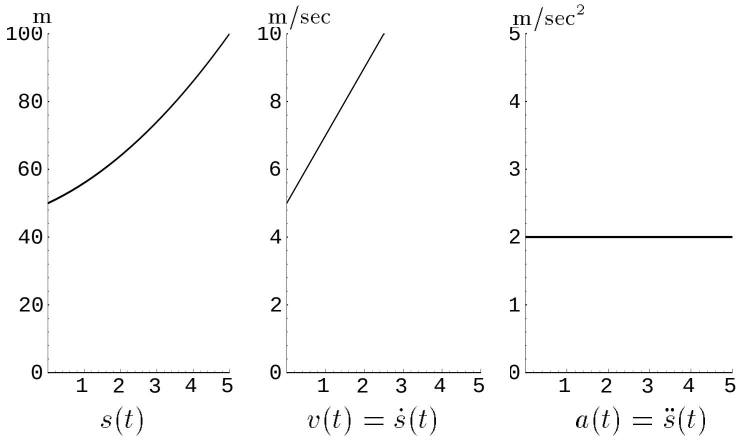
In the case of a distance function $s = s(t)$, both the first derivative function $\dot{s}(t)$ and the second derivative function $\ddot{s}(t)$ have a familiar practical meaning. The first derivative $\dot{s}(t)$ is, as we've seen, the velocity function, which is often denoted $v(t)$: $v(t) = \dot{s}(t) = \dfrac{d}{dt} s(t)$. The derivative of that — the instantaneous rate at which the velocity is changing — is what is called the **acceleration** , often denoted $a(t)$:

$$a(t) = \frac{d}{dt} v(t) = \frac{d^2}{dt^2} s(t) = \ddot{s}(t).$$

Earlier we saw that the velocity at a certain instant t is the slope of the tangent line to the graph of the distance function at the point $(t, s(t))$. Similarly, the acceleration at a certain instant t is the slope of the tangent line to the graph of the velocity function at the point $(t, v(t))$.

Example 6.8. A bicycle is traveling along a road (which we take to be the x-axis). Its distance in meters from the reference point (0 on the x-axis) at time t seconds is given by the formula $x(t) = 50 + 5t + t^2$. Find the derivative function $\dot{x}(t)$ and the second derivative function $\ddot{x}(t)$, graph all three functions (distance, velocity, and acceleration), and explain in words what is going on.

First $v(t) = \dot{x}(t) = \dfrac{d}{dt}(50+5t+t^2) = 5+2t$. Next, $a(t) = \ddot{x}(t) = \dfrac{d}{dt}v(t) = \dfrac{d}{dt}(5+2t) = 2$. The distance function $x(t)$ tells us that the bicycle is at $x(0) = 50$ meters from the reference point at time $t = 0$ and is moving to the right at a certain velocity $v(t)$. The formula for this velocity function $v(t) = 5 + 2t$ says that the speed of the bicycle is 5 m/sec at first ($t = 0$), but it increases steadily. The rate at which the velocity increases is the acceleration function $a(t) = 2$, which happens to be a constant function in this example. The acceleration is measured in "meters per second per second" or "meters per second squared," written 2 m/sec^2. Here are the graphs of the distance, velocity, and acceleration functions:



Example 6.9. (Free fall without air resistance)

This is a good place to discuss the vertical motion of an object moving without air resistance under the influence of gravity. Let y denote the height (say in meters) of the object above ground level and let t be the time (in seconds). Suppose that at time $t = 0$ an object is located at height $y = y_0$ and has velocity v_0. (Notice that v_0 is positive means that the object is initially moving upward.) Then y and t are related by the formula

(Falling Body Formula) $$y = y(t) = -\frac{1}{2}gt^2 + v_0t + y_0 \,.$$

Taking derivatives gives the formula for the velocity

$$v(t) = \frac{d}{dt}y(t) = -gt + v_0 \,,$$

and taking one more derivative gives the acceleration

$$a(t) = \frac{d}{dt}v(t) = -g \,.$$

In particular, the acceleration of a falling body is constant, it is called the **acceleration due to gravity** . On the surface of the earth $g = 9.8$m/sec^2. (Notice that the acceleration is *negative* because the force of gravity points down.)

Deducing the formulas; We stated the above formulas without proofs. However, if we start with the formula $a(t) = -9$, then the formulas for $v(t)$ and $s(t)$ can be deduced by guesswork and the differentiation formulas that we already know. For instance, the formula for $v(t)$ must be $-gt+C$ for some constant C (think why!). Checking the values at $t=0$, we can see why $C=v_0$. You should attempt to finish the formula for $y(t)$ yourself.

This is the process of finding antiderivatives and will be developed later.

5. The Chain Rule

The chain rule is the way to find the derivative of a function that is built up from simpler functions by the **composition of two functions** , i.e. by taking a "function of a function." For example, the function $\sqrt{1+x^4}$ is built up from the square root function and the function $1+x^4$. The function $1/\sqrt{5-\sqrt{x}}$ is built up from the reciprocal square root function (i.e., the $(-1/2)$-power function) and the function $5-\sqrt{x}$. We shall write $y=f(x)=g(u(x))$ to indicate that y is obtained by taking a function $g(u)$ (which we sometimes call the "outside function") and applying it to another function $u(x)$ (which we call the "inside function"). In the two examples just mentioned:

$$g(u)=\sqrt{u} \text{ and } u(x)=1+x^4 \quad \Longrightarrow \quad y=f(x)=g(u(x))=\sqrt{1+x^4};$$
$$g(u)=1/\sqrt{u} \text{ and } u(x)=5-\sqrt{x} \quad \Longrightarrow \quad y=f(x)=g(u(x))=1/\sqrt{5-\sqrt{x}}.$$

The chain rule enables us to find dy/dx once we know the derivative of the outside function $dy/du=g'(u)$ and the derivative of the inside function $du/dx=u'(x)$.

We shall derive the chain rule using the tangent line approximation. The tangent line approximation formula for the function $y=g(u)$ can be written in the form $\Delta y \approx g'(u)\Delta u$. Similarly, the tangent line approximation formula for the function $u(x)$ can be written $\Delta u \approx u'(x)\Delta x$. Putting these two facts together, we see that a small change Δx produces a change $u'(x)$ times as much in u, which, in turn, produces a change $g'(u)$ times as much in y, i.e., $\Delta y \approx g'(u)\Delta u \approx g'(u)u'(x)h$. Since we also have $\Delta y \approx f'(x)\Delta x$, it follows that the two "proportionality factors" between Δx and Δy must be equal, i.e., $f'(x)=g'(u)u'(x)$. This is the chain rule. It is especially easy to remember when it is written in the dy/dx notation:

(Chain Rule) $$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}.$$

It is as if the "dy," "du," and "dx" can be treated individually as algebraic quantities, where we cancel the two "du" on the right.

Indeed, another way of formulating the above calculations is to write

$$\frac{\Delta y}{\Delta x}=\frac{\Delta y}{\Delta u}\cdot\frac{\Delta u}{\Delta x}$$

which is precise. Now we take the limits as Δx goes to 0. We have to see why Δu also goes to zero and then the formula will be proved. We leave the precise treatment for the appendix.

Schematically, we can regard a function of a function as a 2-step procedure, first going from x to u and then from u to y. The chain rule says that the derivative from x to u must be multiplied by the derivative from u to y to get the derivative from x to y.

Before giving some computational examples, we shall give a story problem to show that the chain rule makes sense from a practical point of view.

Example 6.10. Suppose your income is in the 28% tax bracket, and it is increasing at the rate of \$1000 per year. At what rate is your tax increasing?

As before, let us use x to denote income and y to denote the income tax. Recall that "28% tax bracket" means that y as a function of x has slope 0.28 over the range of values of x (your income) in question. In this problem y depends on x, which in turn depends on the time t. We could write that $y = g(x)$ and $x = x(t)$ together give $y = f(t) = g(x(t))$. (Thus, x is playing the rule of u in the chain rule, and t is playing the role of x.) The chain rule says that

$$\begin{aligned}\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} &= (28\%) \cdot (1000 \text{ dollars/ year}) \\ &= 0.28 \cdot 1000 \text{ dollars/year} = 280 \text{ dollars/year}.\end{aligned}$$

That is, tax is increasing at the rate of \$0.28 for every dollar increase in income, and income is increasing at the rate of \$1000 for every year increase in time; hence, tax is increasing at the rate $0.28 \cdot 1000 = 280$ dollars every year.

EXAMPLE 6.11. (a) Use the chain rule to find the derivative of the upper semicircle function $y = \sqrt{r^2 - x^2}$ (where r is a constant), and see that the answer agrees with our earlier formula for the derivative obtained using geometry. Also find the derivative of the two examples mentioned above: (b) $y = \sqrt{1 + x^4}$, (c) $y = 1/\sqrt{5 - \sqrt{x}}$.

(a) We take $y = g(u) = \sqrt{u}$ for the outside function and $u = r^2 - x^2$ for the inside function. Then, since $du/dx = -2x$, we have

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{u}} \cdot (-2x) = -\frac{x}{\sqrt{u}} = -\frac{x}{\sqrt{r^2 - x^2}},$$

as expected.

(b) Here the outside function is $y = g(u) = \sqrt{u}$, and the inside function is $u = 1 + x^4$. So

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{u}} \cdot 4x^3 = \frac{4x^3}{2\sqrt{1 + x^4}}$$

(the answer should be written in terms of x, i.e., in $g'(u)$ you should substitute $u = u(x)$, which in this case is $1 + x^4$).

(c) Here the outside function is $y = g(u) = u^{-1/2}$, and the inside function is $u = 5 - \sqrt{x}$. So

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\frac{1}{2}u^{-3/2}\left(-\frac{1}{2\sqrt{x}}\right) = \frac{1}{4(5 - \sqrt{x})^{3/2}\sqrt{x}}.$$

When the outside function $g(u)$ is the power function u^n, we can write the chain rule in the form

$$\frac{d}{dx}\left(u(x)^n\right) = n\, u(x)^{n-1}\, \frac{du}{dx}.$$

For example, the derivative of the square of a function is $\frac{d}{dx}u(x)^2 = 2u(x)u'(x)$.

EXAMPLE 6.12. Find the derivative of $(1 + 3x)^2$ in two ways: (a) by the chain rule; (b) by expanding the square.

(a) In the above formula for the derivative of $u(x)^2$, we take $u(x) = 1 + 3x$, so that $u'(x) = 3$. Then $\frac{d}{dx}(1 + 3x)^2 = 2(1 + 3x) \cdot 3 = 6(1 + 3x)$.

(b) We write $(1+3x)^2 = 1+6x+9x^2$, and then $\frac{d}{dx}(1+6x+9x^2) = 6+9\cdot 2x = 6+18x = 6(1 + 3x)$, the same answer as in part (a).

We next use the chain rule to show how we get the quotient rule. We write the quotient $u(x)/v(x)$ in the form of a product $u(x)(v(x)^{-1})$. Then we apply the chain rule to find the derivative of $v(x)^{-1}$, where the outside function is the reciprocal function and the inside function is $v(x)$. According to the chain rule,

$$\frac{d}{dx}\left(v(x)^{-1}\right) = -\frac{1}{v(x)^2}v'(x).$$

We are now ready to apply the product rule:

$$\begin{aligned}\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) &= \frac{d}{dx}\left(u(x)(v(x)^{-1})\right) = u(x)\frac{d}{dx}\left(v(x)^{-1}\right) + u'(x)(v(x)^{-1}) \\ &= u(x)\left(-\frac{1}{v(x)^2}v'(x)\right) + \frac{u'(x)}{v(x)} = \frac{-u(x)v'(x)}{v(x)^2} + \frac{v(x)}{v(x)}\cdot\frac{u'(x)}{v(x)} \\ &= \frac{v(x)u'(x) - u(x)v'(x)}{v(x)^2},\end{aligned}$$

which is the quotient rule.

Another comment to make about the chain rule is that it can be used to find the derivative of a function of a function of a function, or even a function of a function of a function of a function, and so on, where in each case we build up a function by applying a chain of simple functions. Here is an example to illustrate this.

Example 6.13. Find the derivative of $\sqrt{1+\sqrt{1+\sqrt{x}}}$.

The outermost function is $g(u) = \sqrt{u}$, where $u(x) = 1 + \sqrt{1+\sqrt{x}}$. In the chain rule we need $u'(x)$, which in this case we find by using the chain rule again:

$$u'(x) = \frac{1}{2\sqrt{1+\sqrt{x}}}\cdot\frac{d}{dx}(1+\sqrt{x}) = \frac{1}{4\sqrt{1+\sqrt{x}}\sqrt{x}}.$$

Finally,

$$\begin{aligned}\frac{d}{dx}\sqrt{1+\sqrt{1+\sqrt{x}}} &= \frac{d}{dx}\sqrt{u} = \frac{1}{2\sqrt{u}}u'(x) = \frac{1}{2\sqrt{1+\sqrt{1+\sqrt{x}}}}\cdot\frac{1}{4\sqrt{1+\sqrt{x}}\sqrt{x}} \\ &= \frac{1}{8\sqrt{x}\sqrt{1+\sqrt{x}}\sqrt{1+\sqrt{1+\sqrt{x}}}}.\end{aligned}$$

HOMEWORK 6.

1. Find d/dx of

(a) $x^4 - 3x^3 + 0.5x^2 + 7x - \pi$, (b) $x^3 - 2x^2 + 4\sqrt{x}$,
(c) $(x^2+1)^3$ (expand out the cube first),
(d) $(x^2+1)^3$ (use the chain rule), (e) $x\sqrt{169 - x^2}$,
(f) $(x^2 - 4x + 5)\sqrt{25 - x^2}$, (g) $(x^2 + x + 1)/(1 - x)$,
(h) $\sqrt{25 - x^2}/x$, (i) $\sqrt{(169/x) - x}$ (use the chain rule),
(j) $\sqrt{(169/x) - x}$ (factor out the $1/x$ and use the product or quotient rule),
(k) $\sqrt{x^3 - x^2 - (1/x)}$, (l) $100/(100 - x^2)^{3/2}$,
(m) $\sqrt[3]{x + x^3}$, (n) $\sqrt{(x^2+1)^2 + \sqrt{1 + (x^2+1)^2}}$.

2. Find the first derivative d/dt and also the second derivative d^2/dt^2 of the following functions of t (here a, b, and c are constants): (a) $-4.9t^2 + 20t + 50$, (b) $at + b$, (c) $\sqrt{at + b}$, (d) $\sqrt{at^2 + bt + c}$.

3. Suppose that the bicycle in Example 6.8 has distance function $x(t) = 50 - 5t + t^2$. Find $v(t) = \dot{x}(t)$ and $a(t) = \ddot{x}(t)$. Graph $x(t)$ and $v(t)$, and explain in words what the bicycle is doing.

4. If $f(x) = \sqrt{3 + x^n}$, then $f(1) = 2$. Use the tangent line approximation to find $f(1 + h)$. Express your answer in terms of n and Δx.

5. You want a ladder to reach a window which is about 24 feet above the ground. You place the bottom of the ladder exactly 7 feet away from the side of the building. If the window were exactly 24 feet high, you compute that you would need 25 feet of ladder to reach it. Suppose that the error in your value of 24 for the height of the window is $\pm h$. Find the error in the figure of 25 feet for the length of ladder.

6. In the situation of Problem 6 of the last homework, suppose that you want to fit the triangle in the smallest possible *sphere* (rather than circle). In that case the triangle's hypotenuse becomes a diameter of the sphere. Find the volume of that smallest sphere, and include an error estimate obtained using the tangent line approximation.

7. Suppose that an object falling from a height of 80 meters (see Problem 4 of Chapter 3) is viewed from a height of 2 meters by a person standing 10 meters from the side of the building where the object is falling. (a) Find a formula in terms of t for the distance from the person's eye to the object. (b) Find a formula in terms of t for the rate at which that distance is changing.

CHAPTER 7

Implicit Functions and Implicit Differentiation

1. Implicit Functions

We have seen that not all functions $y = f(x)$ are given by formulas. Sometimes a function is determined by interpolating a table of experimentally determined data (drawing a smooth curve between the points). Another way a function can be determined is *implicitly*. That means that we have a formula which x and y together must satisfy. The equation may take the form $F(x, y)$ =constant, or perhaps $F(x, y) = G(x, y)$.

EXAMPLE 7.1. (a) The circle can be written $x^2 + y^2 = r^2$, which is actually the implicit equation of two functions at the same time: the upper semicircle function $y = \sqrt{r^2 - x^2}$ and the lower semicircle function $y = -\sqrt{r^2 - x^2}$.

(b) We can avoid fractional powers $y = x^{l/m}$ by raising both sides to the m-th power and writing the implicit equation $y^m = x^l$.

(c) We can avoid negative powers by writing $y = x^{-l/m}$ in the form $y^m x^l = 1$.

(d) The points (x, y) on an ellipse with focii at (x_1, y_1) and (x_2, y_2) and semimajor axis a satisfy the geometrical property:

$$(\text{ distance from } (x, y) \text{ to } (x_1, y_1)) + (\text{ distance from } (x, y) \text{ to } (x_2, y_2)) = 2a,$$

which leads to the following implicit equation for the ellipse:

$$\sqrt{(x - x_1)^2 + (y - y_1)^2} + \sqrt{(x - x_2)^2 + (y - y_2)^2} = 2a.$$

(e) In the case when the ellipse's axes of symmetry are parallel to the x- and y-axes, a simpler implicit equation can be given, namely $(x - h)^2/a^2 + (y - k)^2/b^2 = 1$.

With many implicit equations, it is possible to solve for y in terms of x. For example, we can do this for the circle (Example 7.1(a)). In that case we have the choice of whether to use the **explicit form** $y = f(x)$ or the **implicit form** $F(x, y)$ =constant. However, there are other examples where it is impossible to solve algebraically for y in terms of x, and we have no choice but to work with the implicit equation.

2. Implicit Differentiation

Suppose we want to find the derivative y' at a point (x, y) of an implicitly defined curve. What you do is: (1) take d/dx of both sides of the equation; (2) separate the terms involving y' to one side of the equation and the terms without y' to the other side; and (3) solve for y'. In step (1), it is important to remember that you are differentiating with respect to x (or whatever letter is playing the role of the x-variable), and so the chain rule must be used for expressions like y^2. In other words, the letter y stands for a certain function of x, and so taking d/dx of y^2 is just like taking d/dx of $u(x)^2$: $\frac{d}{dx}y^2 = 2y\frac{dy}{dx} = 2y\,y'$.

EXAMPLE 7.2. Find y' at the point (x, y) for each implicitly defined function in Example 7.1.

(a) When we take d/dx of both sides of $x^2 + y^2 = r^2$, remember that (1) the derivative of y^2 is *not* $2y$ but rather $2yy'$, and (2) the derivative of r^2 is *not* $2r$ but rather $2rr' = 0$, since $r' = 0$ as r is a constant.

Thus, taking d/dx of both sides gives: $2x + 2yy' = 0$. We next move the $2x$ to the right, and divide both sides by $2y$ to solve for y': $y' = -2x/(2y) = -x/y$. This answer agrees with our earlier formula for the derivative of $y = \sqrt{r^2 - x^2}$, namely, $y' = -x/\sqrt{r^2 - x^2}$, since y and $\sqrt{r^2 - x^2}$ are the same. Notice that when you differentiate an explicit function $y = f(x)$ your answer is a formula in terms of x, whereas implicit differentiation generally leads to a formula involving *both* y *and* x.

Note that if we take the lower semicircle function $y = -\sqrt{r^2 - x^2}$, the expression for y' becomes $-\frac{x}{\sqrt{r^2-x^2}}$ which is a different formula. However, the formula $y' = -x/y$ is still the same! Thus the implicit differentiation not only avoids the solution process for y in terms of x, but also gives the right answer for all possible solutions at once.

Parts (b) and (c) of this example will show us how to derive the nx^{n-1} rule when the power of x is a fraction $n = l/m$ or $n = -l/m$, where l and m are positive integers. (In the section on the product rule, we already saw how to get the nx^{n-1} rule for any positive integer n.)

(b) We take d/dx of both sides of $y^m = x^l$, and then solve for y':

$$my^{m-1}y' = lx^{l-1}, \qquad \text{so that} \qquad y' = \frac{l}{m}\frac{x^{l-1}}{y^{m-1}}.$$

The last ratio simplifies algebraically, if we use the fact that $y^{m-1} = y^m/y = x^l/x^{l/m}$. That is,

$$y' = \frac{l}{m}\frac{x^{l-1}}{x^l/x^{l/m}} = \frac{l}{m}x^{l/m-1} = nx^{n-1} \qquad (\text{here } n = \frac{l}{m}).$$

(c) We take d/dx of both sides of $y^m x^l = 1$ (using the product rule), and then solve for y':

$$y^m\left(lx^{l-1}\right) + my^{m-1}y'x^l = 0, \qquad \text{so that} \qquad y' = -\frac{ly^m x^{l-1}}{my^{m-1}x^l}.$$

Canceling $y^m x^l$ from the numerator and denominator in the last expression, we obtain:

$$-\frac{l}{m}\frac{x^{-1}}{y^{-1}} = -\frac{l}{m}\frac{y}{x} = -\frac{l}{m}\frac{x^{-l/m}}{x} = -\frac{l}{m}x^{-\frac{l}{m}-1} = nx^{n-1}, \qquad \text{where} \qquad n = -\frac{l}{m}.$$

(d) When we take d/dx of both sides, we must use the chain rule several times, first with the square root as the outside function and then when we find the derivative of $(x - x_1)^2$, $(y - y_2)^2$, etc. The result is:

$$\frac{(x - x_1) + (y - y_1)y'}{\sqrt{(x - x_1)^2 + (y - y_1)^2}} + \frac{(x - x_2) + (y - y_2)y'}{\sqrt{(x - x_2)^2 + (y - y_2)^2}} = 0.$$

Solving for y' is not difficult, but the expression we end up with is rather messy, so we shall omit it. If we had specific values for the foci (x_1, y_1) and (x_2, y_2), and if we wanted to find the slope y' at a particular point (x, y), then we could substitute all of these numerical values in place of x, y, x_1, y_1, x_2, y_2, ending up with an equation of the form $\alpha + \beta y' = 0$ (where β comes from bringing together all of the terms with y' and α comes from bringing together all the terms without y'). Such an equation has solution simply $y' = -\alpha/\beta$. We will see such situations later in the story problems.

(e) Here we get: $2(x-h)/a^2 + 2(y-k)y'/b^2 = 0$. Solving for y', we obtain

$$y' = -\frac{b^2}{a^2}\frac{x-h}{y-k}.$$

EXAMPLE 7.3. If you know that the curve $(xy)^4 + x^{-2} + y^{-1} = C$ (where C is a constant) passes through the point $(1,2)$, find the y-coordinate of the point $(1.01, ?)$ on the curve near $(1,2)$. Use the tangent line approximation.

(**Note:** The constant C is not given. We could easily compute it from the fact that the point $(1,2)$ satisfies the equation. However, we will never need to know the value of C, so we won't bother to compute it.)

In this problem there is no way you can find an exact expression for the y-coordinate corresponding to $x = 1.01$, because when you substitute $x = 1.01$ you get an equation for y that cannot be solved algebraically. The equation for y could be solved by an approximation method that we'll learn later (Newton's method). In the meantime, the best we can do is to use the tangent line approximation to find the y-coordinate. In this problem we know that the implicitly defined $y = f(x)$ has value 2 when $x = 1$ (this is what it means to say that the curve passes through the point (1,2)). We want to know $f(1.01)$. According to the tangent line approximation, $f(1.01) \approx f(1) + 0.01f'(1) = 2 + 0.01y'$, where y' denotes the derivative at the point $(1,2)$. To find that derivative we first take d/dx of both sides of our implicit equation. We use the chain rule and the product rule to find $\frac{d}{dx}(xy)^4$, and on the right we get 0 (the derivative of any constant is 0):

$$4(xy)^3(xy' + 1\cdot y) + (-2x^{-3}) + (-y^{-2}y') = 0.$$

Once we've taken d/dx we can substitute the particular x and y values of the point where we want the derivative. After that our equation will look a lot simpler — just some numbers together with the unknown y':

$$\begin{aligned} & 4(1\cdot 2)^3(y'+2) + (-2\cdot 1) + (-2^{-2}y') = 0, \\ \text{i.e.,}\quad & y'(32 - 0.25) + 64 - 2 = 0, \\ \text{i.e.,}\quad & y' = -62/31.75 = -1.953. \end{aligned}$$

Finally, substituting this value of $y' = f'(1)$ in the tangent line approximation gives the y-coodinate as $2 + 0.01\cdot(-1.953) = 1.98047$.

EXAMPLE 7.4. Suppose that the ellipse in Example 7.1(d) has foci at the points (0,1) and (3,0), and passes through the point (2,3). Find the y-coordinate of the point $(1.99, ?)$ through which it passes. Use the tangent line approximation.

The tangent line approximation tells us that $f(1.99) \approx f(2) - 0.01f'(2) = 3 - 0.01y'$, where y' is the derivative at the point $(2,3)$. The implicit differentiation of the equation of this ellipse has already been carried out, so it remains just to substitute the particular values we're given: $x_1 = 0$, $y_1 = 1$, $x_2 = 3$, $y_2 = 0$, $x = 2$, $y = 3$:

$$\frac{2+2y'}{\sqrt{8}} + \frac{-1+3y'}{\sqrt{10}} = 0, \qquad \text{so that} \qquad \left(\frac{2}{\sqrt{8}} - \frac{1}{\sqrt{10}}\right) + y'\left(\frac{2}{\sqrt{8}} + \frac{3}{\sqrt{10}}\right) = 0,$$

i.e., $0.39088 + 1.65579y' = 0$. Thus, $y' = -0.39088/1.65579 = -0.23607$, and so $f(1.99) \approx 3 - 0.01y' = 3.00236$.

EXAMPLE 7.5. You are standing at (0,1) and receiving light from two sources: source A located at $(-1, 0)$ and source B located at $(1, 0)$. You want to start walking in the direction which keeps the total light intensity from the two sources as constant as possible. Assume that light intensity from a source is proportional to the strength of the source and inversely proportional to the square of the distance from the source. (a) If A and B are equally strong, show that you should walk horizontally, i.e., set out along the line $y = 1$. (b) If B is twice as strong as A, find the line along which you should move.

We are interested here in the curve of constant total light intensity passing through the point (0,1). That is, we want an equation satisfied by all points (x, y) where the light intensity from A plus the light intensity from B remains constant. Since light intensity is inversely proportional to the square of the distance from the source, this means that

$$\frac{K}{(\text{dist of } (x,y) \text{ from A})^2} + \frac{K}{(\text{dist of } (x,y) \text{ from B})^2} = C.$$

Here the proportionality constant K depends on the strength of the source; since A and B have the same strength in part (a), the constants are the same. In part (b) we change the second K to $2K$, because B is twice as strong as A.

Next, we use the distance formula in the denominators: the square of the distance from (x, y) to $(-1, 0)$ is $(x + 1)^2 + y^2$, and the square of the distance from (x, y) to $(1, 0)$ is $(x - 1)^2 + y^2$. For convenience, we also divide through by K, so that there are no unknown constants on the left. The result is

$$\frac{1}{(x+1)^2 + y^2} + \frac{1}{(x-1)^2 + y^2} = \frac{C}{K}.$$

The question asks for the direction to go in from the point $(0, 1)$ that best keeps to the curve of constant total light intensity. This amounts to asking for the tangent line to the curve at $(0, 1)$. To find y', we take d/dx of both sides, and then substitute the point $(x, y) = (0, 1)$ where we want to find y'. It is important to note that the fact that you happen to be walking from the point $(0, 1)$ is not used until *after* performing the implicit differentiation. When we take d/dx, we use the chain rule, writing the first term on the left as $\left((x + 1)^2 + y^2\right)^{-1}$, and similarly for the second term on the left (it is more efficient to use the rule for $\frac{d}{dx}u^{-1}$ than to use the quotient rule):

$$-\left((x+1)^2 + y^2\right)^{-2}(2(x+1) + 2yy') - \left((x-1)^2 + y^2\right)^{-2}(2(x-1) + 2yy') = 0.$$

(Notice that the constants C and K have disappeared, because the derivative of any constant is 0.) Substituting $x = 0$, $y = 1$ gives:

$$-(1+1)^{-2}(2 + 2y') - (1+1)^{-2}(-2 + 2y') = 0.$$

Solving this, we find that $y' = 0$, i.e., the curve of constant total light intensity has horizontal tangent line at the point $(0, 1)$.

(b) When the source B is twice as strong as the source A, the effect is to replace the second K by $2K$, and this means having a 2 rather than 1 always in the numerator of the second term on the left. Our last equation is now

$$-(1+1)^{-2}(2 + 2y') - 2(1+1)^{-2}(-2 + 2y') = 0.$$

We can clear denominators by multiplying through by 2, obtaining: $-1 - y' + 2 - 2y' = 0$, i.e., $y' = 1/3$. Thus, the tangent line has slope $1/3$ and y-intercept 1, so its equation is $y = \frac{1}{3}x + 1$.

HOMEWORK 7.

1. In each case use the tangent line approximation to estimate the y-coordinate of the point on the curve whose horizontal distance is 0.01 (to the right) from the given point P:

$$\text{(a) } x^2 + xy + y^2 = 7, \; P = (1, 2);$$

$$\text{(b) } \frac{x}{y} + \frac{y}{x} + \frac{xy^2}{100} = \text{const } , P = (2, 5);$$

$$\text{(c) } \sqrt{x} + \frac{3}{x} = y - 1 + \frac{1}{\sqrt{y} + 1}, \; P = (9, 4).$$

2. Find a formula for the derivative y' at the point (x, y):

$$\text{(a) } y^2 = 1 + x^2; \qquad \text{(b) } x^2 + xy + y^2 = 7; \qquad \text{(c) } x^3 + xy^2 = y^3 + yx^2.$$

3. A hyperbola passing through $(8, 6)$ consists of all points whose distance from the origin is a constant more than its distance from the point (5,2). Find the slope of the tangent line to the hyperbola at $(8, 6)$.

4. In Example 7.5(a), suppose that you are starting from the point $(1, 1)$ rather than $(0, 1)$. Find the equation of the line from this point which is the closest approximation to the curve of constant total light intensity. After you've walked a horizontal distance of 0.1 (to the right), what is your y-coordinate?

5. In the situation of Example 7.5, suppose that you don't know the relative strengths of the sources A and B, but you know that the line $y = 0.6x + 1$ is the line that you should start walking along from the point (0,1) if you want the total light intensity to be as constant as possible. Find how much stronger B is than A.

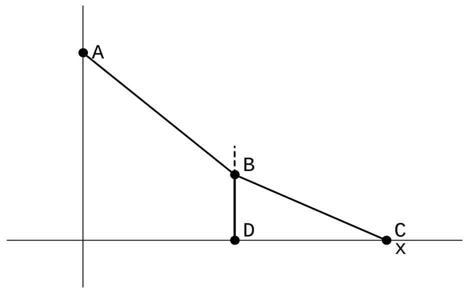
6. Suppose that a light source A is at the origin, and a source B twice as strong is located at $(3, 0)$. Find the slope of the direction you move in from (1,2) in order for the total light intensity to remain as constant as possible.

7. When crude oil flows from a well, water is frequently mixed with it in an emulsion. To remove the water the crude is piped to a device called a heater–treater, which is simply a large tank in which the oil is warmed and the water is allowed to settle out. Operating experience in a particular oil field indicates that the concentration C of water in the treater's output can be modeled by the following equation in the neighborhood of the usual operating point of 135°F and a 2-hour holding time:

$$C = 0.04 - 0.0032\, h^2 - 10^{-6}\, T^2 + 4.1 \cdot 10^{-5} h\, T,$$

where h is the holding time in hours and T is the operating temperature in degrees F. (a) Due to random fluctuations in the well's flow rate, the holding time actually varies slightly around 2 hours. Suppose you are given a simple control device that can change the tank temperature proportionally to the measured change in holding time. What constant of proportionality would best compensate for small holding time fluctuations and keep the water concentration as constant as possible? (b) Now as field equipment ages, its maximum operating settings are generally decreased. Find the equation of the line that best approximates the way in which the holding time would have to be increased as the maximum temperature rating falls slightly below the usual operating temperature.

8. One end of a string (the point A) is attached to the y-axis at a distance 20 meters from the origin; the string is threaded through a hole (the point B) at the end of a pole situated along the line $x = 10$; and the other end of the string (the point C) is attached to a hook that is free to slide along the x-axis and is being pulled by a force to the right that is sufficient to keep the string taut. See the diagram at the right. At time $t = 0$, when the top of the pole is at the point (10,10), the pole starts lowering at the steady rate of 30 cm/sec (=0.30 m/sec). The sliding end of the string reaches the point $x = 20$ at time $t = 10$.



(a) Write an equation relating t and x (the time is an implicit function of the location of the sliding end of the string). (b) Use the tangent line approximation to find the time when the sliding end of the string has moved another 20 cm to the left from the point $x = 20$.

CHAPTER 8

Trig Functions and Sinusoidal Functions

We start by reviewing trig functions of an angle measured in radians. Because this is review material, we will go through it rapidly.

We first draw the unit circle in the xy-plane. The standard way of measuring an angle is to start with the positive x-axis and rotate counterclockwise. As we go through the angle, we sweep around the circumference of the unit circle. The radian measure of the angle is defined to be the distance around the circumference that we travel. That is, a full revolution of $360°$ is equal to 2π radians, half a revolution is $180° = \pi$ radians, and so on. Other frequently used equivalences are: $90° = \pi/2$ rad, $60° = \pi/3$ rad, $45° = \pi/4$ rad, $30° = \pi/6$ rad. *All angles in this course will be assumed to be measured in radians unless otherwise stated. The designation "radian" will often be omitted, e.g., if we write "an angle of $\pi/6$," we mean a $30°$ angle.*

By a *negative angle* we mean a clockwise angle. For example, an angle of $-\pi/2$ (i.e., $-90°$) brings us to the same position as an angle of $3\pi/2$ (which is $270°$).

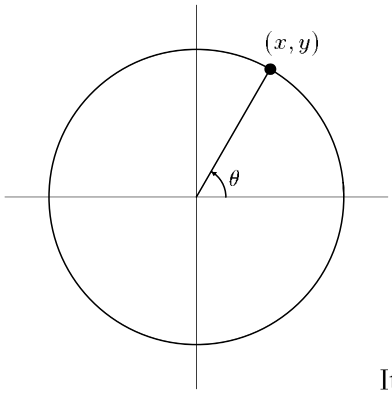
In general, if we have a circle of radius r and sweep out an angle of θ radians, then the distance around the circumference through which we have traveled is simply θr. Of course, one full circumference is $2\pi r$, corresponding to rotation through 2π radians.

Returning to the unit circle (see the drawing below), after rotating through an angle of θ radians let (x, y) be the point where we end up on the circumference. Then the six trig functions are defined as follows:

$$\sin\theta = y \qquad \text{cosecant}\,\theta = \frac{1}{\sin\theta} = \frac{1}{y}$$

$$\cos\theta = x \qquad \text{secant}\,\theta = \frac{1}{\cos\theta} = \frac{1}{x}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} = \frac{x}{y}$$



It is also customary to use the names csc θ for cosecant θ and sec θ for secant θ.

From the definition of $\sin\theta$ we see that as θ increases from 0 to 2π (i.e., as we go once around the unit circle), $\sin\theta$ goes from 0 to 1 to 0 to -1 and back to 0. Then as θ goes from 2π to 4π (i.e., as we go once again around the circle), the same values of $\sin\theta$ are repeated. We say that the function $\sin\theta$ has **period** 2π, i.e., we have $\sin(\theta + 2\pi) = \sin\theta$. Here are some other relations for the trig functions that follow easily from the above definitions: $\cos(\theta+2\pi) = \cos\theta$, $\sin(\theta+\pi) = -\sin\theta$, $\cos(\theta+\pi) = -\cos\theta$, $\tan(\theta+\pi) = \tan\theta$, $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$, $\tan(-\theta) = -\tan\theta$, $\sin^2\theta + \cos^2\theta = 1$, and $\cos\theta = \sin(\frac{\pi}{2} - \theta)$. The last of these relations is familiar from high school trigonometry: the cosine of an angle is equal to the sine of the complementary angle.

It can also be seen from our definitions as follows: the x-coordinate of the point on the unit circle you arrive at when you rotate through θ counterclockwise starting at the positive x-axis is the same as the y-coordinate of the point you arrive at when you rotate through θ clockwise starting at the positive y-axis. Sometimes one also needs the following formulas for the trig functions of a sum of two angles: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, and $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$. As a special case we have the double angle formulas $\sin(2\alpha) = 2\sin\alpha\cos\alpha$ and $\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$.

A simple geometric explanation of this will be given in class.

Also notice the sign of the trig functions in the different quadrants: $\sin x$ is positive in the 1st and 2nd quadrants (for angles between 0 and π), $\cos x$ is positive in the 1st and 4th, and $\tan x$ is positive in the 1st and 3rd.

1. Graphs of Trig Functions

The most important graphs to be familiar with are the graphs of the functions $y = \sin x$, $y = \cos x$, and $y = \tan x$. (We're now using x for the angle in radians; x no longer denotes a coordinate of a point on the unit circle.) For certain x the trig functions have easily stated exact values, namely, when x is a multiple of $\pi/6$ or of $\pi/4$. Here it is useful to draw the $30° - 60° - 90°$ triangle with hypotenuse 1 and the $45° - 45° - 90°$ triangle with hypotenuse 1. Using these triangles, we arrive at a table of values of the trig functions, from which we can sketch the graphs on the next page.

2. Sinusoidal Functions

A sinusoidal function is a function made up from the function $y = \sin x$ (or $y = \sin t$, if time is playing the role of the x-variable) by inserting constants in various places. The reason for inserting these constants is that the pure sine function $y = \sin x$ almost never occurs in the real world, but curves with a sine-like appearance (this is what the word "sinusoidal" means) arise frequently.

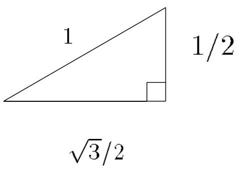
At this point you should review the material at the end of the section on functions: horizontal and vertical shifts, horizontal and vertical expansion, and the last batch of homework problems in that section.

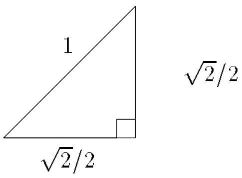
EXAMPLE 8.1. Write the cosine function as a shift of the sine function.

The graphs show that the cosine function is obtained by shifting the sine function $\pi/2$ to the left. Thus, $\cos x = \sin(x - (-\frac{\pi}{2})) = \sin(x + \frac{\pi}{2})$. This relation can also be obtained using the identities listed above: $\cos x = \cos(-x) = \sin(\frac{\pi}{2} - (-x)) = \sin(x + \frac{\pi}{2})$.

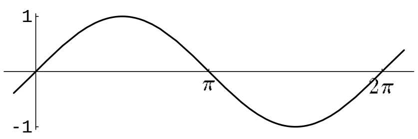
We now give the general form for a sinusoidal function $y = f(x)$. We start with $y = \sin x$.

Step 1. Expansion. Vertically, we expand by a factor of A, which is called the *amplitude* . We do this by replacing $\sin x$ by $A\sin x$. This makes the function go to a maximum of A and a minimum of $-A$ (rather than a maximum of 1 and a minimum of -1 as was the case for $\sin x$). Next, we expand horizontally in such a way that the function repeats its pattern from x to $x + B$ (rather than from x to $x + 2\pi$ as with $\sin x$). B is called the *period*. That is, we want to expand horizontally by a factor of $B/2\pi$. We do this by replacing x by $x/(B/2\pi) = \frac{2\pi}{B}x$ inside the sine. To summarize, the result of the vertical and horizontal expansion is to replace $\sin x$ by the function $A\sin\left(\frac{2\pi}{B}x\right)$. The graph of this

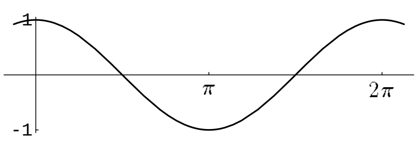




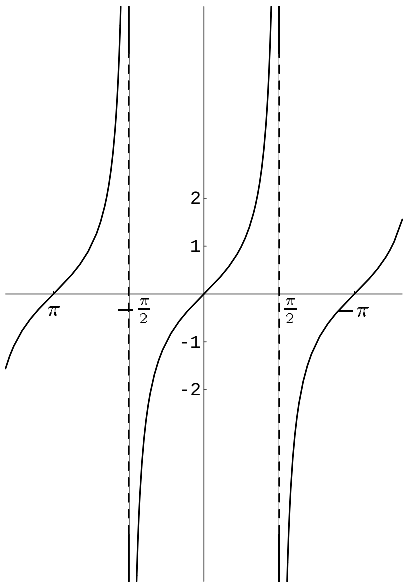
x	$\sin x$	$\cos x$	$\tan x$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	∞
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$
π	0	-1	0
$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1
$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$
$\frac{3\pi}{2}$	-1	0	∞
$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1
$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$
2π	0	1	0



Graph of $y = \sin(x)$

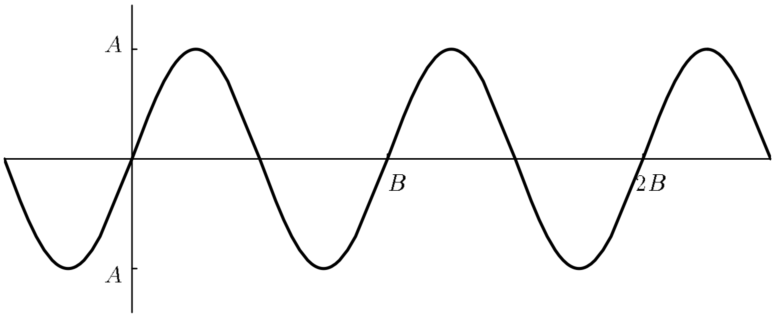


Graph of $y = \cos(x)$



Graph of $y = \tan(x)$

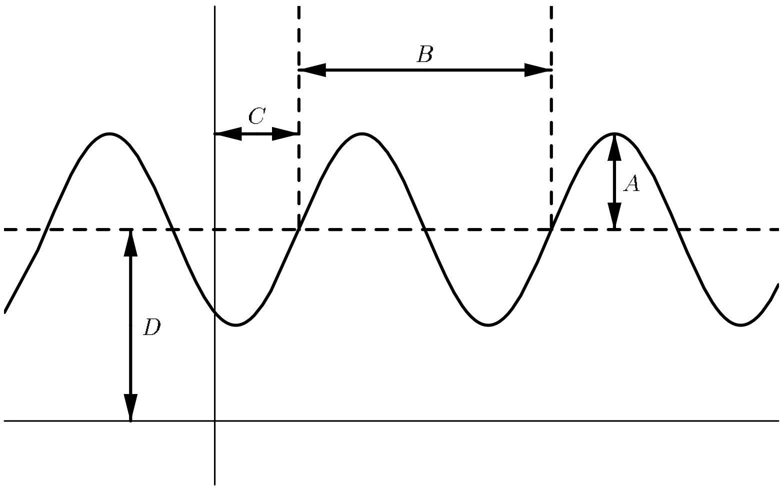
new function is shown below. Notice that as x goes from 0 to B, the value of the function goes from 0 to A to 0 to $-A$ and back to 0.



Step 2. Shifting. We now want to move the whole sine wave upward a certain amount D (called the *vertical shift*) and over to the right a certain amount C (called the *phase shift*). We do this by adding D outside the function, and replacing x by $x - C$ inside the function. The result is:

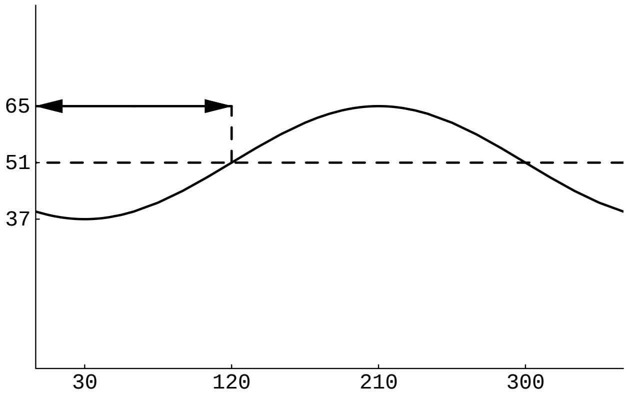
$$y = A \sin\left(\frac{2\pi}{B}(x - C)\right) + D.$$

In the graph of this function below, we indicate the meaning of the four constants A, B, C, and D.



EXAMPLE 8.2. Suppose that the mean daily temperature (i.e., the temperature in °F averaged over the 24-hour period) on the t-th day of the year in Seattle is tabulated over a period of many years, and the results for each day are averaged. This is what we call the " normal mean temperature " on the t-th day. For simplicity, suppose that the year consists of 12 months of 30 days each. Take $t = 1$ to be January 1, and so on, until $t = 360$ for December 30. Further assume: (1) the normal mean temperature function $y(t)$ is sinusoidal; (2) the coldest mean temperature is 37° on January 30; and (3) the warmest mean temperature is 65° on July 30. Find a formula for $y(t)$.

We graph $y(t)$ by first plotting the two points $t = 30$, $y = 37$ and $t = 210$ (i.e., July 30), $y = 65$. We then draw a sine wave in such a way that its minimum is at the first point, its maximum is at the second, and its period is a year, i.e., 360 days. We also draw a horizontal dotted line half-way between the coldest and warmest temperature, i.e., at $y = 51$.



Now we read the four constants A, B, C, D off the graph. The amplitude A is 14, the period B is 360, and the vertical shift (the distance up to the dotted line) is 51. To find C we want to know when the curve crosses the dotted line on its way from a minimum to a maximum, i.e., when the curve looks like $\sin x$ looks when it crosses the origin. This can be found by taking the t-value half-way between the coldest day (which is $t = 30$) and the warmest day (which is $t = 210$). This t-value is $(30 + 210)/2 = 120$ (i.e., April 30). Thus, the phase shift is $C = 120$, and our equation is

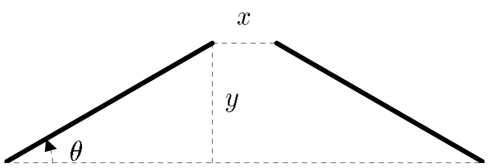
$$y(t) = 14 \sin \left(\frac{2\pi}{360}(t - 120) \right) + 51.$$

HOMEWORK 8.

1. Convert to radians: (a) $15°$, (b) $120°$, (c) a quarter revolution, (d) 10 revolutions.

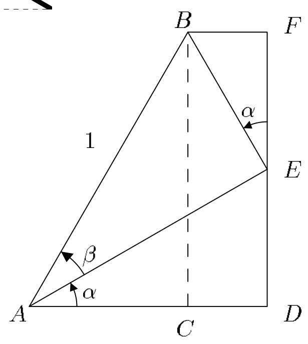
2. A bicycle wheel has radius 40 cm. Find the rate in rad/sec at which the wheel is turning if the bicycle is traveling at (a) 10 m/sec, (b) 10 km/hr.

3. A drawbridge whose two spans are each 30 meters long is opened as shown below. Express in terms of θ: (a) the height y of the end of a span, (b) the slope of the left span, (c) the distance x between the ends of the spans.



4. Using the diagram at the right, show that

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta\,.$$



5. Sketch the graph of each of the following functions:

(a) $y = 1+\sin(\pi x)$, (b) $y = 2\sin\left(\frac{\pi}{3}t\right)$,
(c) $y = 2+2\sin\left(\frac{\pi}{2}(t-1)\right)$,
(d) $y = 3+2\sin(2t+1)$.

6. The mean daily temperature in Fairbanks, Alaska was tabulated over a 30-year period, the average was taken over the 30 values for each day, and the result was found to be very close to the sinusoidal function $37\sin\left(\frac{2\pi}{365}(t-101)\right)+25$, where t is the day of the year.[1] (Here we are NOT assuming 30 days per month.) Sketch the graph of this temperature function, find the maximum and minimum mean daily temperatures, and find the warmest and coolest days of the year.

7. Studies are made of the variation in population of foxes and rabbits in a certain large forest. The number changes from year to year. Roughly speaking, when there are a lot of rabbits, the foxes will start to increase in number, because they can eat rabbits plentifully; but then this decreases the number of rabbits, causing the foxes to be short of food, and this causes a decrease in the fox population; and, finally, when there are fewer foxes the rabbits will be able to increase in number without being eaten, so we are back to where we started. Thus, we might expect a sinusoidal oscillation in the number of foxes and in the number of rabbits. (This is the so-called " predator-prey" problem in mathematical biology.) Suppose that t stands for the number of years since the study began, F stands for the number of foxes, and R stands for the number of rabbits. Suppose that F and R are each a sinusoidal function of t. Use the following table of data (which gives the maximum and minimum population of each) to find a formula for F and for R in terms of t:

t	0	1	2	3	4
F	150	200	150	100	150
R	10000	7500	5000	7500	10000

[1]This example is from an article by B. M. Lando and C. A. Lando in *The Mathematics Teacher* , Sept. 1977.

8. Write a formula for each of the following functions of t, assuming that the function is sinusoidal (also suppose that a year consists of 12 months of 30 days each):

(a) the mean low daily temperature, if the coldest is 20°F on January 19 and the highest low temperature reading is 60°F on July 19;

(b) the time of sunset each day, if the sun sets at 9 p.m. on June 21 and at 5 p.m. on December 21;

(c) the temperature in a well-insulated building, which varies between 25°C at 5 p.m. (which is 17:00 on the 24-hour clock) and 15°C at 5 a.m;

(d) the velocity of a weight attached to the end of a spring, which is extended and released at time $t = 0$, if the weight reaches its maximum velocity of 10 cm/sec after 0.1 sec;

(e) the height above the ground of a pebble that is picked up in the tread of a bicycle tire at time $t = 0$, if the tire has radius 40 cm and the bicycle is traveling at 10 m/sec.

CHAPTER 9

Derivative of Trig Functions

1. The Derivative of sin x

To find the slope of the tangent line to the curve $y = f(x) = \sin x$, we must go back to the definition of the derivative as the limit of the difference quotient:

$$\frac{dy}{dx} = \lim_{\Delta x \longrightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \longrightarrow 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x}.$$

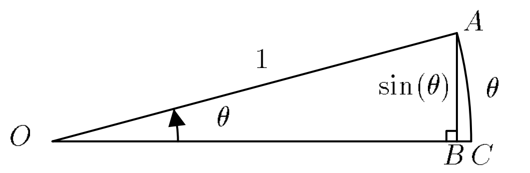
Recall the procedure used to find the derivative of the much simpler function x^2: expand $(x + \Delta x)^2$, simplify algebraically, and then take the limit as Δx gets closer and closer to 0. We do essentially the same thing for $\sin x$, but we need the following three basic facts in order to do this:

$$\text{Fact 1. } \sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta;$$

$$\text{Fact 2. } \lim_{\theta \longrightarrow 0} \frac{\sin\theta}{\theta} = 1;$$

$$\text{Fact 3. } \lim_{\theta \longrightarrow 0} \frac{1 - \cos\theta}{\theta} = 0.$$

Fact (1) was proved in the last section (Problem 4 of the homework). Facts (2) and (3) can be seen from the diagram below, where θ is a small angle in radians. By the definition of the radian measurement of an angle, θ is the distance along the circumference from A to C. Meanwhile, $\sin\theta = \overline{AB}$ and $1 - \cos\theta = \overline{BC}$.



Thus, Fact (2) says that although the line $\overline{AB}$ is smaller than the arc AC, the ratio of the two lengths gets closer and closer to 1 as θ gets smaller. Fact (3) says that the ratio of the line $\overline{BC}$ to the arc AC gets closer and closer to 0 as θ gets smaller. We are now ready to simplify the above difference quotient.

We have:

$$\frac{\sin(x + \Delta x) - \sin x}{\Delta x} = \frac{\sin x\cos(\Delta x) + \cos x\sin(\Delta x) - \sin x}{\Delta x}$$

(by Fact (1) with $\alpha = x$ and $\beta = \Delta x$)

$$= \frac{\cos x\sin(\Delta x) - \sin x(1 - \cos\Delta x)}{\Delta x}$$

(putting $\sin x \cos(\Delta x)$ last on top and factoring out $-\sin x$)

$$\begin{aligned}&= \cos x \frac{\sin(\Delta x)}{\Delta x} - \sin x \frac{1-\cos(\Delta x)}{\Delta x}\\ &\longrightarrow (\cos x)\cdot 1 - (\sin x)\cdot 0 = \cos x \qquad \text{as} \qquad \Delta x \longrightarrow 0.\end{aligned}$$

The last step comes from Facts (2) and (3) with θ replaced by Δx. We therefore conclude the following basic derivative formula:

$$\frac{d}{dx}\sin x = \cos x.$$

EXAMPLE 9.1. You know the exact value $\sin(30°) = 0.5$. Find $\sin(31°)$ using the tangent line approximation, and compare with the exact value (found by calculator).

In the tangent line approximation we choose $f(x) = \sin x$, $x = 30° = \pi/6$ and $\Delta x = 1° = \pi/180$. Here it was important that we changed Δx from degrees to radians. By the tangent line approximation,

$$\begin{aligned}\sin(31°) &= f(\frac{\pi}{6} + \frac{\pi}{180}) \approx 0.5 + \frac{\pi}{180} f'(\frac{\pi}{6})\\ &= 0.5 + \frac{\pi}{180}\cos(\frac{\pi}{6}) = 0.5 + \frac{\pi}{180}\cdot\frac{\sqrt{3}}{2} = 0.515115;\end{aligned}$$

the exact value is $0.515038\ldots$.

When we have a derivative formula for a new function, we can combine it with other rules to find the derivative of a more complicated function that is built up from the basic function.

EXAMPLE 9.2. Find the derivative of $\sin^2(\sqrt{x})$.

The notation $\sin^2$(something) means $\Big(\sin(\text{something})\Big)^2$. To evaluate this derivative, we have to use the chain rule twice:

$$\begin{aligned}\frac{d}{dx}\Big(\sin\sqrt{x}\Big)^2 &= 2\sin\sqrt{x}\,\frac{d}{dx}\sin\sqrt{x} = 2\sin\sqrt{x}\cos\sqrt{x}\,\frac{d}{dx}\sqrt{x}\\ &= 2\sin\sqrt{x}\cos\sqrt{x}\,(1/2\sqrt{x}) = \frac{\sin\sqrt{x}\cos\sqrt{x}}{\sqrt{x}}.\end{aligned}$$

2. Other Trig Derivatives

All of the other trig functions can be expressed in terms of the sine, and so their derivatives can easily be calculated. The most important ones are:

$$\frac{d}{dx}\cos x = \frac{d}{dx}\sin(x+\frac{\pi}{2}) = \cos(x+\frac{\pi}{2}) = \sin((x+\frac{\pi}{2})+\frac{\pi}{2}) = \sin(x+\pi) = -\sin x;$$

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos x\frac{d}{dx}\sin x - \sin x\frac{d}{dx}\cos x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x;$$

$$\frac{d}{dx}\sec x = \frac{d}{dx}\Big(\cos x\Big)^{-1} = (-1)\Big(\cos x\Big)^{-2}(-\sin x) = \frac{\sin x}{\cos^2 x} = \tan x \sec x.$$

HOMEWORK 9.

1. You know the exact value $\tan(45°) = 1$. Find $\tan(44°)$ using the tangent line approximation, and compare with the exact value (found by calculator).

2. Find the derivative of (a) $y = \cot x$, (b) $y = \csc x$ (csc is the abbreviation of cosecant).

3. Use the tangent line approximation to find (a) $\cos(62°)$, (b) $\sin(29°)$, (c) $\cot(47°)$.

4. Find the derivative of each of the functions in Problem 8(a)–(e) of the last homework set. In each case find the first positive time t for which the derivative is 0 and the first positive t for which the derivative reaches its maximum value; then explain in practical terms what is happening at each of these moments.

5. Find the derivative of (a) $y = \sin x \cos x$, (b) $y = \sin(\cos x)$, (c) $y = \sqrt{x \tan x}$, (d) $y = \dfrac{\tan x}{1 + \sin x}$.

6. Suppose that the point $(\pi/3, \pi/4)$ is on the implicitly defined curve $\dfrac{\sin x}{x} + \dfrac{\sin y}{y} = \text{const.}$ Use the tangent line approximation to determine the y-coordinate of the point on the curve with x-coordinate $\dfrac{\pi}{3} + \dfrac{\pi}{180}$.

7. Suppose that the point $(1, 1.5)$ is on the curve $x\sin(\pi y/3) + y\sin(\pi x/4) = \text{const}$. Use the tangent line approximation to determine the y-coordinate of the point on the curve with x-coordinate 1.1.

8. A rope is tied at its left end, and its right end is moved up and down to create a sinusoidal wave of period 6 ft. The equation of the shape of the rope ("standing wave") is then

$$y = A\sin\left(\frac{2\pi}{6}x\right) = A\sin\left(\frac{\pi}{3}x\right),$$

where we are taking the origin to be the point where the rope is fastened, and A is the amplitude. Now actually, the amplitude A varies with time as the rope oscillates. Suppose that the maximum value of A is 2 ft, and it varies sinusoidally with time t with period 2 sec:

$$A = 2\sin(\pi t).$$

Thus, the vertical coordinate y of a point on the rope depends on both its x-coordinate and the time t:

$$y = 2\sin(\pi t) \cdot \sin\left(\frac{\pi}{3}x\right).$$

(a) Sketch the appearance of the rope at quarter-second intervals from $t = 0$ to $t = 2$ sec (all on the same graph).

(b) Suppose your vision is blocked by a horizontal ledge at a height of 6 inches above the x-axis, so you see only the tips of the waves. Write an equation that gives the x-value where the wave emerges above the ledge implicitly as a function of t (i.e., write an equation of the form $F(x,t) = \text{const}$).

(c) What is the first x-value where the rope emerges above the ledge when $t = 1/6$ sec?

(d) How rapidly is that first x-value moving leftward at the instant $t = 1/6$ sec?

(e) Approximately where will that first point where you see the rope be located when $t = 0.2$ sec? (Use parts (c) and (d) and the tangent line approximation.)

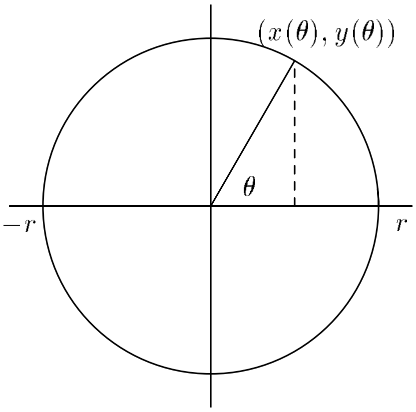
CHAPTER 10

Curves Given Parametrically

So far we have had two different ways of giving a curve. The first was **explicitly**, i.e., in the form $y = f(x)$. For example, the equation $y = \sqrt{r^2 - x^2}$ gives us the upper semicircle of radius r centered at the origin. The second way was **implicitly**, usually in the form $F(x, y)$ =constant. For example, the equation $x^2 + y^2 = r^2$ gives us the circle (both upper and lower semicircles) of radius r centered at the origin.

There is a third fundamental way a curve can be given: **parametrically**. This means that we have a third variable, called the **parameter**, which is used to determine where we are on the curve in the xy-plane. To give a curve parametrically means to give *two* formulas, one for x and one for y, each in terms of the parameter. This parameter might have a geometrical meaning — for example, an angle denoted θ — or it may be the time t. In the case of an angle parameter, our two formulas take the form $x = g(\theta)$, $y = h(\theta)$; in the case of a time parameter, the formulas take the form $x = g(t)$, $y = h(t)$. (Often we don't bother to use separate letters like g and h, instead writing simply $x(t)$ for the formula for x in terms of t and $y(t)$ for the formula for y in terms of t.) The best way to understand parameterized curves is through examples.

Example 10.1. (Circles) Suppose we want to describe the points on the circumference of the circle of radius r centered at the origin. A common way to do this is in terms of the angle θ which the line from the point to the origin makes with the positive x-axis (measured counterclockwise). That is, as θ goes from 0 to 2π radians, we sweep once around the circle:



What we want to do is express the x- and y-coordinates of our points on the circumference in terms of θ. To do this, we drop the perpendicular from the point to the x-axis, obtaining a triangle with hypotenuse r and with legs x and y. So we can use the trig functions *sin* and *cos* to express x and y in terms of θ:

$$x = x(\theta) = r\cos\theta, \qquad y = y(\theta) = r\sin\theta, \qquad 0 \le \theta \le 2\pi.$$

This says that, as θ goes from 0 to 2π, if we determine x and y by these two equations, then the corresponding point (x, y) goes exactly once around the circumference of the circle.

A special case of this is the **unit circle** — this is simply the case $r = 1$. Thus, the points on the unit circle can be described parametrically by the equations

$$x = x(\theta) = \cos\theta, \qquad y = y(\theta) = \sin\theta, \qquad 0 \le \theta \le 2\pi.$$

The fact that the x-coordinate is simply the cosine and the y-coordinate is simply the sine is not surprising, since we *defined* the cosine and sine of an angle to be the x- and y-coordinates of the point that you get by going around the unit circle through the given angle.

If we want to shift the circle of radius r so that its center is at the point (h, k), we can do this by adding h to the x-formula and k to the y-formula:

$$x = x(\theta) = h + r\cos\theta, \qquad y = y(\theta) = k + r\sin\theta, \qquad 0 \le \theta \le 2\pi.$$

This is the circle which is given implicitly by the equation $(x - h)^2 + (y - k)^2 = r^2$.

If instead of $0 \le \theta \le 2\pi$ we had written, say, $0 \le \theta \le \pi$, then we would have described only half of the circle (the upper semicircle). If we had written $0 \le \theta \le 4\pi$, then we would have described going around the circle twice.

EXAMPLE 10.2. (Ellipses) Let us suppose that we have put our ellipse on the axes in the following convenient way: its center is at the origin, its longer axis of symmetry — of length $2a$ — is along the x-axis, and its shorter axis of symmetry — of length $2b$ — is along the y-axis. Such an ellipse can be obtained by taking the unit circle and stretching it by a factor of a in the x-direction and by a factor of b in the y-direction. (If a or b is less than 1, the "stretching" is actually a contraction.) In terms of the parametric equations, we can accomplish this stretching simply by multiplying the x-equation for the unit circle by a and multiplying the y-equation for the unit circle by b:

$$x = x(\theta) = a\cos\theta, \qquad y = y(\theta) = b\sin\theta, \qquad 0 \le \theta \le 2\pi.$$

This is the parametric form for our ellipse — the ellipse which can be given implicitly in the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Note: A word of warning, though, about the parameter θ for an ellipse. It no longer has the meaning of the angle formed by the line from the point (x, y) to the origin. For example, suppose $a = 2$, $b = 1$. Then the point on the ellipse corresponding to the parameter value $\theta = \pi/4 = 45^\circ$ is the point $x = 2\cos(\pi/4) = \sqrt{2}$, $y = \sin(\pi/4) = \sqrt{2}/2$; and it is easy to see that the line joining the point $(1.414, 0.707)$ to the origin does not make a 45° angle with the x-axis.

There is a geometric way to interpret the variable θ as a suitable angle anyway and this is best illustrated from the following picture of an ellipse with its auxiliary circle.

1. Trajectories

It is very common to want to describe a curve in terms of the time parameter t. In that case we sometimes call the curve the "path" or "trajectory" of our point (which we think of as a small object or particle). As t goes from an initial time a to a final time b, the equations $x(t)$, $y(t)$ tell us where the particle is at that particular time t.

EXAMPLE 10.3. Suppose a particle is traveling counterclockwise at 3 rev/sec around the circle of radius r centered at the origin. It starts at the point $(r, 0)$ at time $t = 0$. We want to give a parametric description of the motion in terms of t.

To do this, we find an expression in terms of t for the angle θ through which the particle has rotated, and then we substitute this expression in the equations in Example 10.1.

Because we shall be measuring angles in radians, we first translate the angular velocity of 3 rev/sec into units of rad/sec. Since 1 rev = 2π rad, we obtain: $d\theta/dt = 3 \cdot 2\pi = 6\pi$ rad/sec. Thus, θ starts at 0 at time $t = 0$ (because our point starts at $(r, 0)$ on the positive x-axis), and it increases at the constant rate of 6π. Hence, after t seconds it has value $\theta = 6\pi t$. Finally, substituting this expression for θ in the equations in Example 10.1, we obtain:

$$x = x(t) = r\cos(6\pi t), \qquad y = y(t) = r\sin(6\pi t), \qquad 0 \leq t \leq 1/3,$$

where we chose to take t between 0 and 1/3 so that we would describe exactly one rotation of the particle around the circle.

Example 10.4. (Motion at constant velocity between two points)
Suppose that at time $t = 0$ you start at the point $P = (2, 5)$ and travel at constant velocity along the line from P to the point $Q = (6, 3)$, arriving 5 seconds later at Q. Find parametric formulas for this motion.

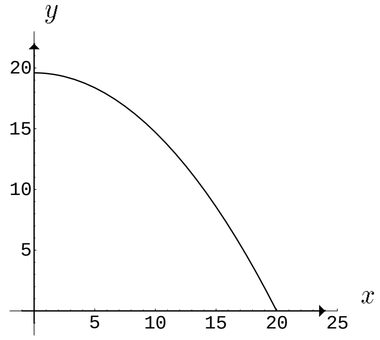
To do this, we use a general principle for setting up parametric equations: **work with the x- and y-directions separately.** In the x-direction in this example we are starting at $x = 2$ and traveling to $x = 6$ in 5 seconds, moving at constant speed. This constant speed — called the "horizontal component of velocity" and denoted $\dot{x} = \frac{dx}{dt}$ — is the distance traveled divided by time: $(6-2)/5 = 0.8$. Thus, the formula for $x(t)$ is $2 + 0.8t$ (i.e., starting x-value plus speed times time). Similarly, in the y-direction we compute the vertical component of velocity $\dot{y} = \frac{dy}{dt}$ to be $(3-5)/5 = -0.4$, so that $y(t)$ is given by the formula $5 - 0.4t$. This is all going on between $t = 0$ and $t = 5$. Thus, our parametric description of the motion is:

$$x = x(t) = 2 + 0.8t, \qquad y = y(t) = 5 - 0.4t, \qquad 0 \leq t \leq 5.$$

We can check these formulas by noting that when $t = 0$ they give us $x(0) = 2$, $y(0) = 5$ (i.e., we're at the point P), and when $t = 5$ they give us $x(5) = 6$, $y(5) = 3$ (i.e., we're at the point Q).

Example 10.5. (Falling body in two dimensions)
Again we are interested in an object falling under the influence of gravity (we neglect air resistance). In the past (see Example 6.9) our object was moving only in one dimension — up and down. For example, if we drop a ball from a window 19.6 meters high, its height y at time t is given by the formula $y = -\frac{1}{2}gt^2 + y_0 = -4.9t^2 + 19.6$ (here $g = 9.8$ m/sec^2; the acceleration is negative, i.e., it is $-g$, because we are choosing up to be the positive direction).

Now suppose that, instead of dropping the ball from the window, we throw it horizontally out from the window at 10 m/sec. The ball then moves in the xy-plane, where we have chosen the side of the building as the y-axis and the line along the ground under the ball's path as the x-axis. Its path is a parabola, as we shall soon see. Our task in this example is to write parametric equations for this motion.



To do this, we again consider the x- and y-directions separately. In the x-direction, there is no force affecting the velocity, and so the 10 m/sec horizontal component of velocity of the ball remains unchanged. That is, as in Example 10.4, in the x-direction we are simply traveling at constant speed. Since we start at initial location $x = 0$, the formula for x at time t is: $x(t) = 10t$.

In the vertical direction we have a falling body problem, and so can use the formula $y = -\frac{1}{2}gt^2 + v_0 t + y_0$. Here it is important to understand what v_0 means. v_0 is the initial velocity in the y-direction. In this example v_0 is 0, since the ball's initial velocity is entirely horizontal. In other words, in the y-direction the problem is exactly as if we had simply dropped the ball from a height of 19.6. Thus, we have $y(t) = -4.9t^2 + 19.6$.

Finally, we want to know the time interval where the formulas for x and y apply. The motion starts at time $t = 0$, and it ends when the ball hits the ground. The ball hits the ground when $y = 0$, and so to find when this happens we set $y = -4.9t^2 + 19.6$ equal to 0 and solve for t. We obtain $t^2 = 4$, $t = \pm 2$, and so $t = 2$ (since -2 does not make practical sense). To summarize, we have the following parametric description of the motion:

$$x = x(t) = 10t, \qquad y = y(t) = -4.9t^2 + 19.6, \qquad 0 \le t \le 2.$$

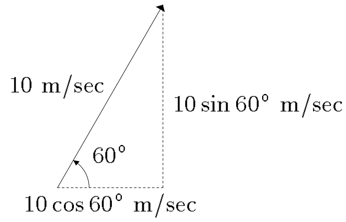
The actual trajectory in Example 10.4 (see the picture above) is a graph in the xy-plane, that is, a function of the form $y = f(x)$. In this example, we can easily find the formula for this function by *eliminating* t in the formulas for $x(t)$ and $y(t)$. That is, since $x = 10t$, we can write $t = 0.1x$, and then substitute this in place of t in the formula for y: $y = -4.9(0.1x)^2 + 19.6 = -0.049x^2 + 19.6$. Thus, the path $f(x)$ is a parabola (more precisely, part of a downward-opening parabola).

2. The Velocity Vector

The velocity of an object moving in the plane is an example of a vector. A **vector** is something which has magnitude and direction. It can be visualized as an arrow of a certain length (or **magnitude**) pointing in a certain direction. The magnitude of the velocity vector is called the **speed** of the object.

Example 10.6. The same as Example 10.5, except that at time $t = 0$ the ball is thrown at an angle of 60° above the horizontal.

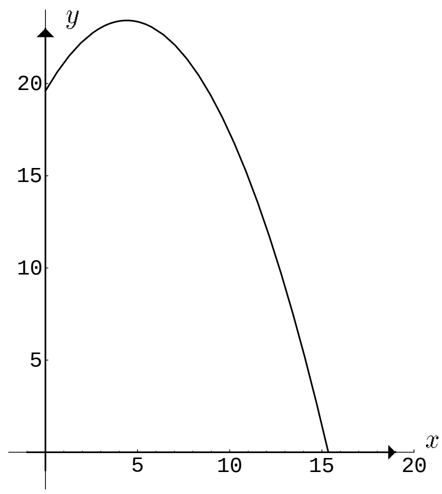
Here the initial velocity vector makes an angle of 60° above the horizontal and has a magnitude of 10 m/sec. That is, it is partly horizontal and partly vertical. The first step in solving this problem is to **resolve the initial velocity vector into its horizontal and vertical components**. To resolve the velocity vector, we draw the **velocity triangle** having the velocity vector as its hypotenuse (see figure). This triangle has hypotenuse 10 m/sec, making an angle of 60° above the horizontal. Then the horizontal leg of the triangle — the initial x-velocity $v_{0,\text{horiz}} = \dot{x}(0)$ — can be found using the cosine, and the vertical leg of the triangle — the initial y-velocity $v_{0,\text{vert}} = \dot{y}(0)$ — can be found using the sine. We thus have:



$$v_{0,\text{horiz}} = \dot{x}(0) = 10\cos 60^\circ = 10 \cdot 0.5 = 5\,\text{m/sec},$$
$$v_{0,\text{vert}} = \dot{y}(0) = 10\sin 60^\circ = 10 \cdot 0.866 = 8.66\,\text{m/sec}.$$

We are now ready to write the equations for $x(t)$ and $y(t)$, as we did in Example 10.5. The horizontal component of velocity (which remains constant) is not 10 m/sec, but rather only the horizontal part, namely, 5 m/sec. So $x = 5t$. In the falling body formula $y = -\frac{1}{2}gt^2 + v_0t + y_0$, we now have a nonzero initial vertical component of velocity $v_0 = v_{0,\text{vert}} = 8.66$ m/sec. Thus, we obtain:

$$x = x(t) = 5t,$$
$$y = y(t) = -4.9t^2 + 8.66t + 19.6.$$



The trajectory of the ball is shown in the figure at the right.

To find the time interval until the ball hits the ground, we again set $y = 0$, and use the quadratic formula to find the time t when we're on the ground. There will be 2 roots, of which we want the positive one:

$$(-8.66 - \sqrt{8.66^2 - 4 \cdot (-4.9) \cdot 19.6})/(2 \cdot (-4.9)) = 3.07 \text{ sec}.$$

So the time interval is $0 \le t \le 3.07$.

Example 10.7. The same as Example 10.6, except that the ball is thrown at an angle of 60° *below* the horizontal.

Here the only change is that the tip of the velocity triangle points downward, i.e., its vertical leg $\dot{y}(0)$ points in the *negative* direction. Thus, in the equations of motion the only thing that changes is the sign of $v_{0,\text{vert}}$ in the falling body formula:

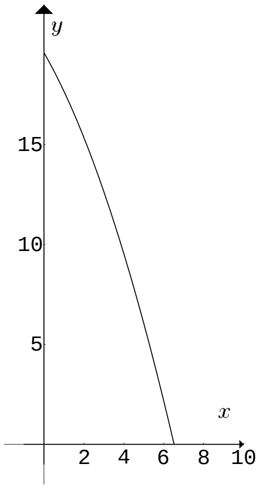
$$x = x(t) = 5t, \qquad y = y(t) = -4.9t^2 - 8.66t + 19.6.$$

Of course, the ball hits the ground much sooner (see figure on the following page).

In all three cases in Examples 10.5–10.7, the trajectory is part of a parabola, as we can see by eliminating t and writing y in terms of x rather than in terms of t.

Suppose that we have a trajectory $x(t)$, $y(t)$, as in Examples 10.3–10.7. There are three types of derivatives we might be interested in. One is the **horizontal component of velocity** $dx/dt = \dot{x}(t)$. The second is the **vertical component of velocity**

$dy/dt = \dot{y}(t)$. The third is not a velocity at all, but rather the slope of the tangent to the curve in the xy-plane: dy/dx. If we find dy/dx at some instant, we can determine the direction in which the object is headed at that instant.



How do we find dy/dx if we do not have y written in terms of x, i.e., if we do not have a formula for the function $y = f(x)$? Well, according to the chain rule, at any instant our three derivatives are connected by the relation

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

In the past we have used the chain rule to find the left hand side. Now we want to find dy/dx. So the form of the chain rule we need is:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dot{y}}{\dot{x}}.$$

In words, this says that *the slope of the tangent to the path is equal to the vertical component of velocity divided by the horizontal component of velocity.* This makes sense: if the horizontal component of velocity is small and the vertical component of velocity is large, we are traveling at a steep angle; while if the horizontal component of velocity is large and the vertical component of velocity is small, we are traveling at a shallow angle.

In finding dy/dx it is always important to remember that you take the ratio of *velocities* $\dot{y}$ over $\dot{x}$, **NOT** simply the ratio of y to x. In other words, you must take the derivatives (with respect to t) of the $y(t)$ and $x(t)$ formulas before taking their ratio.

EXAMPLE 10.8. Suppose we want to know the instant when the ball in Example 10.6 reaches the peak of its trajectory. At that instant the path has a horizontal tangent line, i.e., $dy/dx = 0$. Since $dy/dx = \dot{y}/\dot{x}$, this slope is zero if and only if the numerator $\dot{y}$ is 0. Thus, to find the instant when the ball reaches its highest point we set the vertical velocity $\dot{y}$ equal to 0 and solve for t. This is the same principle that we saw in our earlier one-dimensional falling body problems: the ball is at the peak when its vertical velocity is zero. (Warning: Be sure not to confuse this procedure, where we set $\dot{y}$ equal to zero, with setting y itself equal to 0, which we do to find the time when the ball hits the ground.) Since $\dot{y} = \frac{d}{dt}(-4.9t^2 + 8.66t + 19.6) = -9.8t + 8.66$, we have $-9.8t + 8.66 = 0$, i.e., $t = 8.66/9.8 = 0.88$ sec. If we also want to know the maximum height reached by the ball, we take this value of t and substitute it in the formula for the height y, i.e.,

$$y_{\max} = y(0.88) = -4.9 \cdot (0.88)^2 + 8.66 \cdot 0.88 + 19.6 = 23.4 \text{ m}.$$

EXAMPLE 10.9. Suppose that we want to know the direction in which the ball in Example 10.6 is headed after 0.5 sec, and also at the instant when it hits the ground. We first take the time derivatives of y and x:

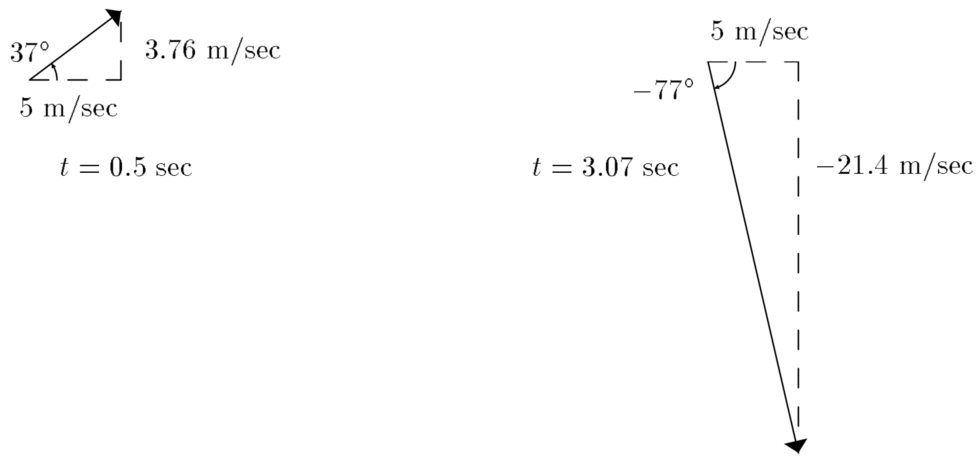
$$\frac{dy}{dt} = \frac{d}{dt}(-4.9t^2 + 8.66t + 19.6) = -9.8t + 8.66 \text{ and} \frac{dx}{dt} = \frac{d}{dt}(5t) = 5\,.$$

Then we have:

$$\frac{dy}{dx} = \frac{\dot{y}(t)}{\dot{x}(t)} = (-9.8t + 8.66)/5\,.$$

Thus, when $t = 0.5$ we have $dy/dx = (-9.8 \cdot 0.5 + 8.66)/5 = 0.752$. At that instant the ball's location is $(x(0.5), y(0.5)) = (2.5, 22.7)$, so that the tangent line to the path at the point $(2.5, 22.7)$ has slope 0.752. We can also determine the angle at which the ball is traveling at time $t = 0.5$. By this we mean the angle above the horizontal of the tangent line to the path. To do that we observe that the slope of a line $\Delta y/\Delta x$ is opposite over adjacent, i.e., the tan of the angle that the line makes with the horizontal. So to determine the angle we use the inverse-tan on our calculator: $\tan^{-1} 0.752 = 37°$. At the same time $t = 0.5$ we could also construct the velocity triangle, having horizontal leg $\dot{x}(0.5) = 5$ and vertical leg $\dot{y}(0.5) = 3.76$, and compute the hypotenuse (which is the speed):

$$\text{speed at time } 0.5 = \sqrt{5^2 + 3.76^2} = 6.26 \text{ m/sec.}$$



Next, we make the same computations at the instant when the ball hits the ground, i.e., at $t = 3.07$ sec. We have $\dot{x}(3.07) = 5$, $\dot{y}(3.07) = -9.8 \cdot 3.07 + 8.66 = -21.4$, and so speed$=\sqrt{5^2 + (-21.4)^2} = 22.0$. The angle at which the ball hits the ground is $\tan^{-1}(-21.4/5) = -77°$. Thus, the ball hits the ground traveling at 22 m/sec at an angle of 77° below the horizontal.

Homework 10.

1. Find parametric equations $x(t)$, $y(t)$ for the point P if:

(a) P travels at constant velocity from (2,3) to (11,−3) between $t = 0$ and $t = 3$;

(b) P travels at constant speed 6.5 units/sec from (1,2) in the direction of (13,7);

(c) P travels at constant speed 10 units/sec from the origin at an angle of 30° above the x-axis.

2. A bicycle has wheels of radius 0.4 meter. Find:

(a) the angular velocity of the wheels in rad/sec if the bicycle is traveling at 10 m/sec;

(b) the angular velocity of the wheels in rev/sec if the bicycle is traveling at 10 m/sec;

(c) the speed of the bicycle in m/sec if the wheels are turning at 5 rev/sec;

(d) the speed of the bicycle in km/hr if the wheels are turning at 15 rad/sec.

3. A truck has wheels of radius 1.6 ft. Find:

(a) the speed of the truck in mph if the wheels are turning at 5 rev/sec;

(b) the angular velocity of the wheels in rad/sec and in rev/sec if the truck is traveling at 60 mph.

4. Find a formula for the distance in feet that a car has traveled during the first t sec if its 10-inch-radius wheels:

(a) are turning at the constant rate of 24 rad/sec;

(b) are turning at the constant rate of 9 rev/sec;

(c) start from rest, and accelerate at the rate of 18 rad/sec per second.

5. A wheel of radius 3 is centered at the origin. Find parameteric equations for the point P on the rim if:

(a) the wheel is spinning counterclockwise at 0.5 rad/sec, and P is at (3,0) at time $t = 0$;

(b) the wheel is spinning clockwise at 2 rev/sec, and P is at (0,3) at time $t = 0$;

(c) the wheel is spinning clockwise at 1 revolution each 5 seconds, and P is at $(0, -3)$ at time $t = 0$;

(d) the wheel is spinning counterclockwise at 10 rev/sec, and P is at $(0, -3)$ at time $t = 0$.

6. A girl is playing on a flatcar which is moving in the positive x-direction at 10 m/sec. At time $t = 0$ she throws the ball straight upward (as viewed from the moving train) with an initial velocity 24.5 m/sec from a height 1 meter above the flatcar. From her point of view, the ball goes straight up and down, and lands on the flatcar directly below the point from which she threw it. However, an observer on the ground sees the ball travel in an arc.

Set up a coordinate system which is fixed to the earth (i.e. one which is not moving along with the train), with the y-axis pointing up and the x-axis pointing in the direction of motion of the train. Choose the origin to be the point in space which is 1 meter below the point from which the ball was released. Neglect air resistance, and take $g = 9.8$ m/sec^2.

(a) Find parametric equations for the motion of the ball in this fixed coordinate system.

(b) Find the instant when the ball reaches the peak of its trajectory.

(c) Find the instant when the ball hits the flatcar.

(d) Describe in words how you would find the angle at which the ball hits the flatcar in these coordinates.

7. Suppose that an object is thrown at 70 ft/sec at an angle of 45° *below* the horizontal from a window 163 ft above the ground.

(a) Express both the horizontal distance x from the window and the height y above the ground as functions of t.

(b) Find dy/dx in terms of t.

(c) Find the **angle** below the horizontal that the object is pointed when $t = 0.5$ sec and when $t = 1$ sec.

(d) Find the **velocity vector** (its magnitude and angle below the horizontal) of the object at the instant when it hits the ground.

8. Suppose that a ball is thrown from ground level at v m/sec at an angle of α above the horizontal. Take the origin to be the point from which the ball is thrown.

(a) Find formulas for $x(t)$ and $y(t)$. The constants v and α should appear in your formulas.

(b) Find the time t when the ball hits the ground, expressing your answer in terms of v and α. There are two ways to do this: (i) Set $y(t) = 0$ and solve for t. (ii) Find the time when the ball reaches its peak (see Example 10.8), and double it (because it takes an equal time to fall back to the ground). Use both methods, and see that your answers agree.

(c) Find the horizontal distance traveled by the time the ball hits the ground, expressing your answer in terms of v and α.

Supplement: Differentiation Formulas

In the following formulas, $f(x)$ and $g(x)$ are differentiable functions and c is a constant:

$$\begin{aligned}
(f+g)'(x) &= f'(x)+g'(x) \\
(f-g)'(x) &= f'(x)-g'(x) \\
(fg)'(x) &= f(x)g'(x)+f'(x)g(x) \\
(cf)'(x) &= cf'(x) \\
(f/g)'(x) &= \frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2} \\
(f(g(x))' &= f'(g(x))g'(x) \\
g'(x) &= 1/f'(g(x)) \text{ if } g(x) \text{ is the inverse of } f(x)
\end{aligned}$$

Formulas for common functions:

$\frac{d}{dx}x^a = ax^{a-1}$	$\frac{d}{dx}a^x = (\ln a)a^x$
$\frac{d}{dx}\ln x = 1/x$	$\frac{d}{dx}\log_a x = \frac{1}{(\ln a)x} = \frac{\log_a e}{x}$
$\frac{d}{dx}\sin x = \cos x$	$\frac{d}{dx}\cos x = -\sin x$
$\frac{d}{dx}\tan x = \sec^2 x$	$\frac{d}{dx}\cot x = -\csc^2 x$
$\frac{d}{dx}\sec x = \sec(x)\tan(x)$	$\frac{d}{dx}\csc x = -\csc(x)\cot(x)$
$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}$
$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$	$\frac{d}{dx}\text{arccot } x = \frac{-1}{1+x^2}$
$\frac{d}{dx}\text{arcsec } x = \frac{1}{x\sqrt{x^2-1}}$	$\frac{d}{dx}\text{arccsc } x = \frac{-1}{x\sqrt{x^2-1}}$
$\frac{d}{dx}\sinh x = \cosh x$	$\frac{d}{dx}\cosh x = \sinh x$

Supplement: Derivative Skills Exercises

1. $f(x) = x^{100} + 50x + 1$
2. $f(x) = x^e$
3. $f(x) = (16x)^3$
4. $f(x) = (x^2 + 1)(2x - 7)$
5. $f(x) = x^2 + \frac{1}{x^2}$
6. $f(x) = \frac{1-x^2}{1+x^2}$
7. $f(x) = \cos(x) - 2\ \tan(x)$
8. $f(x) = \frac{\sin(x)}{1+\cos(x)}$
9. $f(x) = x\ \csc(x)$
10. $f(x) = \csc(x)\ \cot(x)$
11. $f(x) = x\ \sin(x) \cos(x)$
12. $f(x) = e^{\sqrt{x}}$
13. $f(x) = x\ e^{2x}$
14. $f(x) = e^{x\ \cos(x)}$
15. $f(x) = \frac{e^{-x^2}}{x}$
16. $f(x) = \frac{e^{3x}}{1+e^x}$
17. $f(x) = \ln(2 - x - x^2)$
18. $f(x) = \ln(\ln(x))$
19. $f(x) = \ln(\cos(x))$
20. $f(x) = \cos(\ln(x))$
21. $f(x) = \log_3(x^2 - 4)$
22. $f(x) = (x + 2)^8(x + 3)^6$
23. $f(x) = \frac{1}{(x^2-2x-5)^4}$
24. $f(x) = \cos^3(x)$
25. $f(x) = \cos(x^3)$
26. $f(x) = \cos(\tan(x))$
27. $f(x) = x\ \sin(\frac{1}{x})$
28. $f(x) = \frac{\sin^2(x)}{\cos(x)}$
29. $f(x) = -\frac{2}{\sqrt[4]{x^3}}$
30. $f(x) = \sqrt[5]{x\ \tan(x)}$
31. $f(x) = \tan(\sqrt{1 - x})$
32. $f(x) = \frac{\sin(mx)}{x}$
33. $f(x) = \cos^2(\tan\ x)$
34. $f(x) = \sqrt{x\sqrt{x\sqrt{x}}}$
35. $f(x) = \arcsin(2x - 1)$
36. $f(x) = \arctan(x^3)$
37. $f(x) = (\arcsin\ x)^2$
38. $f(x) = \frac{\arccos(x)}{x}$

39. $f(x) = (1 + x^2)\ \arctan(x)$
40. $f(x) = \log_{10}(x^2 - x)$
41. $f(x) = \arcsin(e^x)$
42. $f(x) = \ln(x^2\, e^x)$
43. $f(x) = x^{\sin(x)}$
44. $f(x) = x^{e^x}$
45. $f(x) = \sin(\tan\sqrt{1 + x^3})$
46. $f(x) = \ln(\sqrt[4]{\frac{x^2+x+1}{x^2-x+1}})$

Use implicit differentiation:

47. $x^2 - xy + y^3 = 8$
48. $x^4 + y^4 = 16$
49. $\sqrt{xy} - 2x = \sqrt{y}$
50. $\cos(x - y) = y\ \sin(x)$

CHAPTER 11

Combined Parametric Motions

Sometimes we are interested in the motion of a point P which has a simple motion relative to some other point C, which, in turn, also is moving in a particular way. It is not hard to write parametric formulas $x(t)$, $y(t)$ for the point P, if we proceed carefully in a step-by-step manner.

EXAMPLE 11.1. (The baton)

Suppose we want to describe the motion of a tip of a baton as it rotates after being thrown into the air. More precisely, suppose that at time $t = 0$ a 2-foot long baton is situated vertically with the tip we are interested in at height 6 ft above ground level and the other tip at height 4 ft. At that moment the baton is tossed at 20 ft/sec at an angle of 45° above the horizontal, and it is also set spinning at 3 rev/sec clockwise in the plane of its trajectory. (It could have been set spinning in a different plane, but that would have made this into a *three*-dimensional problem, and that would be too complicated for our present purposes.)

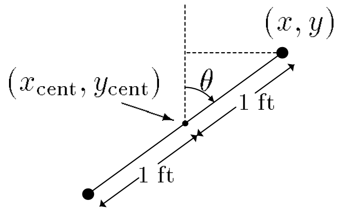
To set up the parametric equations for the motion of the tip of the baton, the basic procedure is to divide the motion into two parts: **the motion of the center of the baton** and **the rotation of the tip around the center**. The first of these is a falling body problem, and the second is a circular motion problem. We shall take the y-axis to be the vertical line along which the baton is situated at time $t = 0$, and we shall take the x-axis to be the line on the ground under the baton's trajectory.

According to a principle of physics, the center of the baton follows the same path as a small object (say, a ball) that is thrown with the same initial velocity. In other words, the first half of our procedure — the motion of the center of the baton — is like Example 10.6 of the last chapter. Let $(x_{\text{cent}}(t), y_{\text{cent}}(t))$ denote the x- and y-coordinates of the center of the baton at time t. Repeating the procedure used in Example 10.6 — with $g = 32$ ft/sec^2, $v_{0,\text{horiz}} = 20\cos 45° = 14$ ft/sec (rounded to the nearest ft/sec), $v_{0,\text{vert}} = 20\sin 45° = 14$ ft/sec — we obtain:

$$x_{\text{cent}}(t) = 14t, \qquad y_{\text{cent}} = -16t^2 + 14t + 5.$$

(Notice that $y_0 = 5$ ft, because the initial location of the center is half-way between 4 and 6.)

Now let $(x(t), y(t))$ denote the x- and y-coordinates of the tip of the baton at time t. The second half of our procedure consists in determining how much $x(t)$ differs from $x_{\text{cent}}(t)$ and how much $y(t)$ differs from $y_{\text{cent}}(t)$, i.e., how far the tip is from the center in the x- and y-directions. Let θ be the angle in radians through which the baton has spun at time t. We have the following diagram:



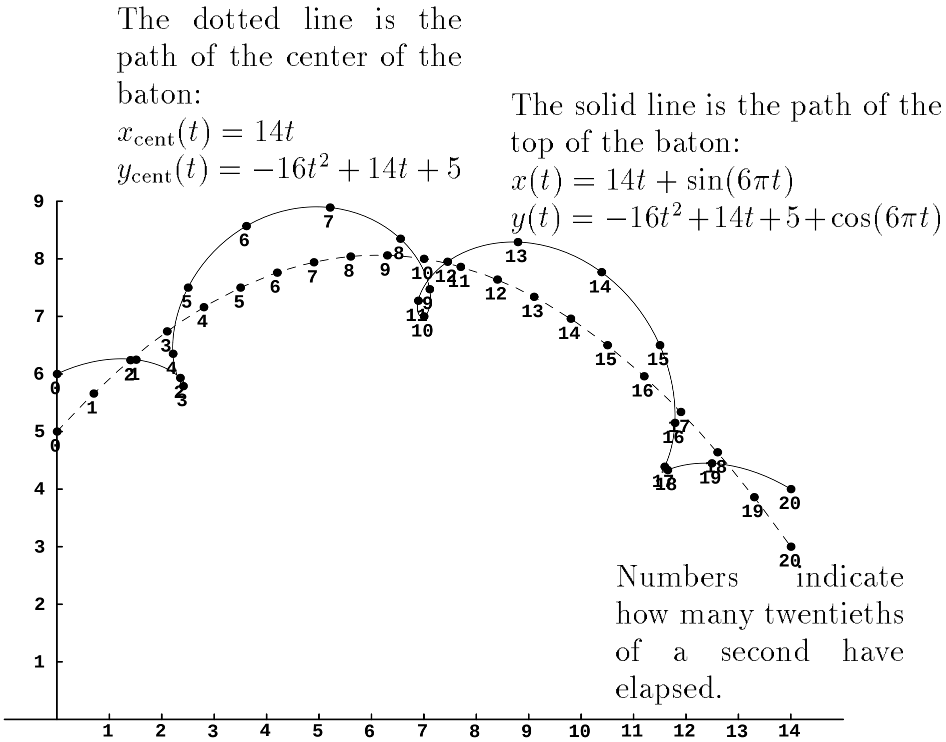
Since the tip is 1 ft from the center, the hypotenuse of this triangle is 1. Hence $x - x_{\text{cent}} = \sin\theta$ and $y - y_{\text{cent}} = \cos\theta$, i.e.,

$$x(t) = 14t + \sin\theta, \qquad y(t) = -16t^2 + 14t + 5 + \cos\theta.$$

It remains to write θ in terms of t. We do this as in Example 10.3 of the last chapter, obtaining $\theta = 6\pi t$. We conclude that

$$x(t) = 14t + \sin(6\pi t), \qquad y(t) = -16t^2 + 14t + 5 + \cos(6\pi t).$$

(Notice that here it is the x-coordinate that involves the sine and the y-coordinate that involves the cosine, the reverse of what we got in our earlier examples of circular motion. The reason is that in this problem it was convenient to measure the angle θ clockwise from the positive y-direction rather than counterclockwise from the positive x-direction. This different way of defining the angle accounts for the reversal of the sine and cosine.)



The graph above shows the trajectory of the center of the baton (dotted line) and the trajectory of the tip of the baton (solid line) between $t = 0$ and $t = 1$ sec. The numbered points from 0 to 20 give the location at intervals of 0.05 seconds. If you join the numbers on the dotted line with the corresponding numbers on the solid line, you will see the spinning of (the top half of) the baton.

Even though the equations for $x(t)$ and $y(t)$ are not tremendously complicated, the path of the tip of the baton is a rather elaborate curve. In this example we cannot find an algebraic or trig formula for y in terms of x, because we cannot eliminate t. So the parametric method is the only reasonable way to describe this curve.

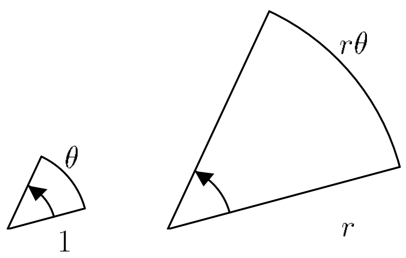
EXAMPLE 11.2. (The cycloid)

Suppose a bicycle wheel of radius r is rolling along the x-axis. A pebble lying at the origin is picked up in the tread of the tire as it rolls over the origin. We would like to describe the path of the pebble as it moves along the x-axis, spinning around the center of the wheel as the wheel turns. This path is called a **cycloid**.

As in Example 11.1, the procedure is to split the problem into two parts: describe the motion of the center of the wheel, and then determine the location of the pebble relative to the center. The first part is easy, because the center of the wheel is simply moving along the straight line $y = r$ (parallel to the x-axis at a height of r); the second part is a circular motion problem.

We shall express the coordinates of the center $(x_{\text{cent}}, y_{\text{cent}})$ and the coordinates of the pebble (x, y) in terms of the following parameter: the angle θ in radians through which the wheel has rotated after picking up the pebble at the origin. We shall also introduce another letter s to denote the linear distance the wheel has traveled. s is the distance along the x-axis from the origin to the point of contact of the wheel, and s is also the x-coordinate of the center of the wheel.

There is a simple relation between the distance s the wheel has traveled and the angle θ in radians through which it has rotated. Namely, the distance traveled along the x-axis is also equal to the distance measured around the circumference through which the wheel has turned (here we are assuming that the wheel rolls without slipping or sliding). For example, every time the wheel rotates through 1 revolution (i.e., 2π radians), the wheel moves a distance equal to the circumference $2\pi r$. More generally, if the wheel rotates through an angle of θ radians, the wheel moves a distance equal to θr. This is because the distance along the circumference of the unit circle of the part of the circumference corresponding to an angle of θ radians is simply θ, by the definition of the measure of an angle in radians. In a circle of radius r — i.e., the unit circle magnified by a factor of r — the distance along the circumference will be r times as great:



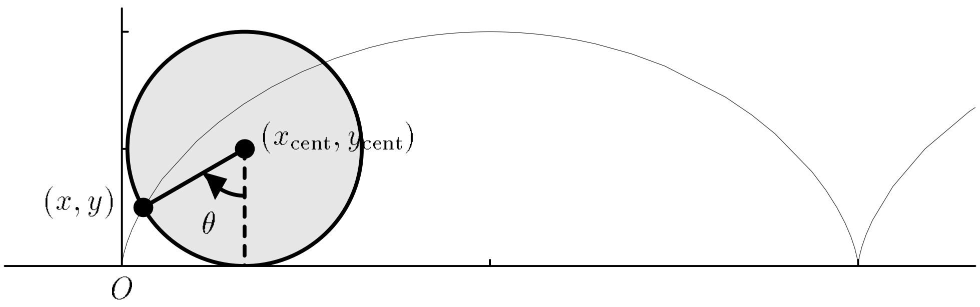
To summarize, we have the relation:

$$s = r\,\theta.$$

The center of the wheel always has the same y-coordinate r (a constant), and its x-coordinate is $s = r\,\theta$. Thus, we have

$$x_{\text{cent}}(\theta) = r\theta, \qquad y_{\text{cent}}(\theta) = r.$$

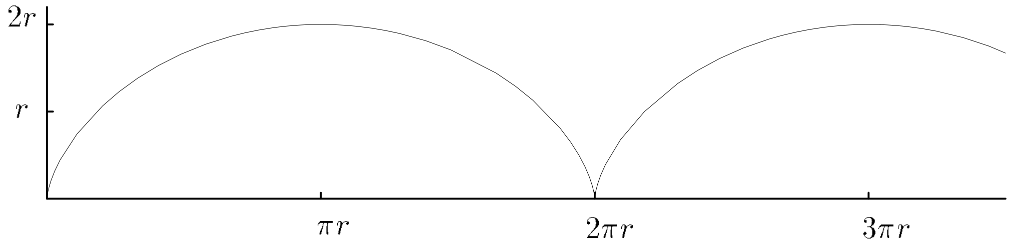
The second part of our procedure is to find the difference between the coordinates (x, y) of the pebble and the coordinates $(x_{\text{cent}}, y_{\text{cent}})$ of the center. Since the wheel is rolling from left to right, and the radius from the center to the pebble is pointing down when we start out at the origin, it is convenient to measure the angle θ clockwise from the negative y-direction. We have the following diagram:



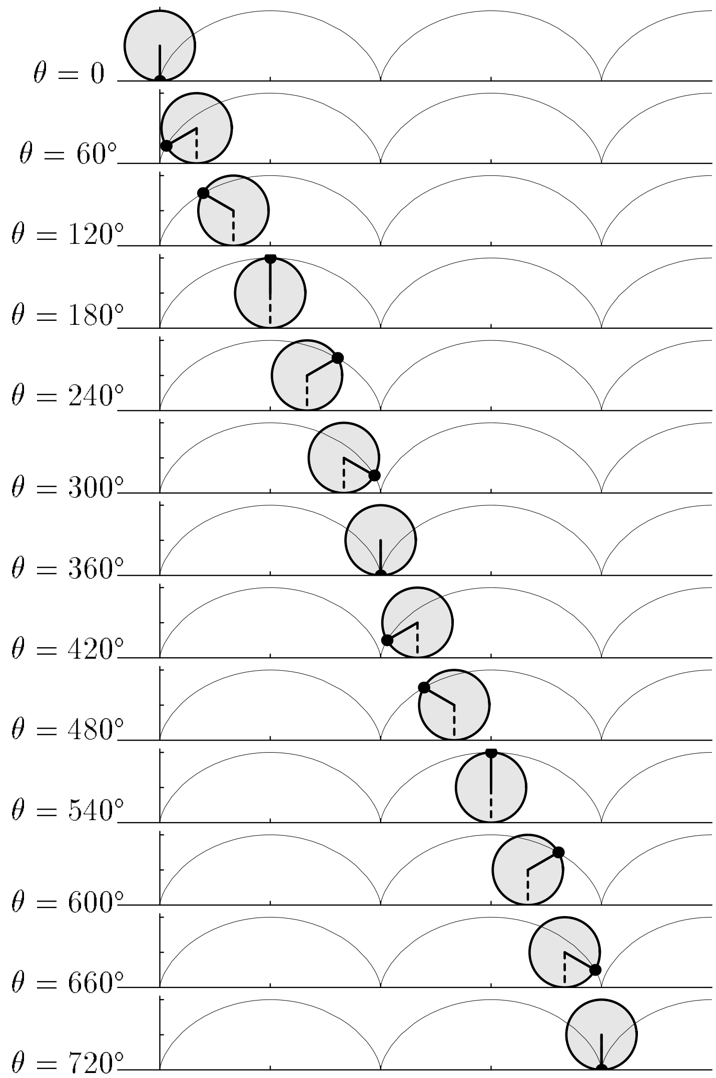
If θ is an acute angle, as in the diagram, then $\sin\theta$ and $\cos\theta$ are positive. Since the pebble is to the *left* and *below* the center, we must *subtract* the horizontal leg of the triangle from x_{cent} to get x, and we must *subtract* the vertical leg of the triangle from y_{cent} to get y. Using the trig functions of θ to get the horizontal and vertical legs of the triangle (whose hypotenuse is equal to r), we end up with

$$x(\theta) = x_{\text{cent}}(\theta) - r\sin\theta = r\theta - r\sin\theta, \qquad y(\theta) = y_{\text{cent}}(\theta) - r\cos\theta = r - r\cos\theta.$$

This parametric curve is drawn below. Each time θ goes through 2π radians, the wheel turns around once, and the pebble goes through one motion from cusp to cusp.



The figure below shows the point on the curve for several values of θ.



Now suppose that we know how the bicycle is moving with time. For example, suppose that it is traveling at constant speed v, and it passes the origin at time $t = 0$. That is, $ds/dt = v$. To express the position of the pebble in terms of t, all we have to do is express θ in terms of t, and then substitute in the above equations for (x, y). Since θ has a simple relation to s — namely, $\theta = \frac{1}{r}\,s$ — it follows that $d\theta/dt$ has a simple relation to ds/dt, namely (taking d/dt of both sides):

$$\frac{d\theta}{dt} = \frac{1}{r}\,\frac{ds}{dt} = \frac{1}{r}\,v.$$

Because θ is 0 at time $t = 0$, and it increases as the constant rate v/r, it follows that we have: $\theta = (v/r)t$. Thus, when the bicycle is traveling at constant velocity v, we obtain the following equations for the motion of the pebble:

$$x(t) = r(v/r)t - r\sin((v/r)t) = vt - r\sin(vt/r), \qquad y(t) = r - r\cos(vt/r).$$

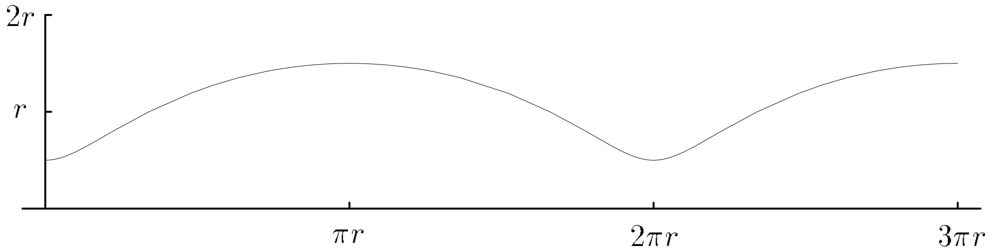
EXAMPLE 11.3. (The trochoid)

The same as in Example 11.2, except that when the wheel rolls over the origin, instead of getting stuck in the tread on the outside of the tire, the pebble flies up and gets stuck in a spoke at a distance a directly below the center (where $a < r$). Thus, the pebble again rotates around the center, but at a distance a rather than r from the center. The path of this pebble is called a **trochoid**.

In this problem everything is just like in Example 11.2, except that in the circular motion part the hypotenuse of the triangle (which is the line from the center to the pebble after the wheel has rotated through an angle of θ radians) is now a rather than r. This leads to the formulas

$$x(\theta) = x_{\text{cent}}(\theta) - a\sin\theta = r\theta - a\sin\theta, \qquad y(\theta) = y_{\text{cent}}(\theta) - a\cos\theta = r - a\cos\theta.$$

Here is a picture of the trochoid path in the case when $a = r/2$:



HOMEWORK 11.

1. Suppose that the wheel in Example 11.2 has radius 1.

(a) Graph the cycloid by plotting the points where the pebble is located after the wheel rotates through all multiples of 30° from 0 to 360° (i.e., θ goes through all multiples of $\pi/6$ from 0 to 2π).

(b) Use the chain rule in the form

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

to find a formula for dy/dx in terms of θ.

(c) Draw the tangent lines to the curve in part (a) at all of the plotted points from $\theta = \pi/6$ to $\theta = 11\pi/6$, and use part (b) to find the slope of each tangent line.

2. A spring is causing an object to oscillate up and down sinusoidally between $y = 3$ and $y = -3$. At $t = 0$ the object is at maximum height $y = 3$, and it makes one full motion (from 3 to -3 and back to 3) twice a second. Meanwhile, the whole apparatus is moving at constant velocity 2 units/sec down the x-axis, starting at 0 at time $t = 0$.

(a) Find $x(t)$.

(b) Find $y(t)$. Namely, use the information given to (i) graph y as a sinusoidal function of time in the ty-plane, and (ii) find A, B, C and D in the formula $y = A\sin\left(\frac{2\pi}{B}(t - C)\right) + D$.

(c) Graph the path of the object in the xy-plane. Find the formula for $y = f(x)$ in two ways: (i) Use the graph of the path to find a, b, c and d in $y = a\sin\left(\frac{2\pi}{b}(x - c)\right) + d$ (where we're using small letters for the constants, so as to avoid confusion with the sinusoidal function in part (b)). (ii) Eliminate t, and use part (b) to write y directly in terms of x. Use both methods, and check that your answers agree.

3. The same as Problem 2, except suppose that the whole apparatus is moving down the x-axis according to the formula $x(t) = t^2$. Notice that the vertical motion $y(t)$ as a function of time is unaffected by this change. However, in the horizontal direction the object is **accelerating**, i.e., its velocity, rather than being constant, is steadily increasing according to the formula $v(t) = \dot{x}(t) = 2t$; and the acceleration $a(t) = \ddot{x}(t) = 2$ is constant. Here $x(t) = t^2$ can be thought of as $\frac{1}{2}gt^2$, where the constant acceleration g here is 2 units/sec^2 (and of course, unlike gravity, which pulls in the negative y-direction, here the constant acceleration is pulling in the positive x-direction). Also notice that your graph of the path $y = f(x)$ is no longer sinusoidal: the distance between peaks becomes greater as the object accelerates to the right. The formula for $y = f(x)$ can be obtained by substituting $t = \sqrt{x}$ in place of t in the formula in part (b).

4. A wheel of radius 2 ft is centered at the point $(0, 100$ ft$)$ and is rotating clockwise at the constant rate of one revolution every 2 seconds. At time $t = 0$ a pebble on the perimeter of the wheel is located at the point $(0, 102)$. At time $t = 0$ the wheel is dropped. Write down formulas for the x- and y-coordinates of the pebble as functions of time, and graph its trajectory for t between 0 and 2.5 sec (plot points at intervals of 1/4 sec).

5. A wheel rolls down the x-axis from left to right without slipping at constant speed 6 m/sec. The diameter of the wheel is 1.5 m. As it passes over the origin at time $t = 0$ a pebble gets stuck in a spoke right under the center at a distance 25 cm from the rim. Find $x(t)$ and $y(t)$ for the pebble.

6. In an episode of the TV program "Northern Exposure," the hero Chris and a mathematician named Amy send his motorcycle careening over a cliff. We want to describe the path of the point P at the top of one of the wheels at the instant $t = 0$ when the wheel starts over the cliff. Suppose that the motorcycle wheel has a radius 0.25 m, and until time $t = 0$ the motorcycle is traveling at 10 m/sec. Neglect air resistance, and take $g = 9.8$ m/sec^2. Suppose that the wheel is traveling in the xy-plane, the motorcycle is moving horizontally (along the negative x-axis) until time $t = 0$, and the origin is the point at the edge of the cliff where the wheel is touching at the instant $t = 0$. Find the coordinates $x(t)$, $y(t)$ of P at time t (while the motorcycle is falling).

CHAPTER 12

Related Rates

Suppose we have two variables x and y (in most story problems the letters will be different, but for now let's use x and y) which are both changing with time. A "related rates" problem is a problem where we know one of the rates of change at a given instant — say, dx/dt — and we want to find the other rate dy/dt at that instant.

If y is written in terms of x, i.e., $y = f(x)$, then this is easy to do using the chain rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

That is, find the derivative of $f(x)$, plug in the value of x at the instant in question, and multiply by the given value of $\dot{x} = dx/dt$ to get $\dot{y} = dy/dt$.

But often x and y will be related in some other way, for example $x = f(y)$, or $F(x,y) =$const, or perhaps $F(x,y) = G(x,y)$, where $F(x,y)$ and $G(x,y)$ are expressions involving both variables. In all cases, you can solve the related rates problem by taking $\frac{d}{dt}$ of both sides, plugging in all the known values (namely, x, y, and $\dot{x}$), and then solving for $\dot{y}$.

To summarize, here are the steps in doing a related rates problem when there is a simple direct relation in the two variables x, y.

1. Decide what the two variables are.
2. Find an equation relating them.
3. Take d/dt of both sides.
4. Plug in all known values at the instant in question.
5. Solve for the unknown rate.

Sometimes it may not be convenient or desirable to find the equation between x, y directly. They might in turn be related to each other by a chain of variables and conditions. We will explain such cases in the examples below.

Example 12.1. The radius of a circular puddle is 3 m, and it is increasing at the rate of 1 cm/min. How fast is the puddle's area increasing?

Here the two things that change with time are: the radius r and the area A. We know $dr/dt = 1$ cm/min $= 0.01$ m/min. We want to find dA/dt. Thus, in this problem r is playing the role of x, and A is playing the role of y. The relation between the two variables is: $A = \pi r^2$. Taking d/dt of both sides, we obtain

$$\frac{dA}{dt} = \pi \cdot 2r\frac{dr}{dt}.$$

Notice that, since we are taking derivatives with respect to t, the derivative of r^2 is $2r\frac{dr}{dt}$, by the rule for the derivative of a power of a function (the u^n-rule). In this case we alternately could have used the chain rule for the variables A, r and t:

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt} = 2\pi r \cdot \frac{dr}{dt}.$$

We now plug in the information given. At the instant in question, $r = 3$, and $dr/dt = 0.01$. Thus, $A = 2\pi \cdot 3 \cdot 0.01 = 0.1885$ m^2/min.

Note: A crucial point to notice in this and similar problems is that, even though a numerical value is given for r, r must be regarded as a variable, not a constant. *The value of the variable at the instant in question cannot be plugged into the equation until* **after** *taking d/dt of the equation.*

EXAMPLE 12.2. You are inflating a spherical balloon at the rate of 7 cm^3/sec. How fast is its radius increasing at the instant when $r = 4$? How fast is the area increasing at the same time?

First we answer the first question.

Here the variables are the radius r and the volume V. We know dV/dt, and we want dr/dt. The two variables are related by means of the equation $V = \frac{4}{3}\pi r^3$. Taking d/dt of both sides gives $dV/dt = 4\pi r^2 \cdot dr/dt$. We now substitute the values we know at the instant in question: $7 = 4\pi 4^2 \cdot dr/dt$. Solving for dr/dt, we obtain $dr/dt = 7/(64\pi) = 0.0348$ cm/sec.

Now the second question.

The formula for the area is $A = 4\pi r^2$. Taking d/dt of both sides, we get $dA/dt = 4\pi 2r dr/dt = 8\pi r dr/dt$.

Now that we have worked out the answer to dr/dt, we can easily deduce that

$$dA/dt = 8\pi(4)\frac{7}{64\pi} = 7/2 = 3.5.$$

Note: If only the second question was asked, then we would have first written down the two well known formulas:

$$V = 4\pi/3r^3 \ , \ A = 4\pi r^2.$$

and tried to find a direct relation between V and A. The relation is $36\pi v^2 = a^3$. Then we would try to solve this for A and take derivatives, or take d/dt of both sides and finally arrive at the same result.

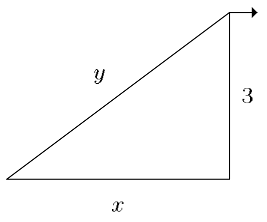
The point is that it is not necessary to eliminate r right away. It is far easier to take d/dt of the two relations first and get equations

$$dV/dt = 4\pi r^2 dr/dt \ , \ dA/dt = 8\pi r dr/dt \text{ so } 2dV/dt = r dA/dt \text{ or } dA/dt = (2/r)dV/dt.$$

Our final answer is now clear!

EXAMPLE 12.3. A plane is flying at 500 mph at an altitude of 3 miles in a direction away from where you're standing (i.e., the point on the ground directly beneath the plane is moving away from you). How fast is the plane's distance from you increasing at the moment when the plane is flying over a point on the ground 4 miles from you?

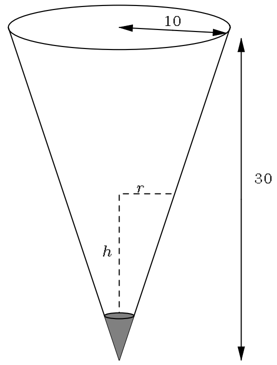
To see what's going on, we first draw a triangle whose hypotenuse is the line from you to the plane (see figure at right). The vertical leg of the triangle is a constant — 3 mi — which does not change in the course of the problem. The horizontal leg x is the variable whose rate we know — its rate $\dot{x}$ is the speed of the plane. The hypotenuse y is the variable whose rate we want. The equation relating x and y is the Pythagorean theorem: $x^2 + 3^2 = y^2$. We next take d/dt of both sides, getting $2x \cdot \dot{x} = 2y \cdot \dot{y}$. At the instant in question



we're told that $x = 4$ and $\dot{x} = 500$. What about y? From the Pythagorean theorem, we have: $y = \sqrt{4^2 + 3^2} = 5$ at the instant in question. Thus, we obtain: $2 \cdot 4 \cdot 500 = 2 \cdot 5 \cdot \dot{y}$, and solving for $\dot{y}$ gives the value 400 mph. That is, the plane is receding from you at 400 mph. (Until you think about this for a while, it might seem counterintuitive that, although the side y is greater than the side x, its rate of increase is less.)

Example 12.4. Water is poured into a conical container at the rate of 10 cm^3/sec. The cone points directly down (see side view at right), and it has a height of 30 cm and a base radius of 10 cm. How fast is the water level rising when the water is 4 cm deep (at its deepest point)?

In this problem, the water forms a conical shape within the big cone whose height and base radius — and hence also its volume — are all increasing as water is poured into the container. This means that we actually have three things varying with time: the water level h (the height of the cone of water), the radius r of the circular top surface of water (the base radius of the cone of water), and the volume of water V. And we have the relation $V = \frac{1}{3}\pi r^2 h$. We know dV/dt, and we want dh/dt. At first something seems to be wrong: we have a third variable r whose rate we don't know.



But the dimensions of the cone of water must have the same proportions as those of the container. In other words, r and h fit into the above diagram of similar triangles, from which it follows that $\frac{r}{h} = \frac{10}{30}$, i.e., $r = h/3$. So we can eliminate r from the problem entirely: $V = \frac{1}{3}\pi(h/3)^2 h = \frac{\pi}{27}h^3$. We now take d/dt of both sides and then plug in $h = 4$ and $dV/dt = 10$, obtaining $10 = \frac{\pi}{27} 3 \cdot 4^2 \cdot \frac{dh}{dt}$. Thus, $dh/dt = 90/16\pi = 1.79$ cm/sec.

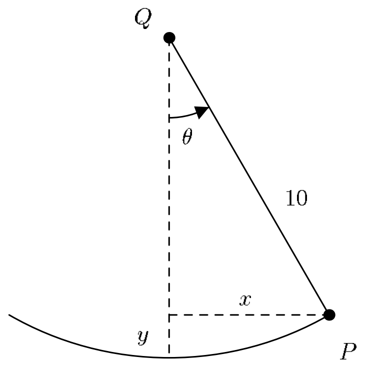
Note: Again we can take the lazy approach and write the equations:

$$V = \frac{1}{3}\pi r^2 h \text{ and } r = h/3$$

that we noticed above. Instead of eliminating the variable r we could have just taken separate derivatives first and then eliminated dr/dt. Of course, here the second equation is so easy to solve (it is already solved for r), we did eliminate r right away in our solution.

In general, the lazy method is better, unless the elimination is very easy.

EXAMPLE 12.5. A swing consists of a board at the end of a 10 ft long rope. Think of the board as a point P at the end of the rope, and let Q be the point of attachment at the other end. Suppose that the swing is directly below Q at time $t = 0$, and is being pushed by someone who walks at the speed of 6 ft/sec from left to right. Find (a) how fast the swing is rising after 1 sec; (b) the angular speed of the rope in deg/sec after 1 sec.



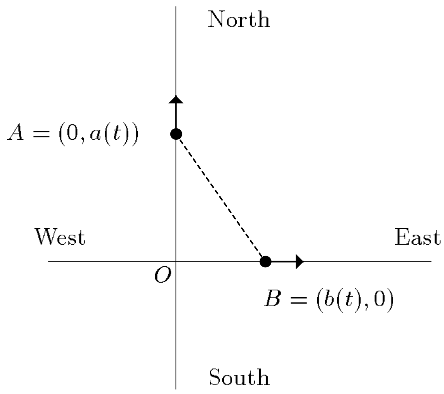
In doing this problem, we must start out by asking: What is the geometric quantity whose rate of change we know, and what is the geometric quantity whose rate of change we're being asked about? Note that the person pushing the swing is moving **horizontally** at a rate we know. In other words, the **horizontal** coordinate of P is increasing at 6 ft/sec. In the xy-plane let us make the convenient choice of putting the origin at the location of P at time $t = 0$, i.e., a distance 10 directly below the point Q of attachment. Then the rate we know is dx/dt, and in part (a) the rate we want is dy/dt (the rate at which P is rising). In part (b) the rate we want is $\dot{\theta} = d\theta/dt$, where θ stands for the angle in radians through which the swing has swung from the vertical. (Actually, since we want our answer in deg/sec, at the end we must convert $d\theta/dt$ from rad/sec by multiplying by $180/\pi$.)

(a) From the diagram above we see that we have a right triangle whose legs are x and $10 - y$, and whose hypotenuse is 10. Hence $x^2 + (10 - y)^2 = 100$. Taking d/dt of both sides — and recalling that the derivative of the square of a function of t is twice the function being squared times the time derivative of that function — we obtain: $2x\dot{x} + 2(10 - y)(0 - \dot{y}) = 0$. We now look at what we know after 1 sec, namely $x = 6$ (because x started at 0 and has been increasing at the rate of 6 ft/sec for 1 sec), $y = 2$ (because we get $10 - y = 8$ from the Pythagorean theorem applied to the triangle with hypotenuse 10 and leg 6), and $\dot{x} = 6$. Putting in these values leaves us with $2 \cdot 6 \cdot 6 - 2 \cdot 8\dot{y} = 0$, from which we can easily solve for $\dot{y}$, getting $\dot{y} = 4.5$ ft/sec.

(b) Here our two variables are x and θ, so we want to use the same right triangle as in part (a), but this time to relate θ to x. Since the hypotenuse is constant (equal to 10), the best way to do this is to use the sine: $\sin\theta = x/10$. Taking d/dt of both sides, we obtain $(\cos\theta)\dot{\theta} = 0.1\dot{x}$. At the instant in question ($t = 1$ sec), when we have a right triangle with sides 6-8-10, the $\cos\theta$ in this equation is $8/10$. We also substitute $\dot{x} = 6$ on the right. We get $0.8\dot{\theta} = 0.6$, i.e., $\dot{\theta} = 0.6/0.8 = 0.75$ rad/sec $= 43$ deg/sec.

Sometimes there are several variables that change with time, you know the rates of all but one of them, and you want to know the remaining rate. As in the case when there are just two variables, take d/dt of both sides of the equation relating all of the variables, and then plug in all of the known values and solve for the unknown rate. In other words, our lazy method is the recommended method!

EXAMPLE 12.6. A road going South–North crosses a road going West–East at the point O. Car A is driving North along the first road, and car B is driving East along the second road. At time t car A is $a(t)$ to the North of O, and car B is $b(t)$ to the East of O. Suppose that at some instant you know the values of $a(t)$, $b(t)$, the velocity $v_1(t) = \dot{a}(t)$ of car A, and the velocity $v_2(t) = \dot{b}(t)$ of car B. Find a formula in terms of these known values for the rate at which the distance between the two cars is increasing.



Let c be the distance from car A to car B. By the Pythagorean Theorem, $c^2 = a^2 + b^2$. We now take d/dt of both sides of the Pythagorean Theorem, obtaining $2c\dot{c} = 2a\dot{a} + 2b\dot{b}$. Dividing by $2c$ gives the following formula for the unknown rate:

$$\dot{c} = \frac{a\dot{a} + b\dot{b}}{c} = \frac{av_1 + bv_2}{\sqrt{a^2 + b^2}}.$$

Notice how this problem differs from Example 12.3. In both cases we took d/dt of the Pythagorean Theorem. However, in Example 12.3 one of the sides was a constant (the altitude of the plane), and so d/dt of the square of that side was simply zero. In Example 12.6, on the other hand, all three sides of the right triangle are variables. As always in a story problem, it's important to read the problem carefully enough to determine at the start what are the variables and what are the constants.

HOMEWORK 12.

Part A. Basic problems:

1. A cylindrical tank has radius 20 cm. How fast does the water level in the tank drop when the water is being drained at 25 cm^3/sec?

2. A ladder 13 m long rests on horizontal ground and leans against a vertical wall. The foot of the ladder is pulled away from the wall at the rate of 0.6 m/sec. How fast is the top sliding down the wall when the foot of the ladder is 5 m from the wall?

3. A rotating beacon is located 2 miles out in the water. Let A be the point on the shore that is closest to the beacon. As the beacon rotates at 10 rev/min, the beam sweeps down the shore once each time it revolves. Assume that the shore is straight. How fast is the point where the beam hits the shore moving at an instant when the beam is lighting up a point 2 miles downshore from the point A?

4. A baseball diamond is a square 90 ft on a side. A player runs from first base to second base at 15 ft/sec. At what rate is the player's distance from third base decreasing when she is half way from first to second base?

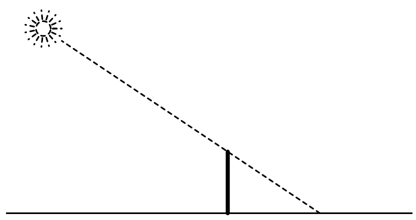
5. Sand is poured onto a surface at 15 cm^3/sec, forming a conical pile whose base radius is always half its altitude. How fast is the altitude of the pile increasing when the pile is 3 cm high?

6. A boat is pulled in to a dock by a rope with one end attached to the front of the boat and the other end passing through a ring attached to the dock at a point 5 ft higher than the front of the boat. The rope is being pulled through the ring at the rate of 0.6 ft/sec. How fast is the boat approaching the dock when 13 ft of rope are out?

7. A balloon is at a height of 50 meters, and is rising at the constant rate of 5 m/sec. At that instant a bicycle passes beneath it, traveling in a straight line at the constant speed of 10 m/sec. How fast is the distance between them increasing 2 seconds later?

8. A cone-like vat has square cross-section (rather than circular cross-section, as in the case of the usual cone). The dimensions at the top are 2 m $\times$ 2 m, and the depth is 5 m. If water is flowing into the vat at 3 m^3/min, how fast is the water level rising when the depth of water (at the deepest point) is 4 m? Note: the volume of any conical shape is $(1/3) \times$ height $\times$ area of base.

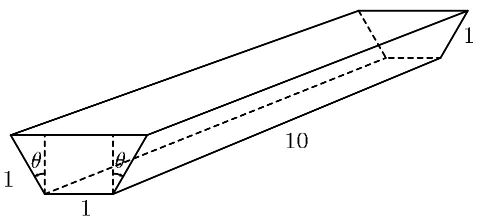
9. The sun is setting at the rate of 1/4 deg/min on a day when the sun passed directly overhead. How fast is the shadow of a 25 meter wall lengthening at the moment when the shadow is 50 meters? (See the picture.)



10. A woman 5 ft tall walks at the rate of 3.5 ft/sec away from a streetlight that is 12 ft above the ground. At what rate is the tip of her shadow moving? At what rate is her shadow lengthening? Notice that in this problem these rates do not depend on the instant in question — both rates are constant, provided that the woman's speed is constant.

11. A police helicopter is flying at 150 mph at a constant altitude of 0.5 mile above a straight road. The pilot uses radar to determine that an oncoming car is at a distance of exactly 1 mile from the helicopter, and that this distance is decreasing at 190 mph. Find the speed of the car.

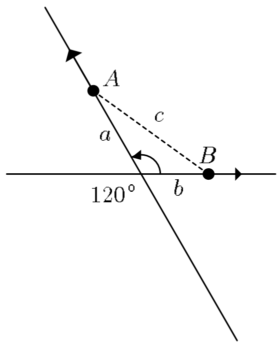
12. The trough shown on the right is constructed by fastening together three slabs of wood of dimensions 10 ft $\times$ 1 ft, and then attaching the construction to a wooden wall at each end. The angle θ was originally 30°, but because of poor construction the sides are collapsing. The trough is full of water. At what rate (in ft^3/sec) is the water spilling out over the top of the trough if the sides have each fallen to an angle of 45°, and are collapsing at the rate of 1° per second?



13. Suppose that in Example 12.6 the angle between the two roads is 120°, instead of 90° (that is, the South–North road actually goes in a somewhat northwesterly direction from O). In this problem use the law of cosines: $c^2 = a^2 + b^2 - 2ab\cos\theta$.

(a) Express $\dot{c}$ in terms of $a, b, \dot{a}, \dot{b}$.

(b) Notice that the angle θ is constant (and so $\cos\theta$ is also constant). State a story problem (perhaps involving a bug on a door, rather than a car on a road) in which not only a, b, c but also θ are all variables. In that case suppose you know the values of a, b, θ and their rates. Find a formula for $\dot{c}$.



Part B. Additional Practice (do these as soon as possible, certainly before the midterm):

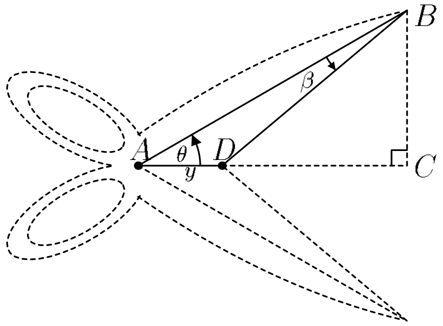
14. A light shines from the top of a pole 20 m high. An object is dropped from the same height from a point 10 m away. How fast is the object's shadow moving on the ground one second later? (Neglect air resistance and take $g = 9.8$ m/sec^2 for the falling object.)

15. The two blades of a pair of scissors are fastened at the point A. Let a denote the distance from A to the tip of the blade (the point B). Let β denote the angle at the tip of the blade that is formed by the line $\overline{AB}$ and the bottom edge of the blade (see the diagram). Suppose that a piece of paper is cut in such a way that the center of the scissors at A is fixed, and the paper is also fixed. As the blades are closed (i.e., the angle θ in the diagram is decreased), the distance y between A and D increases, cutting the paper.

(a) Express y in terms of a, θ, and β.

(b) Suppose you know the rate $d\theta/dt$ at which the angle between the blades is decreasing. Express dy/dt in terms of a, θ, β, and $d\theta/dt$.

(c) Suppose that the distance a is 20 cm, and the angle β is 5°. Further suppose that you are closing the scissors at the rate of 50 deg/sec. At the instant when $\theta = 30°$, find the rate (in cm/sec) at which the paper is being cut.



16. In the previous problem,

(a) express in terms of $d\theta/dt$ the speed of the point on the scissors that moves fastest (this point is B);

(b) if we have $a = 20$ kilometers (i.e., we have scissors of grand proportions) and if $\beta = 0.001$ radians, find a value of $d\theta/dt$ for which dy/dt is a little faster than the speed of light ($c = 300000$ km/sec) when the pair of scissors is almost closed;

(c) would the situation in your answer to part (b) be possible, or would it violate the Theory of Relativity? Explain.

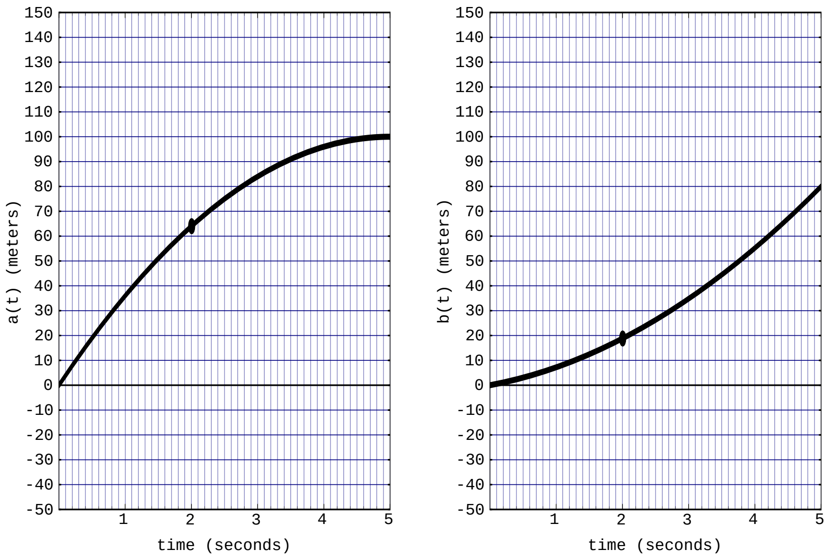
17. In the situation of Example 12.6, find the rate at which the distance between the cars is increasing or decreasing if

(a) car A is 300 meters north of O, car B is 400 meters east of O, both cars are going at constant speed toward O, and the two cars will collide in 10 seconds;

(b) 8 seconds ago car A started from rest at O and has been picking up speed at the steady rate of 5 m/sec^2, and 6 seconds after car A started car B passed O moving east at constant speed 60 m/sec;

(c) the functions $a(t)$ and $b(t)$ are given by the graphs below and you want to find the rate at which the cars are separating at time $t = 2$ sec.

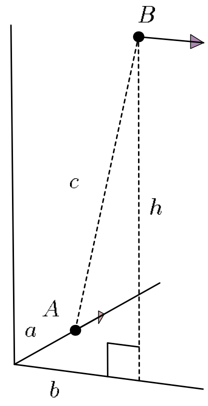
Hint: Estimate derivatives by drawing tangent lines.



18. In Example 12.6 suppose that instead of car B you have a helicopter flying at speed v_2 to the east of O. Let h be the altitude of the helicopter, and let c be the distance from car A to the helicopter. Then the three-dimensional Pythagorean Theorem states: $c^2 = a^2 + b^2 + h^2$. Find a formula for $\dot{c}$ in terms of a, b, h and the various rates if:

(a) the helicopter is flying at speed $\dot{b}$ at constant altitude;

(b) the helicopter is flying at horizontal speed $\dot{b}$ (i.e., the point on the ground under the helicopter is moving at this speed), and at the same time is gaining altitude at rate $\dot{h}$.



CHAPTER 13

Curve Sketching

In this section we discuss how to sketch the graph of a function $y = f(x)$ without plotting many points. Indeed, most parts of a curve are routine and uninteresting and plotting them precisely serves very little purpose. What is really important is to learn the exact points features of the curve change and if we plot these points precisely then we will have a reliable as well as useful graph.

Without having to make up a table of values for the function, we can obtain a good *qualitative* picture of the graph using certain crucial information — local maxima and local minima, inflection points, asymptotes, etc. Our aim is not to draw an exact graph, but rather to get an accurate overall picture of the graph and to pinpoint the points where something special happens.

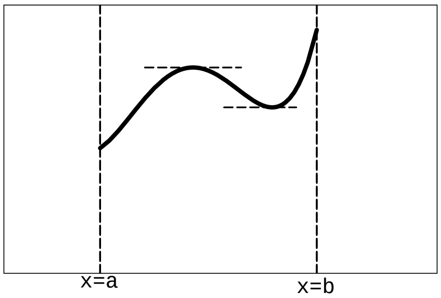
We start by describing the steps to take in curve sketching. For a particular $f(x)$, not all of the steps below will necessarily lead to useful information. The various possibilities will be illustrated later in the examples.

1. Maxima/Minima

If $(p, f(p))$ is a point where $f(p)$ is the highest value as long as the x-values vary in a small interval around p, then $f(p)$ is said to be a **local maximum**value of $f(x)$. Similarly, $f(p)$ is a **local minimum** value, if $f(p)$ is the lowest value as x varies in a small interval near p. Collectively, a local maximum or minimum is called a **local extremum.** The plurals of maximum, minimum and extremum are maxima, minima and extrema respectively.

If the derivative $f'(x)$ exists at $x = p$, or, in other words, if the tangent line to the curve $y = f(x)$ exists at $x = p$, then it can be shown that *the tangent line at that point is horizontal*, i.e., $f'(p) = 0$. The idea of the proof of this claim is easy. If $f'(p) > 0$ then the function is increasing near $x = p$ and there must be values of $f(x)$ bigger than $f(p)$ near (and to the right) of $x = p$. Similarly, if $f'(p) < 0$, then the function is decreasing at $x = p$ and there must be values of $f(x)$ smaller than $f(p)$ near (and to the left) of $x = p$. So, in either case, the function cannot have a local maximum at $x = p$. The case of a local minimum is similar.

Such points can thus be found by setting the derivative equal to zero, and solving for x. (There may be one x for which $f'(x) = 0$, there may be many, or there may be none.) Once you find the x for which $f'(x) = 0$ (and the corresponding y-coordinate of each possible max/min point), you have to determine whether it really is a maximum or minimum. In simple examples it might be obvious. A systematic way to tell is by the second derivative test, which will be described below.



First, we make a few remarks.

1. The reason for using the term **local maximum** point instead of just "maximum " is that it may not actually be a maximum value of the function on the whole domain. As in the above drawing, the local maximum might only be the maximum $f(x)$ for all nearby x — the function might later turn again and go up to a higher value. The reason for the use of the term **local minimum** instead of just "minimum" is similar. Sometimes, local maxima and minima are referred to as "upper turning points" and "lower turning points."

   What is usually understood by the "maximum" will be called the "absolute maximum". Thus $f(p)$ is said to be the absolute maximum value of $f(x)$ on some interval, if it is bigger than or equal to $f(x)$ as x varies over the interval. The term absolute minimum is similar. Some people use the term "global" in place of "absolute".

   If you want to be very precise, you should never use the terms maximum or minimum without some adjective like "local" or "absolute" or "global". If a given word problem does not use these precise terms, then you should use some common sense reasoning to determine what the question must mean!
2. A maximum or minimum might occur at a point where there is no well defined tangent line, i.e., where $f'(x)$ does not exist. For example, $f(x) = |x|$ has its (local as well as absolute) minimum point at $(0, 0)$.

   Sometimes a curve has a vertical tangent. At such a point, the value of $f'(x)$ will come out undefined as the limiting process for the derivative will lead to $\pm\infty$ as limiting values. For our purposes, we can treat this also as a case of $f'(x)$ undefined, even though it is possible to make a reasonable definition and declare it well defined. A simple example of this is the curve $y = {}^{1/3}\!\sqrt[3]{x}$ which has a vertical tangent at $x = 0$. [1]
3. Suppose you want to find the absolute maximum of a function $f(x)$ on the interval from a to b. To do this, you have to try
   (a) all points x where $f'(x) = 0$,
   (b) all points x where $f'(x)$ does not exist (as explained above), and
   (c) the endpoints a and b (for example, in the above drawing the maximum point is at the endpoint b).

   The largest of the function values among these shall come out to be the absolute maximum.

[1] Generally, we forbid taking fractional powers of negative numbers. However, the function that we are talking about here is the "real" cube-root which is well defined even for negative values of x. It can be thought to be the usual cube-root function extended to the negative values by making it an odd function.

4. Later, we will see examples of points where, even though the slope of the tangent line is 0, we have *neither* a maximum nor a minimum there.

2. The Second Derivative Test

Suppose that $x = p$ is a solution of $f'(x) = 0$. The second derivative test is a method that **in most cases** will tell us whether the point $(p, f(p))$ is a local maximum or minimum. The test works as follows.

Compute the *second derivative of the function at the same value* $x = p$, i.e., $f''(p)$. If $f''(p)$ is positive, that means that $f'(x)$ is an increasing function at p, i.e., $f'(x)$ is negative a little to the left of p and is positive a little to the right of p. This is what happens near a *local minimum*. On the other hand, if $f''(p)$ is negative, then $f'(x)$ is positive a little to the left of p and is negative a little to the right of p, which is what happens near a *local maximum*. If it so happens that $f''(p) = 0$ (i.e., if *both* $f'(p)$ and $f''(p)$ are zero at the same x), then the second derivative test doesn't give you any information. To summarize,

$$f''(x)\begin{cases} > 0 & \Longrightarrow \text{min pt;} \\ < 0 & \Longrightarrow \text{max pt;} \\ = 0 & \Longrightarrow \text{can't tell.} \end{cases}$$

To see how and why this test succeeds or fails, you should always consider graphs of the functions x, x^2, x^3, x^4. We have provided a picture below. You should analyze the point $x = 0$ in each case.

3. The First Derivative Test

The Second Derivative Test works like a charm if we can find $f''(x)$ and it is nonzero. If it is difficult to evaluate or zero, then the first derivative case comes in handy. In fact, it can also be used in place of the Second Derivative Test.

Suppose you are testing a point $x = p$ and $f'(p) = 0$. You wish to check if this is a local maximum or local minimum. Evaluate the derivative $f'(x)$ at a point x_1 to the left of p **and near** p. Also evaluate the derivative $f'(x)$ at a point x_2 to the right of p **and near** p.

The test says:

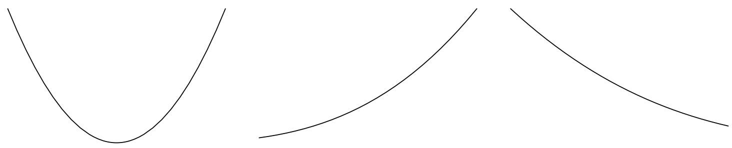
1. If $f'(x_1) > 0$ and $f'(x_2) < 0$, the $x = p$ is a local maximum.
2. If $f'(x_1) < 0$ and $f'(x_2) > 0$, the $x = p$ is a local minimum.
3. If $f'(x_1)$ and $f'(x_2)$ have the same sign, then it is neither a local maximum nor a local minimum. It is an "inflection point" discussed below.

As you can see, this test is definitive. The only problem is to decide how near should the points x_1, x_2 should be. The requirement is that **there should be no critical points of the function** inside $[x_1, p)$ and $(p, x_2]$. If you have only finitely many critical points, then this will be easy to do. If you have infinitely many critical points which cluster around $x = p$, then this test cannot be applied at all!

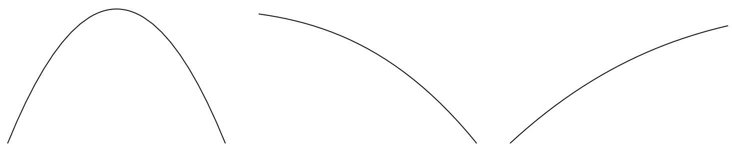
See if you can think of such functions, they don't appear naturally.

4. Concavity and Inflection

We know that the sign of the derivative tells whether a function is increasing or decreasing. For example, any time that $f'(x) > 0$, that means that $f(x)$ is increasing. We now explain the geometrical meaning of the sign of the **second** derivative. Suppose that $f''(x) > 0$ for a certain interval of x. This means that as we go from left to right across the interval, the slope of the tangent line increases. We say that the function is "concave upward" over this interval. The pictures below all show concave upward functions.



Conversely, if $f''(x) < 0$ over a certain interval, then the slope of the tangent line decreases, and we say that the function is "concave downward." The pictures at the top of the next page all show concave downward functions.



There are several ways to describe the difference between upward concavity and downward concavity: (1) An upward concave function looks like part of a right-side-up bowl (holds water), while a concave downward function looks like part of an upside-down bowl. (2) If you draw the tangent line to a concave upward function at a point, then the curve sits on the tangent line, whereas in the case of a concave downward function the tangent line rests on the curve. (3) As you go from left to right, the tangent line to a concave upward curve rotates counterclockwise, while the tangent line to a concave downward curve rotates clockwise.

A point that separates a region of upward concavity and a region of downward concavity is called a **point of inflection**. The geometrical meaning of a point of inflection is that, as you roll the tangent line along the curve from left to right, at a point of inflection it reverses its direction of rotation (from clockwise to counterclockwise, or vice-versa). That is, *the sign of second derivative changes as you roll past a point of inflection.* Thus, inflection points can be found by setting $f''(x) = 0$ and solving for x. Each of the drawings for Examples 13.1 and 13.2 below shows a point of inflection at the origin separating a concave downward part of the curve (on the left) from a concave upward part (on the right).

Warning. Notice that $(x, f(x))$ may not be a point of inflection even though $f''(x) = 0$. After finding the points where $f''(x) = 0$, you then must check that $f''(x)$ changes sign. (See Example 13.5 below.)

5. Asymptotes and Other Things to Look For

An asymptote of a curve is a line which will be a tangent to the curve if the curve can be analyzed near infinity. There are precise ways of discussing points at infinity of a curve and making this discussion rigorous. We will, however, take a naive definition and learn how to recognize asymptotes.

Thus, an asymptote for us will be a line such that the graph of the curve approaches the line along some branch of the curve going out to the infinite region of the plane.

The easiest to analyze is a vertical asymptote. A vertical line $x = p$ is an asymptote to $y = f(x)$, if $f(x)$ becomes infinite as x nears p. It is possible that the function gets closer to plus or minus ∞ depending our approach to the point $x = p$. Usually, the formula for the function has a denominator that becomes zero at $x = p$. For example, the reciprocal function has a vertical asymptote at $x = 0$, and the function $\tan x$ has a vertical asymptote at $x = \pi/2$ (and also at $x = -\pi/2$, $x = 3\pi/2$, etc.).

Note that usually $f(p)$ would not be defined - or at least not naturally defined by the formula. It may be artificially defined in the construction of the function.

Vertical asymptotes are found by asking when the denominator of the formula of a function goes to zero.

A horizontal asymptote is a horizontal line to which $f(x)$ gets closer and closer as x approaches ∞ (or as x approaches $-\infty$). For example, the reciprocal function has the x-axis for a horizontal asymptote. Sometimes one also encounters slanted lines for asymptotes. For example, the function $f(x) = 2x + \frac{1}{x}$ has the line $y = 2x$ as a slanted asymptote, because when x is large $2x + \frac{1}{x}$ and $2x$ are very close together.

In general a line $y = mx + c$ shall be an asymptote to $y = f(x)$ if the expression $g(x) = f(x) - (mx + c)$ approaches 0 as x approaches ∞ (or as x approaches $-\infty$.

If the domain of our function does not extend out to infinity, we should also ask what happens as x approaches the boundary of the domain. For example, the function $y = f(x) = 1/\sqrt{r^2 - x^2}$ has domain $-r < x < r$, and y becomes infinite as x approaches either r or $-r$. Thus we get vertical asymptotes. This reconfirms the fact that **in general**, if $x = p$ is a vertical asymptote then $x = p$ is **not in the domain.**

If there are any points where the derivative fails to exist (a cusp or corner), then we should take special note of what the function does at such a point.

Finally, it is worthwhile to notice any symmetry. A function $f(x)$ that has the same value for $-x$ as for x, i.e., $f(-x) = f(x)$, is called an "even function." **It is crucial that this property holds throughout the domain of the function. It is not enough to check the equation for a few sample points!** Its graph is symmetric with respect to the y-axis. The most important examples of even functions are: x^n when n is an even number, indeed, this is the reason for the use of the word

even. Other examples of even functions are: $\cos x$, and $\sin^2 x$. Expressions formed from even functions are also even, for example $x^2 + 1/x^2 - \cos(2x)$.

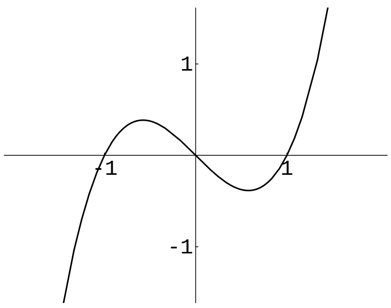
On the other hand, a function that satisfies the property $f(-x) = -f(x)$ is called an "odd function." Its graph is symmetric with respect to the origin. **The warning about checking the whole domain should be applied here also.** As before, the main examples of odd functions are: x^n when n is an odd number, and as before this is the reason for the terminology.

Some more examples of odd functions are: $\sin x$, and $\tan x$. Unlike even functions, expressions in odd functions may fail to stay odd. For example the square of an odd function is always even! However, an odd function stays off if multiplied by a constant and sums of odd functions are odd.

Of course, most functions are neither even nor odd, and do not have any particular symmetry. **A striking fact is that every function $f(x)$ can be written as an odd function $g_1(x)$ and an even function $g_(x)$. You should try to prove this.**

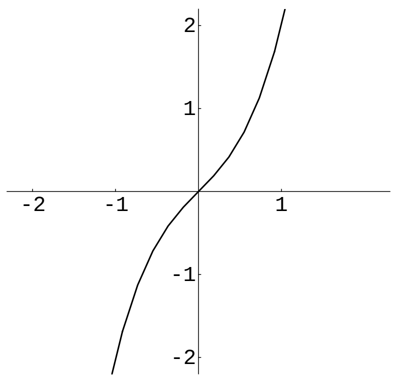
EXAMPLE 13.1. Sketch $y = x^3 - x$.

First, we set $0 = y' = 3x^2 - 1$, which has solutions $x = \pm\sqrt{3}/3 = \pm 0.577$. The corresponding y-coordinates are -0.385 and $+0.385$, i.e., the two "critical points" are $(0.577, -0.385)$ and $(-0.577, 0.385)$. The second derivative test gives $y'' = 6x$, which is positive for the first point and negative for the second. Thus, the first of the two points (in the fourth quadrant) is a local minimum, and the second is a local maximum. Since $y'' > 0$ when $x > 0$ and $y'' < 0$ when $x < 0$, it follows that the curve is concave upward in the right half of the graph and concave downward in the left half. The two halves are separated by the inflection point at the origin. This curve has no asymptotes. It does have symmetry, however, because it is an odd function. Its graph is shown above.



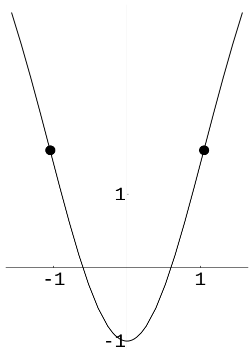
EXAMPLE 13.2. Sketch $y = x^3 + x$.

The only difference with Example 13.1 is the $+$ in front of the x. But this means that the derivative $y' = 3x^2 + 1$ is *never* zero, and hence there are no local maxima or minima. In fact, the function is always increasing, because y' is always positive. The second derivative $y'' = 6x$ is the same as in the last problem, and hence the concavity situation is the same. In particular, this curve also has an inflection point at the origin.



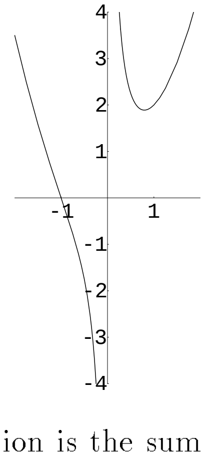
EXAMPLE 13.3. Sketch $y = x^2 - \cos(2x)$ for $-\pi/2 \le x \le \pi/2$.

When we set $0 = y' = 2x + 2\sin(2x)$, we obtain the equality $\sin(2x) = -x$. However, from a quick sketch of the two curves $\sin(2x)$ and $-x$, we immediately see that the only x for which they are equal is $x = 0$. When $x = 0$ the y-coordinate is $0^2 - \cos(2 \cdot 0) = -1$, so our critical point is $(0, -1)$. Since $y'' = 2 + 4\cos(2x)$, which is positive when $x = 0$, the second derivative test tells us that $(0, 1)$ is a local minimum. To find inflection points we set $0 = y'' = 2 + 4\cos(2x)$. This gives $\cos(2x) = -0.5$. Looking at our table in the section on trig functions, we see that in the range from $x = 0$ to $x = \pi/2$ the equality $\cos(2x) = -0.5$ holds when $2x = \frac{2}{3}\pi$, i.e., $x = \pi/3$. Since $\cos(-2x) = \cos(2x)$, the equality also holds when $x = -\pi/3$. Thus, the points $(\pi/3, 1.5966)$ and $(-\pi/3, 1.5966)$ are inflection points. Between these two inflection points the second derivative is positive (concave up), whereas for $x > \pi/3$ and for $x < -\pi/3$ the second derivative is negative (concave downward). Finally, note that $x^2 - \cos(2x)$ is an even function, and so the graph is symmetrical with respect to the y-axis.



EXAMPLE 13.4. Sketch $y = x^2 + \frac{1}{x}$.

Setting $0 = y' = 2x - \frac{1}{x^2}$, we solve this by bringing $1/x^2$ to the left and clearing denominators: $1 = 2x \cdot x^2 = 2x^3$. So $x = \sqrt[3]{0.5} = 0.7937$. The corresponding y-coordinate is 1.8899. Using the second derivative $y'' = 2 + 2 \cdot x^{-3}$, we see that this is a local minimum. To find inflections, we set $2 + 2/x^3 = 0$. Clearing denominators and solving for x gives $x^3 = -1$, and so $x = -1$. Thus, the point $(-1, 0)$ is an inflection. In this example we have an asymptote when $x = 0$. To the right of the asymptote (i.e., for positive x), the second derivative is always positive; whereas for negative x the second derivative is positive when $x < -1$ and negative when x is between -1 and 0. Thus, the interval $-1 < x < 0$ is a region of downward concavity; the graph is concave upward outside of this interval. Putting all this together leads to the graph at the right.

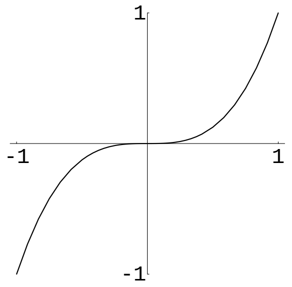


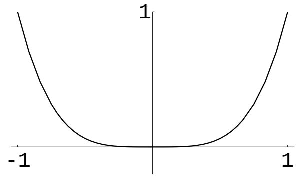
There is another way to think of this example. Our function is the sum of two functions x^2 and $1/x$. The former function is by far the larger of the two when x is large positive or large negative, whereas the reciprocal function is by far the more important when x is near 0. Thus, the graph resembles $1/x$ when x is near 0 and resembles x^2 when x is far from 0. Roughly speaking, the inflection point $(-1, 0)$ and the local minimum $(0.7937, 1.8899)$ mark the transition from behaving like the graph of x^2 to behaving like the graph of $1/x$.

EXAMPLE 13.5. Sketch (a) $y = x^3$ and (b) $y = x^4$.

(a) Setting $0 = y' = 3x^2$, we see that the origin is a possible maximum or minimum. However, the second derivative test tells us nothing, since $y'' = 6x$ also is zero when $x = 0$. In fact, even though $y' = 0$ when $x = 0$, the origin is neither a maximum nor a minimum. Rather, it is a point of inflection, separating the concave downward region in the third quadrant from the concave upward region in the first quadrant.

(b) Again we see that both the first derivative and the second derivative vanish at the origin (and neither derivative is zero anywhere else). This time, however, the origin is a local minimum. Even though the second derivative test doesn't tell us this, we can see directly that, since x^4 is positive for nonzero x, its smallest possible value is when $x = 0$. Note that the origin is *not* a point of inflection, even though $y'' = 0$ there. This is because $y'' = 12x^2 > 0$ both for $x > 0$ and for $x < 0$, so everywhere we have upward concavity. (It is rare for a point where $y'' = 0$ not to be an inflection point; this can occur only when the *third* derivative y''' is also zero at the same point.)

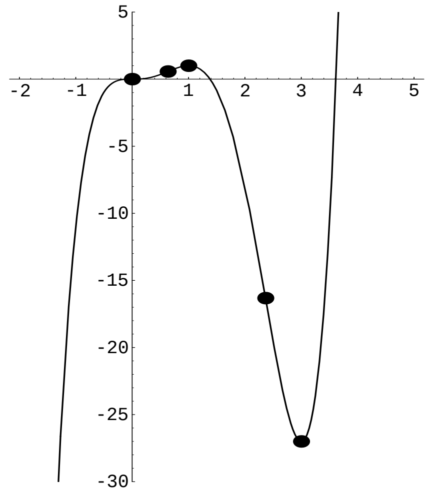




EXAMPLE 13.6. Sketch $y = x^5 - 5x^4 + 5x^3$.

First, we set $0 = y' = 5x^4 - 20x^3 + 15x^2$. To solve this, we factor out what we can, namely $5x^2$. This leaves a quadratic that can be factored either by inspection or by the quadratic formula. The result is $0 = y' = 5x^2(x - 1)(x - 3)$. Thus, the critical points are $(0, 0)$, $(1, 1)$, and $(3, -27)$. Using $y'' = 20x^3 - 60x^2 + 30x = 10x(2x^2 - 6x + 3)$, we see from the second derivative test that $(1, 1)$ is a local maximum and $(3, -27)$ is a local minimum, but we get no information about $(0, 0)$. Setting $0 = y''$ and using the quadratic formula to find the roots of $2x^2 - 6x + 3$, we find the following three points of inflection: $(0, 0)$, $(0.634, 0.569)$, $(2.366, -16.32)$.

In a complicated case like this, it is also worthwhile to see what the function is doing when x is large positive or large negative. If x is large, the x^5 term in our function dominates (is greater in absolute value than all the other terms). Thus, the function heads upward steeply into the first quadrant as $x \longrightarrow +\infty$, and it heads steeply down into the third quadrant as $x \longrightarrow -\infty$. Putting this information together, we obtain the graph shown above. The curve is concave downward in the third quadrant, and also between the two points of inflection $(0.634, 0.569)$ and $(2.366, -16.32)$. For $0 < x < 0.634$ and for $x > 2.366$ the curve is concave upward.



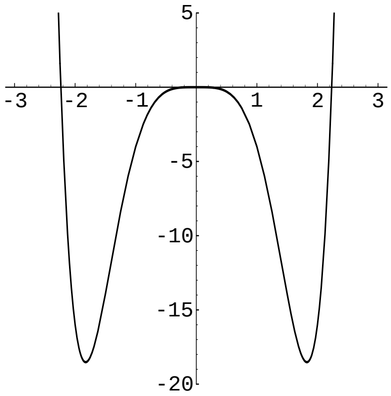
HOMEWORK 13.

1. Suppose that n is an integer greater than 2. On the curve $y = f(x) = x^n$, what sort of point is the origin? Sketch the curve, and indicate the concavity.

2. In parts (a)-(j) below, find each max/min point (using the second derivative test to be sure what type of point it is), point of inflection, and asymptote (vertical, horizontal, or slanted), if there are any. Also indicate where the curve is concave up and concave down. Sketch the graph.
(a) $y = x^2 - x$, (b) $y = 2 + 3x - x^3$, (c) $y = x^3 - 9x^2 + 24x$, (d) $y = x^4 - 2x^2 + 3$, (e) $y = 3x^4 - 4x^3$, (f) $y = (x^2 - 1)/x$, (g) $y = 3x^2 - (1/x^2)$, (h) $y = \cos(2x) - x$, (i) $y = \sin x + \cos x$, (j) $y = \tan(x/2) - x$.

3. Suppose $\begin{cases} f'(x) > 0 & \text{for } |x| > 2; \\ f'(x) < 0 & \text{for } |x| < 2; \\ f''(x) < 0 & \text{for } x < 0; \text{ and} \\ f''(x) > 0 & \text{for } x > 0 \end{cases}$.
From the given information, sketch a possible graph of $f(x)$. How could your answer vary and still be correct?

4. Suppose $f''(x) < 0$ for $x < 1$ and $f''(x) > 0$ for $x > 1$. Suppose also that $f(1) = 1$. Sketch a possible graph of $f(x)$, assuming that
i) $f'(1) = 0$, ii) $f'(1) > 0$ and iii) $f'(1) < 0$.



5. At the right is a sketch of $y = x^6 - 5x^4$.

(a) Find the exact coordinates (x, y) of the local maxima and local minima.

(b) For what values of x is the function concave upward?

(c) Find the exact x-coordinates of all points of inflection. (**Hint:** Color the part of the curve that is concave up blue and color the part that is concave down red. Points of inflection occur *only* where the curve *changes* color!)

(d) Explain how you *know from your calculations* (not from the sketch you were given) which of your answers to part (a) are minima and which are maxima.

CHAPTER 14

Maximum/Minimum Problems

A max/min problem is a story problem where we want to *optimize* something, that is, we want to determine the situation where something is the biggest possible (such as profit, volume of a container, etc.) or where something is the smallest possible (cost, amount of material needed, time of travel, etc.) Solving a max/min problem is the two-step process of (1) reformulating the story problem as a "pure math problem," and (2) solving the "pure math problem". We begin with a brief outline of the procedure, then discuss how to solve the pure math problem, and then present several examples illustrating the procedure.

The **pure math problem** just alluded to was described at the beginning of Chapter 13:

> *Find the maximum (or minimum) value of a function $f(x)$ on the interval from a to b.*

Here is a procedure for solving this type of problem:

1. Find the value of $f(x)$ at each point x between a and b where $f'(x) = 0$.
2. Find the value of $f(x)$ at all points x between a and b where $f'(x)$ does not exist.
3. Find the value of $f(x)$ at the endpoints a and b. (If $a = -\infty$ and/or $b = +\infty$ you have to find the values that $f(x)$ approaches as x approaches $\pm\infty$.)
4. The maximum (minimum) value of $f(x)$ is the largest (smallest) of the values found in steps (1)–(3).

Here are three simple examples which illustrate the procedure.

EXAMPLE 14.1. Let $f(x) = -x^2 + 4x - 3$. Find the maximum value of x on the interval between 0 and 4.

First note that $f'(x) = -2x + 4 = 0$ when $x = 2$, and $f(2) = 1$. Next observe that $f'(x)$ is defined for all x, so step (2) gives no x-values. Finally, $f(0) = -3$ and $f(4) = -3$. The largest value of $f(x)$ on the interval between 0 and 4 is $f(2) = 1$.

EXAMPLE 14.2. Again let $f(x) = -x^2 + 4x - 3$. Find the maximum value of $f(x)$ on the interval between -1 and $+1$.

First note that $f'(x) = -2x + 4 = 0$ when $x = 2$. But $x = 2$ is not in the interval, so we don't use it. Next observe that $f'(x)$ is defined for all x, so step (2) gives no x-values. Finally, $f(-1) = -8$ and $f(1) = 0$. So the largest value of $f(x)$ on the interval between -1 and 1 is $f(1) = 0$.

Note: The maximum value depends not only on the function but on the interval as well. Later in this chapter, we will give examples of story problems where this observation is important.

EXAMPLE 14.3. Find the minimum value of the function $f(x) = 7 + |x - 2|$ for x between 1 and 4.

Note that $|x - 2| = x - 2$ for $x \geq 2$ and $|x - 2| = 2 - x$ for $x < 2$. Thus the function $f(x)$ is given by two separate formulas:

$$f(x) = 7 + 2 - x = 9 - x \text{ for } 1 \leq x < 2 \text{ and } f(x) = 7 + x - 2 = 5 + x \text{ for } 2 \leq x \leq 4.$$

Thus the derivative of $f(x)$ is given by:

$$f'(x) = -1 \text{ for } 1 < x < 2 \text{ and } f'(x) = 1 \text{ for } 2 < x < 4.$$

Thus, $f'(2)$ is undefined and $f'(x)$ is never zero. So we compute $f(2) = 7$. Finally, we check the end points $f(1) = 8$ and $f(4) = 9$. The smallest of these numbers is $f(2) = 7$, which is, therefore, the minimum value of $f(x)$ on the interval $1 \leq x \leq 4$.

Note that we did not include the endpoints in our investigation of $f'(x)$. They are not needed in our procedure and need special handling even for definition since the derivative at such points can only make sense from one direction

EXAMPLE 14.4. Of all rectangles of area 100, which has the smallest perimeter?

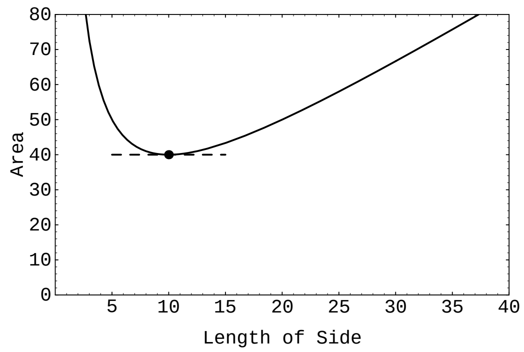
The first step toward solving this problem is to reduce it to a pure math problem. If x denotes one of the sides of the rectangle, then the adjacent side must be $100/x$ (in order that the area be 100). So the function we want to minimize is

$$f(x) = \text{perimeter of } x \text{ by } \frac{100}{x} \text{ rectangle} = 2x + 2\frac{100}{x}.$$

The perimeter function is given by the formula $2x + 200/x$, where the *domain* is $x > 0$ (because it makes no sense to talk of a rectangle with negative side). We have reduced the problem to finding the minimum value of $f(x)$ for x between 0 and $+\infty$.

We next find $f'(x)$ and set it equal to zero: $0 = f'(x) = 2 - 200/x^2$. To solve for x, we clear denominators and then move the constant to the left, obtaining $200 = 2x^2$, which gives us $x = 10$ (actually, $x = \pm 10$, but $x = -10$ does not have practical meaning). Since $f'(x)$ is defined for all $x > 0$, so there are no corners or cusps to check. Our next step is to examine the behavior of $f(x)$ at the endpoints. The sketch this function $f(x) = 2x + 200/x$ in the region of positive x is shown below at the right. We find that as x gets close to 0 the function becomes larger and larger; and the perimeter function also becomes large when x gets very large.

(More precisely, the line $x = 0$ is a vertical asymptote, and the line $y = 2x$ is a slanted asymptote that the function approaches as $x \to \infty$.) Consequently, the minimum value of $f(x)$ occurs at $x = 10$, and the desired rectangle is the 10×10-rectangle, whose perimeter is 40. Any other rectangle of area 100 has perimeter greater than 40. In the same way it is not hard to show that for any fixed area A, the rectangle of area A with smallest perimeter is the square of side $\sqrt{A}$.



Another way to see that $x = 10$ gives a minimum is to notice that $f(x)$ decreases for a while as we get away from 0 ; but if we go too far to the right $f(x)$ is increasing. In other words $f'(x) < 0$ if $x < 10$; while $f'(x) > 0$ if $x > 10$. So it is clear that the point $x = 10$ where $f'(x) = 0$ really is a minimum.

EXAMPLE 14.5. You want to sell a certain number x of items in order to maximize your profit. Market research tells you that if you set the price at \$1.50, you will be able to sell 5000 items; and for every 10 cents you lower the price below \$1.50 you will be able to sell another 1000 items. Suppose that your fixed costs ("start-up costs") total \$2000, and the per item cost of production ("marginal cost") is \$0.50. Find the price to set per item and the number of items sold in order to maximize profit, and also determine the maximum profit you can get.

To answer these questions, we first observe that our profit is equal to the number of items sold times the price per item, minus the total cost to us. If P denotes the price, then this profit is $xP - (2000 + 0.50x)$. (The term in parentheses is fixed cost plus number of items times per item cost.)

To find P in terms of x, we first note that the information given allows us to write x in terms of P: $x = 5000 + 1000(1.50 - P)/0.10$ (i.e., 5000 plus 1000 times the number of multiples of 10 cents that P is below \$1.50). Solving for P and simplifying gives: $P = 2 - 0.0001x$. Thus, the profit function that we want to maximize is:

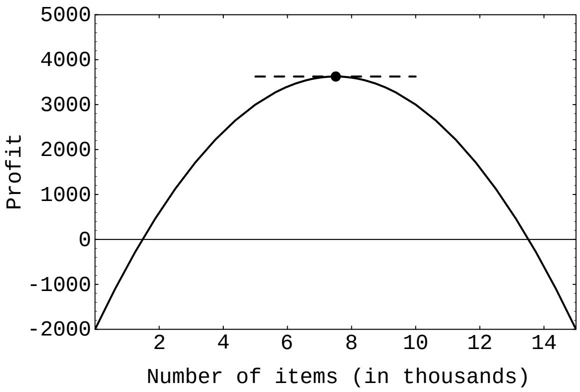
$$\begin{aligned} f(x) &= x\,P - (2000 + 0.50\,x) \\ &= x(2 - 0.0001\,x) - 2000 - 0.50\,x = 1.5x - 0.0001\,x^2 - 2000. \end{aligned}$$

We are now ready to set the derivative equal to zero: $0 = f'(x) = 1.5 - 0.0002x$, from which we obtain $x = 7500$. Thus, we plan to sell 7500 items, which we accomplish by lowering the price to \$1.25. It costs us $\$2000 + 0.50 \cdot 7500 = \$5750$ to produce these items, and we take in $7500 \cdot \$1.25 = \$9375$, leaving us a profit of $f(7500) = \$9375 - \$5750 = \$3625$. This is the largest profit we can get under the assumptions of this problem.

Further analysis. We did not take all the usual steps and we now explain why they do not affect the answer here. For a complete answer, they are necessary!

What is the interval of values for x? Clearly $x > 0$ and x cannot be too large. By looking at the function, clearly x cannot be too large as we cannot be interested in a negative profit function. Thus our interval could be $[0, \infty)$ and neither endpoint will be of interest, since it leads to negative profit. Since we have only one other point (zero of $f'(x)$) to consider, it must be the point with maximum profit.

It is also possible to find a more precise interval for x corresponding to non negative profit, but it is hardly worth the effort!



To be safe, we should check that the value of x which made the profit-function have derivative 0 is really a maximum. It is not hard to compute that $f(x)$ is increasing (i.e., $f'(x) > 0$) when $x < 7500$, and $f(x)$ is decreasing when $x > 7500$, so that $f(x)$ reaches a peak at $x = 7500$. In most of the examples that follow (and when you do the homework problems) it is clear enough from the practical set-up that the value we find really is the maximum point (if we are looking for a maximum) or the minimum point (if we are looking for a minimum). So it is not necessary to belabor the matter. However, we should be cautious, as shown by Example 14.9 below.

EXAMPLE 14.6. Find the largest rectangle that fits inside the graph of the parabola $y = x^2$ below the line $y = a$. (The top side of the rectangle should be on the horizontal line $y = a$ — see the drawing on the following page.)

Here the problem is to find where along the parabola to put the lower-right corner of the rectangle. That corner will determine all the other corners. Thus, if we let (x, x^2) be the point on the parabola where the lower-right corner is put, then the upper-right corner will be (x, a), the lower-left corner will be $(-x, x^2)$, and the upper-left corner will be $(-x, a)$. We can now write the area of the rectangle in terms of x:

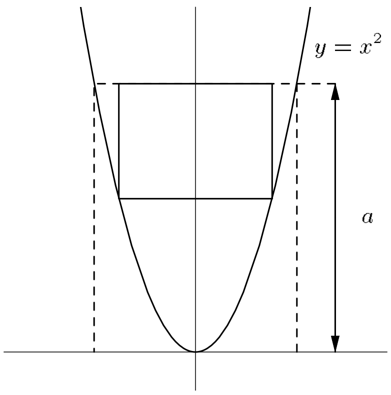
$$f(x) = \text{Area} = (2x)(a - x^2) = 2ax - 2x^3.$$

To maximize this area function, we set

$$0 = f'(x) = 2a - 6x^2,$$

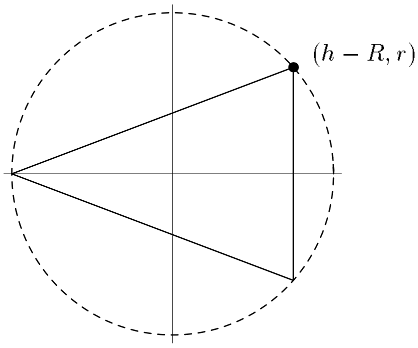
i.e., $x = \sqrt{a/3}$. The resulting rectangle has dimensions $2\sqrt{a/3} \times 2a/3$, and its area is $4(a/3)^{3/2}$.

Additional considerations. It seems reasonable that the biggest possible rectangle should be situated with sides parallel to the x and y axes and corners on the parabola and the line $y = a$ as stipulated. However, this has not been proved. We need to argue this carefully. If the rectangle is situated in a crooked manner, then we should be able to straighten it and then make it slightly bigger! You should think and try to establish these points.



EXAMPLE 14.7. If you fit the largest possible cone inside a sphere, what fraction of the volume of the sphere is occupied by the cone? (Here by “cone” we mean a right circular cone, i.e., a cone for which the base is perpendicular to the axis of symmetry, and for which the cross-section cut perpendicular to the axis of symmetry at any point is a circle.)

Let R be the radius of the sphere, and let r and h be the base radius and height of the cone inside the sphere. What we want to maximize is the volume of the cone: $\frac{1}{3}\pi r^2 h$. Here R is a fixed value, but r and h can vary. Namely, we could choose r to be as large as possible — equal to R — by taking the height equal to R; or we could make the cone's height h larger at the expense of making r a little less than R. See the side view depicted at the right where we have situated the picture in a convenient way relative to the x- and y-axes, namely, with the center of the sphere at the origin and the vertex of the cone at the far left on the x-axis.



Notice that the function we want to maximize — $\frac{1}{3}\pi r^2 h$ — depends on two variables. This happens in many max/min problems. However, the two variables are related in some way — by a certain condition of the form $C(r, h)$ =constant. So our next step is to find such a condition, and use it to solve for one of the variables in terms of the other, so as to have a function of only one variable to maximize. In this problem, the condition can be read off the above drawing: the upper corner of the triangle, whose coordinates are $(h-R, r)$, must be on the circle of radius R. That is,

$$(h-R)^2 + r^2 = R^2.$$

We can solve for h in terms of r or for r in terms of h. Let's do the latter. Then we substitute the result for r in $\frac{1}{3}\pi r^2 h$, obtaining $\frac{1}{3}\pi\big(R^2 - (h-R)^2\big)h$.

Notice a simple trick that we used. We do not actually solve for r, but write $r^2 = R^2 - (h-R)^2$ and then plug in our result into our function. If you try to solve for r, then you get two roots and you will have to simplify the expression again.

Recall that R is just a constant — the only variable now is h. Let us take $f(h) = \big(R^2 - (h-R)^2\big)h = (R^2 - h^2 + 2hR - R^2)h = 2h^2R - h^3$. What happened to the $\frac{1}{3}\pi$? Well, we dropped this constant factor for convenience. The value of h that makes $\frac{1}{3}\pi f(h)$ maximum is the same as the value of h that makes $f(h)$ maximum, so we don't need to keep that constant factor when we take the derivative and set it equal to zero.

We're now ready to set $0 = f'(h) = 4hR - 3h^2 = h(4R - 3h)$. This gives us $h = 4R/3$. In other words, we should, in fact, choose h a little larger than R, at the expense of having r somewhat less than R. The exact value of r is determined by the condition above: $(h-R)^2 + r^2 = R^2$. Solving for r here (after setting $h = 4R/3$) gives: $r = (\sqrt{8/9})R$. Finally, the fraction of the sphere occupied by this cone is:

$$\frac{\text{vol of cone}}{\text{vol of sphere}} = \frac{\frac{1}{3}\pi r^2 h}{\frac{4}{3}\pi R^3} = \frac{(8R^2/9)(4R/3)}{4R^3} = \frac{8}{27}.$$

The cone takes up a little less than 30% of the sphere.

Alternative Method: We can use the technique of implicit differentiation to avoid solving for h or r as follows.

We wish to find a maximum value for $F(h,r) = r^2h$ (dropping $\pi/3$ as before), subject to the condition $(h-R)^2 + r^2 = R^2$. Let us think of everything as a function of h and take derivatives with respect to h.

From the derivative of the relation, we get:

$$2(h-R) + 2r\,dr/dh = 0$$

By calculation of the derivative of $F(h,r)$ and setting it to zero, we get:

$$2rh(dr/dh) + r^2h = 0$$

Example 14.8. You are making cylindrical containers to contain a given volume. Suppose that the top and bottom are made of a material that is N times as expensive (cost per unit area) as the material used for the lateral side of the cylinder. Find (in terms of N) the ratio of height to base radius of the cylinder that minimizes the cost of making the containers.

Let us first choose letters for various things: h for the height, r for the base radius, and V for the volume of the cylinder, and c for the cost per unit area of the lateral side of the cylinder. Using the area formulas $2\pi r^2$ for the top and bottom together and $2\pi rh$ for the lateral side, we find that the total cost of the material is

$$c(2\pi rh) + Nc(2\pi r^2).$$

Here c and N are constants that are given to us, but r and h are variables. To eliminate one of the two variables, we use the *condition* that the volume must be V, i.e., $\pi r^2 h = V$. We use this relation to eliminate h (we could eliminate r, but it's a little easier if we eliminate h, which appears in only one place in the above formula for cost). Thus we solve $h = V/(\pi r^2)$ and substitute in the cost function. As a result, we consider the function:

$$f(r) = \pi r(V/\pi r^2) + N\pi r^2 = \frac{V}{r} + N\pi r^2.$$

(What happened to the c and the 2 in the above formula? For convenience, we can factor out the constant $2c$ and drop it from the expression to be minimized, as in the previous example.) We now set $0 = f'(r) = -V/r^2 + 2N\pi r$, giving $r = \sqrt[3]{V/(2N\pi)}$. Next, we know that $h = V/\pi r^2$, and so the ratio h/r is equal to

$$\frac{h}{r} = \frac{V}{\pi r^3} = \frac{V}{\pi\big(V/(2N\pi)\big)} = 2N,$$

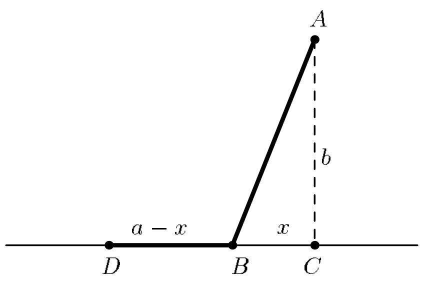
i.e., we choose the height to be $2N$ times the base radius for the most economical container.

EXAMPLE 14.9. Suppose you want to reach a point A that is located across the sand from a nearby road. Suppose that the road is straight, and b is the distance from A to the closest point C on the road. Let v be your speed on the road, and let w, which is less than v, be your speed on the sand. Right now you are at the point D, which is a distance a from C. At what point B should you turn off the road and head across the sand in order to minimize your travel time to A?

Let x be the distance short of C where you turn off, i.e., the distance from B to C. We want to minimize the total travel time. When you're traveling at constant velocity, you have: time=distance/velocity.

In this problem (see the diagram at the right), we travel the distance $\overline{DB}$ at speed v, and then the distance $\overline{BA}$ at speed w. Since $\overline{DB} = a - x$ and, by the Pythagorean theorem, $\overline{BA} = \sqrt{x^2 + b^2}$, we want to minimize the time-of-travel function

$$f(x) = \frac{a-x}{v} + \frac{\sqrt{x^2+b^2}}{w}.$$



We set $f'(x) = 0$ and next simplify by moving $1/v$ to the left and cross-multiplying:

$$0 = f'(x) = -\frac{1}{v} + \frac{x}{w\sqrt{x^2+b^2}}, \qquad \text{so that} \qquad w\sqrt{x^2+b^2} = vx.$$

We now square both sides and solve for x, obtaining

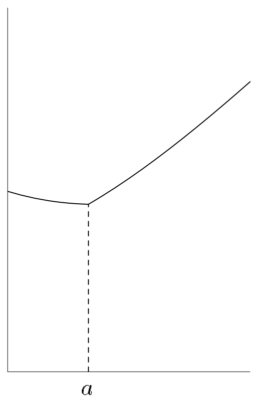
$$w^2b^2 = (v^2 - w^2)x^2, \qquad \text{i.e.,} \qquad x = \frac{wb}{\sqrt{v^2 - w^2}}.$$

This is our solution.

Notice, by the way, that our answer turns out not to depend on a; that is, the point where you should turn off the road does not depend on how far back you started. But something's not completely right here. What if a is very small, i.e., suppose $a < wb/\sqrt{v^2 - w^2}$. Then our solution is saying that we should go *back* along the road, and then head across the sand. This makes no sense. What have we done wrong in the case of small a? The problem is with the first term $(a - x)/v$ in our function $f(x)$. If $a < x$ that gives us negative time. What we really should have is $|a - x|/v$. In the case when $a < wb/\sqrt{v^2 - w^2}$, the graph of our revised function

$$f(x) = \frac{|a - x|}{v} + \frac{\sqrt{x^2 + b^2}}{w}$$

is given at the right. The minimum of this function cannot be obtained by setting $f'(x) = 0$ (in fact, the derivative of this function is never 0). Rather, it occurs at a point where there is no derivative — namely, at the "corner" at $x = a$. In practical terms, this says that if $a < wb/\sqrt{v^2 - w^2}$, then what you should do is head out immediately across the sand in a straight line to A.



SUMMARY — STEPS IN A MAX/MIN STORY PROBLEM:

1. Decide what the variables are and what the constants are, draw a diagram if appropriate, understand clearly what it is that is to be maximized or minimized.

2. Write a formula for what you are maximizing or minimizing.

3. Express that formula in terms of only one variable, that is, in the form $f(x)$.

4. Set $f'(x) = 0$ and solve.

5. Be sure that your answer makes sense, and that the max/min value is not really somewhere else.

HOMEWORK 14.

Part A. Basic problems:

1. Let $f(x) = \begin{cases} 1 + 4x - x^2 & \text{for } x \leq 3 \\ \dfrac{x+5}{2} & \text{for } x > 3 \end{cases}$

(a) Find the maximum value of $f(x)$ for x between 0 and 4.
(b) Find the minimum value of $f(x)$ for x between 0 and 4.
(c) Graph $f(x)$ to check your answers.

2. Find the dimensions of the rectangle of largest area having fixed perimeter P. (Compare with Example 14.4 above.)

3. A box with square base and no top is to hold a volume V. Find (in terms of V) the dimensions of the box that requires the least material for the five sides. Also find the ratio of height to side of the base. (This ratio will not involve V.)

4. You have l feet of fence to make a rectangular play area alongside the wall of your house. The wall of the house bounds one side. What is the largest size possible (in ft^2) for the play area?

5. Marketing tells you that if you set the price of an item at \$10 then you will be unable to sell it, but that you can sell 500 items for each dollar below \$10 you set the price. Suppose your fixed costs total \$3000, and your marginal cost is \$2 per item. What is the most profit you can make?

6. Find the area of the largest rectangle that fits inside a semicircle of radius r (one side of the rectangle is along the diameter of the semicircle).

7. For a cylinder with given surface area S, find the ratio of height to base radius that maximizes the volume.

8. You want to make cylindrical containers of a given volume V using the least amount of construction material. The lateral side is made from a rectangular piece of material, and this can be done with no material wasted. However, the top and bottom are cut from squares of side $2r$, so that $2(2r)^2 = 8r^2$ of material is needed (rather than $2\pi r^2$, which is the total area of the top and bottom). Find the optimal ratio of height to radius.

9. Given a right circular cone, you put an upside-down cone inside it so that its vertex is at the center of the base of the larger cone and its base is parallel to the base of the larger cone. If you choose the upside-down cone to have the largest possible volume, what fraction of the volume of the larger cone does it occupy? (Let H and R be the height and base radius of the larger cone, and let h and r be the height and base radius of the smaller cone. Hint: Use similar triangles to get an equation relating h and r.)

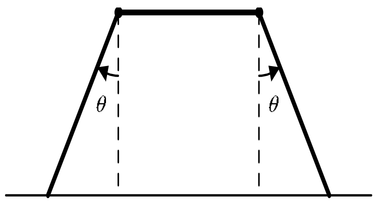
10. In Example 14.9, what happens if $w > v$ (i.e., your speed on sand is greater than your speed on the road)?

11. A container holding a fixed volume is being made in the shape of a cylinder with a hemispherical bottom. (The hemispherical bottom has the same radius as the cylinder.) Find the ratio of height to radius of the cylinder which minimizes the cost of the container if (a) the cost per unit area of the bottom is twice as great as the cost per unit area of the side, and the container is made with no top; (b) the same as in (a), except that the container is made with a circular top, for which the cost per unit area is 1 1/2 times the cost per unit area of the side.

12. (a) A square piece of cardboard of side a is used to make an open-top box by cutting out a small square from each corner and bending up the sides. How large a square should be cut from each corner in order that the box have maximum volume? (b) What if the piece of cardboard used to make the box is a rectangle of sides a and b? Note that the side x of the square to be cut from each corner must be less than $a/2$ and $b/2$, and in this way you can eliminate one of the two solutions of $f'(x) = 0$. In addition, use the second derivative test to be sure your value of x really gives the maximum volume.

13. A window consists of a rectangular piece of clear glass with a semicircular piece of colored glass on top. Suppose that the colored glass transmits only k times as much light per unit area as the clear glass (k is some fraction). If the distance from top to bottom (across both the rectangle and the semicircle) is fixed, find (in terms of k) the ratio of vertical side to horizontal side of the rectangle for which the window lets through the most light. For what range of values of k does this question make sense?

14. In the same situation as Problem 4, suppose that instead of a rectangular shape you decide to make the play area in the form of a trapezoid, using three sections of fence of length $l/3$ each (see the diagram at the right). Find the angle θ that the sides should be turned outward in order for the enclosed area to be maximal, and find the maximum play area of this form. Is the maximum play area greater or less than in Problem 4?



Part B. Additional Practice (do these as soon as possible, certainly before the final exam):

15. A terrorist sees a plane flying away from him at 500 mph at a height of 2 miles. At time $t = 0$ it is flying over a point on the ground which is 1.5 miles from him. At that instant he shoots a missile at the plane, but because of nervousness and perhaps a subconscious sense of morality, he mistakenly aims directly at the present location of the plane, rather than slightly ahead. For this reason he misses the plane. The purpose of this problem is to determine by how much. Suppose the missile travels at 2000 mph, and **neglect both air resistance and gravity**. First find parametric equations for both the plane and the missile. Then find the time at which the two objects are closest. (Hint: Use the distance formula, and save yourself unnecessary algebra by finding the time at which the *square* of the distance is minimal.) By how much does the missile miss the plane?

16. You are designing a poster to contain a fixed amount A of printing (measured in square inches) and have margins of a inches at the top and bottom and b inches at the sides. Find the ratio of vertical dimension to horizontal dimension of the printing on the poster if you want to minimize the amount of posterboard needed.

17. The strength of a rectangular beam is proportional to the product of its width times the square of its depth. Find the dimensions of the strongest beam that can be cut from a cylindrical log of radius r.

18. What fraction of the volume of a sphere is taken up by the largest cylinder that can be fit inside the sphere?

19. The U.S. post office will accept a box for shipment only if the sum of the length and girth (distance around) is at most 108 in. Find the dimensions of the largest acceptable box with square front and back.

20. Find the dimensions of the lightest cylindrical can containing 0.25 liter (=250 cm^3) if the top and bottom are made of a material that is twice as heavy (per unit area) as the material used for the side.

21. A conical paper cup is to hold a fixed volume of water. Find the ratio of height to base radius of the cone which minimizes the amount of paper needed to make the cup. Use the formula $\pi r\sqrt{r^2+h^2}$ for the area of the side of a cone.

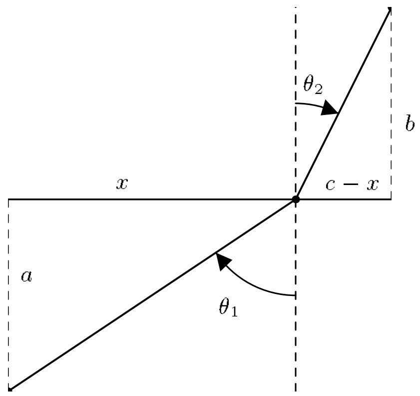
22. If you fit the cone with the largest possible surface area (lateral area plus area of base) into a sphere, what percent of the volume of the sphere is occupied by the cone? Compare your answer with the answer to Example 14.7.

23. Two electrical charges, one a positive charge A of magnitude a and the other a negative charge B of magnitude b, are located a distance c apart. A positively charged particle P is situated on the line between A and B. Find where P should be put so that the pull away from A towards B is minimal. Here assume that the force from each charge is proportional to the strength of the source and inversely proportional to the square of the distance from the source.

24. Find the fraction of the area of a triangle that is occupied by the largest rectangle that can be drawn in the triangle (with one of its sides along a side of the triangle). Show that this fraction does not depend on the dimensions of the given triangle.

25. How are your answers to Problem 5 affected if the cost per item for the x items, instead of being simply \$2, decreases below \$2 in proportion to x (because of economy of scale and volume discounts) by 1 cent for each 25 items produced.

26. You are standing near the side of a large wading pool of uniform depth when you see a child in trouble. You can run at a speed v_1 on land and at a slower speed v_2 in the water. Your perpendicular distance from the side of the pool is a, the child's perpendicular distance is b, and the distance along the side of the pool between the closest point to you and the closest point to the child is c (see the drawing). Without even stopping to do any calculus, you instinctively choose the quickest route (shown in the diagram at the right) and save the child. Our purpose is to derive a relation between the angle θ_1 your path makes with the perpendicular to the side of the pool when you're on land, and the angle θ_2 your path makes with the perpendicular when you're in the water. To do this, let x be the distance between the closest point to you at the side of the pool and the point where you enter the water. Write the total time you run (on land and in the water) in



terms of x (and also the constants a, b, c, v_1, v_2). Then set the derivative equal to zero. The result, called "Snell's law" or the "law of refraction," also governs the bending of light when it goes into water.

CHAPTER 15

Newton's Method

Suppose you have a function $f(x)$, and you want to find as accurately as possible where it crosses the x-axis; in other words, you want to solve $f(x) = 0$. Suppose you know of no way to find an exact solution by any algebraic procedure. Newton's method is a way to find a solution to the equation to as many decimal places as you want. It is what is called an "iterative procedure," meaning that it can be repeated again and again to get an answer of greater and greater accuracy. Iterative procedures like Newton's method are well suited to programming for a computer.

Here are the steps:

1. We start with a rough guess, which we call x_0.
2. Starting at the point x_0 on the x-axis, we go up to the curve, hitting it at the point $(x_0, f(x_0))$.
3. Now we slide down the **tangent line to the curve at** $(x_0, f(x_0))$, and take the point where this tangent line hits the x-axis as our improved estimate for the desired x-value. Call this value x_1. (See the diagram below.)

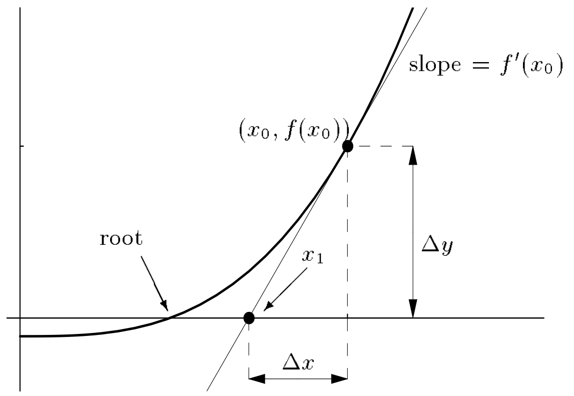
   The tangent line has slope $f'(x_0)$ and passes through the points $(x_0, f(x_0))$ and $(x_1, 0)$. This gives the equation:

$$f'(x_0) = \frac{f(x_0) - 0}{x_0 - x_1} \text{ or solving } x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

4. Replace x_0 by x_1 and repeat as often as needed. Thus we will produce a sequence of guesses $x_0, x_1, x_2, \ldots$ and hopefully these will get closer and closer to a root.

The main questions are how often do we need to do this and will it actually give the right answer. First note that if we land on a root, i.e. $f(x_0) = 0$, then $x_1 = x_0$ and thus, we can quit if we have $x_1 = x_0$. In general, it makes sense to keep track of the difference between x_1 and x_0 and quit when the difference gets smaller than the desired accuracy.

If our function has several roots and if we start close to a root, then the Newton's Method can lead us astray and take us to some other root. Usually this happens because the denominator in our formula $f'(x_0)$ is small and this causes x_1 to be far away from x_0. In an extreme case, we could have $f'(x_0) = 0$ and x_1 running off to infinity! It is also possible to create situations where $x_2 = x_0$ and so we get into an infinite loop! This is easy to avoid, since a slight change in x_0 will usually get us out of this situation. We will illustrate these peculiarities in the exercises below. In general, Newton's method is reliable when the derivative is well defined and does not become too small in the region of interest.



A less wordy description of the process is to use the formulas:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)},\ x_2 = x_1 - \frac{f(x_1)}{f'(x_1)},\ x_3 = x_2 - \frac{f(x_2)}{f'(x_2)},\ x_4 = x_3 - \frac{f(x_3)}{f'(x_3)},\quad \text{etc.}$$

How do we know when we are close enough? Two things happen: (1) when we plug our approximate value x_n into $f(x)$ we get something very near zero, and (2) our successive approximations x_{n-1} and x_n are very close together. For example, if we want three decimal places of accuracy, then once we have two successive approximations which agree out to three decimal places, we need go no further. In the homework and exam problems, you will usually be told how many iterations to perform: "one iteration" means finding x_1, "two iterations" means finding x_2, and so on. If, on the other hand, the directions specify the desired accuracy ("three decimal places," for example), then just keep iterating Newton's method until the only changes in your value from one iteration to the next occur beyond the decimal places needed.

Example 15.1. Find $\sqrt{102}$ without using the square-root function on a calculator. Use Newton's method with 3 iterations.

The first step is to set up the function we want to equal zero. We set $f(x) = x^2 - 102$, since $\sqrt{102}$ is a root of that function. Since we know $\sqrt{100} = 10$, we can make a rough guess of $x_0 = 10$. We are now ready to apply the formula for Newton's method. Here $f'(x)$ is, of course, $2x$. We obtain:

$$x_1 = 10 - \frac{f(10)}{f'(10)} = 10 - \frac{10^2 - 102}{2 \cdot 10} = 10.1.$$

Actually, x_1 is the same value we would have obtained using the tangent line approximation. But with the tangent line approximation, if our approximate value isn't good enough, there is nothing further we can do. However, with Newton's method we can improve this approximate value $10.1 \approx \sqrt{102}$. That is, we successively compute:

$$x_2 = 10.1 - \frac{f(10.1)}{f'(10.1)} = 10.1 - \frac{0.01}{20.2} = 10.09950495;$$

$$x_3 = 10.09950495 - \frac{f(10.09950495)}{f'(10.09950495)} = 10.09950495 - \frac{0.0000002}{20.199} = 10.09950494.$$

Thus, $\sqrt{102} = 10.09950494\cdots$. If we computed x_4, x_5, etc., we would get the same as x_3 out to 8 places past the decimal point.

STEPS IN A NEWTON METHOD STORY PROBLEM:

1. Decide what equation you want to solve, and what variable is playing the role of x.

2. Simplify algebraically if you can (dividing out constants, clearing denominators).

3. Bring everything to one side of the =, so you have $f(x) = 0$.

4. Apply Newton's method for the required number of iterations.

EXAMPLE 15.2. Suppose that the material you are using to manufacture marbles costs 1.0 cent/in^3, and the coating that is painted on the surface costs 0.15 cent/in^2. Find the radius of the marbles if you want to spend 0.3 cent per marble on the material and coating. Use Newton's method with one iteration, and take 0.3 inch as a rough guess for the radius.

We first write out the condition on the total cost, using the formulas $\frac{4}{3}\pi r^3$ for the volume and $4\pi r^2$ for the surface area of a marble: $1.0 \cdot \frac{4}{3}\pi r^3 + 0.15 \cdot 4\pi r^2 = 0.3$. In Newton's method we need our equation in the form $f(x) = 0$ (rather, $f(r) = 0$ in this problem), and this means bringing everything over to the left of the equal sign. In addition, for convenience we'll divide through by the coefficient of r^3, thereby obtaining: $f(r) = r^3 + 0.45r^2 - \frac{0.225}{\pi} = 0$. Notice that this is a third degree equation. A third degree equation can be solved algebraically, but the formula is much more complicated than the quadratic formula. If you want an approximate answer, then Newton's method is much faster. We use the Newton's method formula with x replaced by r, and with $r_0 = 0.3$. Here $f'(r) = 3r^2 + 0.9r$. We obtain:

$$r_1 = r_0 - \frac{f(r_0)}{f'(r_0)} = 0.3 - \frac{0.027 + 0.45 \cdot 0.09 - 0.07162}{3 \cdot 0.09 + 0.9 \cdot 0.3} = 0.3076 \text{ inch.}$$

EXAMPLE 15.3. Recall the example of the path of a baton's tip (Example 11.1). From the picture of the trajectory it's clear that the tip reaches its highest point at roughly $7/20 = 0.35$ sec. Find to several more decimal places the time when this peak is reached, and find what the maximum height is. (Note: In reality, of course, it is pointless to find something like this to such a great accuracy, since our model for the physical situation has some important sources of error — the neglect of air resistance, for example — and our measurement of the constants given in the problem was probably fairly inaccurate.)

In this example we want to find the maximum value of $y(t)$. We do this by setting the vertical component of velocity dy/dt equal to zero:

$$0 = \frac{d}{dt}\Big(-16t^2 + 14t + 5 + \cos(6\pi t)\Big) = -32t + 14 - 6\pi\ \sin(6\pi t).$$

We take $t_0 = 0.35$ as our first approximate value, and apply Newton's method with $f(t)$ taken to be the vertical velocity function $f(t) = -32t + 14 - 6\pi\ \sin(6\pi t)$. Then

$f'(t) = -32 - 36\pi^2 \cos(6\pi t)$ (this function is actually the vertical component of the acceleration of the tip of the baton). So Newton's method gives:

$$t_1 = t_0 - \frac{f(t_0)}{f'(t_0)} = 0.35 - \frac{-32 \cdot 0.35 + 14 - 6\pi\ \sin(6\pi \cdot 0.35)}{-32 - 36\pi^2 \cos(6\pi \cdot 0.35)} = 0.341823;$$

$$t_2 = t_1 - \frac{f(t_1)}{f'(t_1)} = \cdots = 0.341989; \qquad t_3 = \cdots = 0.341975; \qquad t_4 = \cdots = 0.341975.$$

So t_3 is accurate to 6 decimal places (the last digit could conceivably be off by 1 due to rounding). Thus, the baton's tip reaches its maximum height at $t = 0.341975$ sec. Finally, to find the maximum height reached, we substitute this value of t into the height function $y = -16t^2 + 14t + 5 + \cos(6\pi t)$. We obtain 8.90326 ft.

EXAMPLE 15.4. Recall Problem 15 of the max/min homework. Suppose that the pilot was daydreaming, did not notice the missile passing a half-mile away, and so the next day takes the plane again on the exact same route. The terrorist is waiting, and this time he decides to adjust the angle at which he shoots the missile (aiming at a point roughly a half-mile ahead of the plane). Again suppose that the missile travels at 2000 mph, and neglect both air resistance and gravity. Find the angle α above the horizontal at which the terrorist should fire the missile in order to hit the plane. Use Newton's method with three iterations. Also determine how the function in Newton's method would be affected if you took gravity into account in the motion of the missile.

To solve this, we first write the parameterized equations for the plane

$$x_{\text{plane}}(t) = 1.5 + 500t, \qquad y_{\text{plane}}(t) = 2,$$

and for the missile

$$x_{\text{missile}}(t) = (2000 \cos \alpha)t, \qquad y_{\text{missile}}(t) = (2000 \sin \alpha)t.$$

At the time t when the missile hits the plane, the coordinates of the missile must be equal to the corresponding coordinates of the plane. This means that in the horizontal direction we have: $1.5+500t = (2000 \cos \alpha)t$, i.e., $t = 1.5/(2000 \cos \alpha - 500)$; and in the vertical direction $2 = (2000 \sin \alpha)t$. We substitute the value for t (determined from the horizontal motion) into the last equation, clear denominators, and move everything to the left of the equal sign: $2(2000 \cos \alpha - 500) - (2000 \sin \alpha)1.5 = 0$. Dividing through by 1000 for convenience, we arrive at the equation: $f(\alpha) = 4 \cos \alpha - 1 - 3 \sin \alpha = 0$. Since the problem mentioned that a rough guess would be to aim the missile at the point $(2, 2)$ (a half-mile in front of the plane), which means an angle of $\pi/4$ radians, it makes sense to take $\alpha_0 = \pi/4$. Here we have $f'(\alpha) = -4 \sin \alpha - 3 \cos \alpha$, so Newton's method gives:

$$\alpha_1 = \frac{\pi}{4} - \frac{f(\pi/4)}{f'(\pi/4)} = 0.785398 - \frac{-0.29289}{-4.94975} = 0.726225;$$

$$\alpha_2 = \cdots = 0.725937; \qquad \alpha_3 = \cdots = 0.725937 \text{ rad} = 41.5931^\circ.$$

Finally, if we want to take gravity into account, we must add the term $-\frac{1}{2}gt^2$ to $y_{\text{missile}}(t)$, where the gravitational constant g is converted from 9.80665 m/sec^2 to 78980 mi/hr^2 (using the fact that 1 mi = 1609.3 m, and 1 hr =3600 sec). Since nothing changes in the horizontal direction, we still have $t = 1.5/(2000\cos\alpha - 500)$. As before, we equate the y-coordinates of the plane and the missile and clear denominators, which now means multiplying through by $(2000\cos\alpha - 500)^2$. We divide through by 1000000 (for convenience), and bring everything over to the left. The result is:

$$8\cos^2\alpha - 4\cos\alpha - 6\sin\alpha\cos\alpha + 1.5\sin\alpha + 0.58885 = 0.$$

This is what we use for our $f(\alpha)$ in Newton's method.

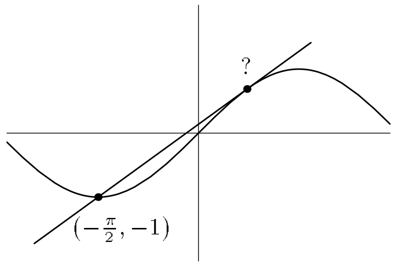
HOMEWORK 15.

1. Find the fifth root of 7, using $x_0 = 1.5$ as a first guess. Apply Newton's method with three iterations.

2. A rectangular piece of cardboard of dimensions 8×17 is used to make an open-top box by cutting out a small square of side x from each corner and bending up the sides. (See Problem 12 of the last homework set.) If $x = 2$, then the volume of the box is $2 \cdot 4 \cdot 13 = 104$. Use Newton's method to find the value of x for which the box has volume 100. Do your computations using at least 3 significant figures of accuracy, and apply Newton's method with enough iterations so that you know that your value of x is accurate to 3 significant figures.

3. A wheel of radius 1 picks up a pebble in its tread. How much farther will the wheel rotate (in radians) before the pebble has traveled a horizontal distance of 1? Use the cycloid formula, and take $\theta = 2$ radians as a rough guess. Use Newton's method with three iterations.

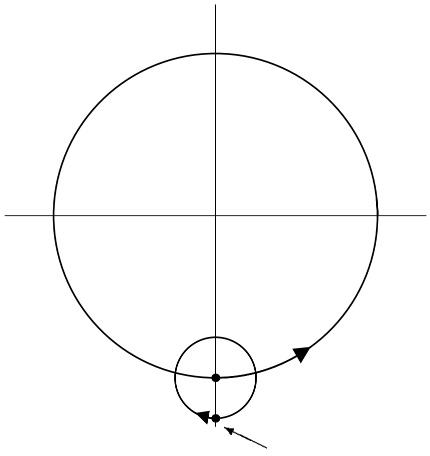
4. Consider a vertical line passing through the point $(-\pi/2, -1)$ on the curve $y = \sin x$. This line is rotated clockwise until it rests against the sine curve at a point somewhere between $(0.7, \sin 0.7)$ and $(0.8, \sin 0.8)$. (See figure at right.) Use Newton's method to find the point where the line rests, with enough iterations to be sure that your answer is accurate to 4 significant figures.



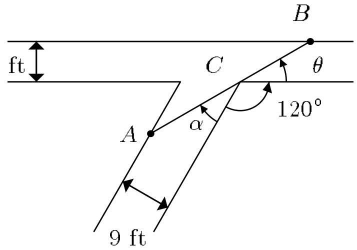
5. In the baton example, notice that the tip of the baton makes a downward loop at around $t = 0.5$ sec. Find the time (to four decimal places) when the tip of the baton is at the bottom of the loop, and find its height at that moment.

6. In the baton example, suppose that there is a wall located at exactly $x = 10$ ft. Use Newton's method to find (to three decimal places) when the tip of the baton hits the wall.

7. You are riding a double ferris wheel at an amusement park. The large circle has radius 40 ft, and the smaller circle has radius 10 ft. (See the diagram below.) Your seat rotates clockwise around the smaller circle once every 6 sec, and the center of the small circle rotates counterclockwise around the big circle once every 20 sec. At time $t = 0$ you are at the lowest possible point (50 ft below the center of the large circle). Call that point the origin. First write parametric equations for your position at time t. Then use Newton's method to find (to at least three places past the decimal point) the instant when you reach the highest point during the first revolution of the large circle. Also find your height at that moment.



8. Two mine shafts meet at an angle of 120°, as shown above. The horizontal shaft has a width of 7 ft, while the entrance shaft is 9 ft wide. What is the longest ladder that can negotiate the turn? Neglect the thickness of the ladder, and assume it is not tipped as it is maneuvered around the corner. At a given moment the ladder makes an angle of θ as shown. First convince yourself that the problem can be rephrased as follows: Find the angle θ at which the length $l = \overline{AB}$ is minimal, and then find the corresponding value of l. To do this, first express $l = \overline{AC} + \overline{CB}$ in terms of θ (notice that α can be expressed in terms of θ). Then set $dl/d\theta = 0$, and use Newton's method with $\theta_0 = 30°$ to find θ and l to at least 3 significant figures of accuracy.



CHAPTER 16

Indefinite Integrals

By an **antiderivative** of a function $f(x)$ we mean a function $F(x)$ whose derivative is $f(x)$. The reason for the term is that we're going in the opposite direction from taking the derivative. For example, an antiderivative of the function $f(x) = x^2$ is $F(x) = \frac{1}{3}x^3$, since the function $F(x) = \frac{1}{3}x^3$ has derivative $F'(x) = x^2$. But note that $\frac{1}{3}x^3 + 5$ is also an antiderivative of x^2, and in fact so is $\frac{1}{3}x^3 + C$ for any constant C. This is because a constant has derivative zero, and so we can add it to our function $F(x)$ without affecting the derivative $F'(x)$.

Notice that a constant C is the *only* thing that can be added to $F(x)$ without affecting the derivative. If it were possible to add some other function $G(x)$, then we would have to have $F'(x) + G'(x) = F'(x)$, i.e., $G'(x) = 0$. But any function that is not constant must either increase or decrease for certain values of x, and at such x it has a nonzero derivative. To put this another way, the *only* function $G(x)$ whose graph is always pointing horizontally is a constant function. **(There is a subtle point involved here. If you consider any interval contained in the domain of the function $f(x)$, then $G'(x)$ exists and is zero on such an interval, so we can argue that $G(x)$ is a constant. However, if the domain of $f(x)$ is split into disjoint intervals, then the function $G(x)$ can take on distinct constant values on them. Thus it becomes a step function rather than a constant. Usually, we have a single interval in mind and this point is not stressed at all.)**

Thus, if we find one antiderivative $F(x)$ for $f(x)$, then we can write the most general antiderivative as $F(x) + C$, where C is an arbitrary constant. (As explained above, this is valid on some interval in the domain of $f(x)$.)

The notation for the general antiderivative of $f(x)$ is $\int f(x)\,dx$, where the *integral sign* $\int$ says "take the antiderivative," and the "dx" indicates what the variable is. Thus, we write

$$\int x^2\,dx = \frac{1}{3}x^3 + C.$$

The expression $\int f(x)\,dx$ is said to be an **indefinite integral** of the function $f(x)$. The term "indefinite" is used because of the constant C that always appears in the answer to an antiderivative problem. C is called the "constant of integration." indexconstant of integration The function $f(x)$ that is being integrated is called the "integrand."

Another way to state an antiderivative problem is to say that we want to find a function $y = F(x)$ that satisfies the equation $y' = \frac{dy}{dx} = f(x)$. Such an equation $y' = f(x)$, in which y stands for an *unknown function*, is called a **differential equation.**

In general, *a differential equation is an equation for an unknown function in which the derivative (and maybe the second or higher derivatives) occur.* Differential equations will be treated at greater length in higher courses.

The above example $f(x) = x^2$ suggests how we can take the antiderivative of any power function x^n. Just increase the power by 1, and divide by $n+1$ in order to cancel the factor of $n+1$ that arises when taking the derivative of x^{n+1}. In other words,

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, \qquad \text{for any } n \text{ except } -1$$

(the formula makes no sense if $n = -1$; the antiderivative of the function x^{-1} is a special case, and it will be dealt with in the exercises below. This formula can be regarded as the reverse of the nx^{n-1} rule for taking derivatives. As in that rule, n is allowed to be negative or fractional.

The same formula can be used with other names for the x-variable, for example,

$$\int t^n\,dt = \frac{1}{n+1}t^{n+1} + C; \qquad \int u^n\,du = \frac{1}{n+1}u^{n+1} + C; \qquad \text{etc.}$$

antiderivatives, like derivatives, are **linear**. This means that if we know the antiderivatives of a bunch of functions, then we can compute the antiderivative of any linear combination of those functions. For example, using the above formula for $\int x^n dx$, we can find the antiderivative of any polynomial.

EXAMPLE 16.1.

(a) $\int x^3 - 6x^2 + 3x + 4\,dx = \frac{1}{4}x^4 - 6\cdot\frac{1}{3}x^3 + 3\cdot\frac{1}{2}x^2 + 4x + C = \frac{1}{4}x^4 - 2x^3 + 1.5x^2 + 4x + C.$

(b) $\int\left(\sqrt{x} - \frac{1}{x^2}\right)dx = \int\left(x^{1/2} - x^{-2}\right)dx = \frac{1}{3/2}x^{3/2} - \frac{1}{-1}x^{-1} + C = \frac{2}{3}x^{3/2} + \frac{1}{x} + C.$

In practical problems, often the time t is playing the role of the x-variable, and the function $f(t)$ to be integrated is the rate of change of something. Just as the derivative procedure takes us from the distance function $s(t)$ to the velocity function $v(t)$ and from the velocity to the acceleration $a(t)$, the antiderivative procedure takes us from $a(t)$ to $v(t)$ and then to $s(t)$. Below we will give examples of this.

Being able to find antiderivatives is at least as important as being able to find derivatives, for the following reason. Very often in the real world we do not start out knowing a formula for something, but rather we know some information about how it is changing, i.e., about its derivative. For example, in physics we have Newton's law of motion that says that Force=Mass$\times$Acceleration. Usually the basic thing we know about is the forces. This immediately gives us the acceleration function $a(t)$. From that we want to "go backwards" (i.e., find antiderivatives) to get the velocity function $v(t)$, and then finally what we are really interested in, which is the location $s(t)$ of the object. A basic example of this is a falling body problem.

1. From Acceleration to Velocity to Distance; Falling Bodies

Earlier in the course, we encountered the falling body formula $s(t) = -\frac{1}{2}gt^2 + v_0t + s_0$ for the height of an object above the ground after t seconds. However, we did not give any explanation of how that formula arises — in fact, at first the whole formula (especially the $\frac{1}{2}$ at the beginning) looks "pulled out of a hat." Now we are ready to derive the formula starting from a basic principle of physics: *near the surface of the earth gravity causes a constant force of acceleration.* That "gravitational acceleration" g is 9.806194 m/sec^2 (we'll use simply 9.8 m/sec^2 or 32 ft/sec^2). More precisely, the gravitational acceleration is $-g$ if we take upward to be the positive direction; it is g if downward is taken to be the positive direction. We shall usually take upward to be the positive direction. If we are near the surface of some other planet, the same principle applies, but with a different value of g.

The basic principle can be restated as follows: if no force other than gravity is affecting the object's motion (in particular, we neglect air resistance), then the acceleration function $a(t)$ is the constant $-g$: $a(t) = -g$. Our object is to derive the formula for the height function $s(t)$. We do this in two steps:

Finding $v(t)$. Since $dv/dt = a(t)$, it follows that $v(t)$ is an antiderivative of $a(t) = -g$. But the antiderivative of the constant $-g$ is $-\int g\,dt = -gt + C$. Thus, $v(t) = \int a(t)\,dt = -\int g\,dt = -gt + C$. How do we find C? Here we have to suppose that we know "initial information," i.e. the value of $v(t)$ when $t = t_0$, which we call v_0. (Often, but not always, the initial time t_0 will be 0.) In order for $v(t_0)$ to equal v_0, we must have $v_0 = -g \cdot t_0 + C$, i.e., our constant of integration is $v_0 + gt_0$. In general, *you solve for the constant of integration using initial information.* Thus, $v(t) = -gt+v_0+gt_0 = -g(t-t_0)+v_0$. (In this simple situation, we could have obtained the same result by common sense: if something starts at value v_0 and decreases at *constant* rate g, then after $t - t_0$ seconds its value is $v_0 - g(t - t_0)$.) If the initial time is $t = 0$, as is often the case, then $v = -gt + v_0$.

Finding $s(t)$. Let us suppose that $t_0 = 0$, i.e., the starting time for the falling body is taken to be $t = 0$. Since $ds/dt = v(t)$, it follows that $s(t)$ is an antiderivative of $v(t) = -gt + v_0$, i.e.,

$$s(t) = \int v(t)\,dt = \int -gt + v_0\,dt = -g\frac{1}{2}t^2 + v_0 t + C_1 = -\frac{1}{2}gt^2 + v_0 t + C_1.$$

(We used C_1 rather than C for the constant of integration because it has no relation to the C in the last paragraph.) To find the constant of integration C_1 we must again have initial information. Suppose we know the height at time 0: $s(0) = s_0$. Substituting this value for $s(t)$ when $t = 0$ gives $s_0 = s(0) = -\frac{1}{2}g \cdot 0^2 + v_0 \cdot 0 + C_1 = C_1$, i.e., $C_1 = s_0$. We obtain our final formula: $s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0$. This completes the derivation of the falling body formula.

Having the derivative relations $ds/dt = v(t)$ and $dv/dt = a(t)$, together with a formula for $a(t)$, we can work back to find $s(t)$, provided we also have two pieces of initial information that allow us to find the two constants of integration that arise. We assumed that the two known values are the initial velocity and the initial height.

EXAMPLE 16.2.

Suppose that, starting at $t = 1$, your acceleration is given by the formula $a(t) = 1/t^3$. Your velocity is 2 units/min at that instant, and your location is 0. Find the distance function $s(t)$.

The first step is to find $v(t) = \int a(t)\,dt = \int t^{-3}dt = \frac{1}{-2}t^{-2} + C = -\frac{1}{2t^2} + C$. The initial information is that $v(1) = 2$. Hence, substituting $t = 1$ gives: $2 = v(1) = -1/2 + C$, and so $C = 2.5$. Thus $v(t) = -\frac{1}{2t^2} + 2.5$. The second step is to find

$$s(t) = \int v(t)\,dt = \int (-\frac{1}{2} \cdot t^{-2} + 2.5)dt = -\frac{1}{2} \cdot \frac{1}{-1}t^{-1} + 2.5t + C_1 = \frac{1}{2t} + 2.5t + C_1.$$

Now we use the initial information that $s(1) = 0$: $0 = s(1) = \frac{1}{2} + 2.5 \cdot 1 + C_1$. Solving for C_1, we obtain $C_1 = -3$. Thus, $s(t) = \frac{1}{2t} + 2.5t - 3$.

We can check this answer by verifying three things: (i) it has the right second derivative $\ddot{s} = 1/t^3$; (ii) its value when $t = 1$ is 0; and (iii) the value of its derivative when $t = 1$ is 2. These computations are easy to do, and they give us the assurance that we did not make any careless errors in the last paragraph.

2. antiderivative of sin$x$ and cos$x$

From the derivative formulas $\frac{d}{dx}\sin x = \cos x$ and $\frac{d}{dx}\cos x = -\sin x$ we obtain the corresponding antiderivative formulas:

$$\int \cos x\,dx = \sin x + C, \qquad \int \sin x\,dx = -\cos x + C.$$

Notice that the derivative of sin is cos, but its antiderivative is *minus* cos.

3. Substitution

In practical applications we rarely encounter the pure sine or cosine function. As we have seen, it is useful to be able to work with the more general sinusoidal function

$$f(t) = A\sin\left(\frac{2\pi}{B}(t - C)\right) + D,$$

where A, B, C, and D stand for various constants, called the amplitude, period, phase shift, and vertical shift, respectively. In order to find antiderivatives of functions of this type and many other functions, we describe a basic method for finding antiderivatives called the **method of substitution.**

In the method of substitution we choose a new variable, which will usually be denoted u, and set u equal to an "inside function" (as in the chain rule), so that the function we are integrating takes on a simpler appearance. Then **EVERYTHING** must be converted to the new variable u. Here "everything" includes the "dx" (or "dt"). We convert dx to du by taking the derivative du/dx. We illustrate by some examples. In each case the reason for having to use du/dx to convert dx to du will become apparent when we check our answer. **(Recall that the answer to an antiderivative problem can always be checked by taking the derivative and verifying that we get our original function back again. It is also strongly recommended that you do this whenever possible.)**

EXAMPLE 16.3. If a and b are constants, find formulas for:

(a) $\int (ax + b)^n\,dx$ (for any $n \neq -1$); (b) $\int \sin(ax + b)dx$; (c) $\int \cos(ax + b)dx$.

(a) We substitute $u = ax + b$. We also find du/dx, which in this case is simply the constant a. We then treat du and dx as if they were algebraic quantities, writing $\frac{du}{dx} = a \Longrightarrow du = a\,dx$, and so $dx = \frac{1}{a}\,du$. Thus, in our integral we change $ax + b$ to u and we change dx to $\frac{1}{a}\,du$, thereby obtaining an integral purely in terms of the new variable u:

$$\int (ax+b)^n dx = \int u^n \frac{1}{a}\,du = \frac{1}{a} \cdot \frac{1}{n+1} u^{n+1} + C = \frac{1}{a(n+1)}(ax+b)^{n+1} + C.$$

(Your final answer should be written in terms of x, not u.)

Explanation: This might be a good place to explain what the substitution means and why it works. To begin with, we have an expression $f(x)\,dx$ inside the integral sign and when we get through with the substitution we convert this to an expression $g(u)\,du$. Where should two such expressions be the same? The answer is that $f(x)$ must be equal to $g(u(x))\frac{du(x)}{dx}$.

First suppose that this is true and we find an antiderivative $G(u)$ for the function $g(u)$. Then we can see that $G(u(x))$ is an antiderivative for $f(x)$, since $\frac{d}{dx}(G(u(x)) = G'(u(x))u'(x) = g(u(x))\frac{du(x)}{dx} = f(x)$, so $G(u(x))$ is an antiderivative of $f(x)$ as needed. Clearly, the answer $G(u)$ that we want, is given by the symbol $\int G(u)\,du$ and this solves the problem $\int f(x)\,dx$. So we convert the equation

$$f(x) = g(u(x))\frac{du(x)}{dx} \text{ to the equation } f(x)dx = g(u(x))d(u(x)) \text{ or } g(u)du.$$

It is possible to give completely rigorous meaning to the symbols dx and du as abstract (universal) derivatives (called differentials) and make precise sense out of the equation $du = \frac{du}{dx}dx$, but we will not attempt to go into the details here. We are only stating that the symbol dx in $f(x)dx$, is more than just a whimsical notation!

It is instructive to check this answer by taking its derivative. We find ourselves using the chain rule — as always happens when checking an antiderivative that was found by a u-substitution:

$$\frac{d}{dx}\left(\frac{1}{a(n+1)}(ax+b)^{n+1}\right) = \frac{1}{a(n+1)}(n+1)(ax+b)^n \cdot a = (ax+b)^n.$$

Thus, the $\frac{1}{a}$ term that we obtained when converting dx to du is needed to cancel the factor of $a = du/dx$ that comes out from the chain rule.

This also explains the importance of the dx (or dt or du, depending on what the variable of integration is) inside the integral. Keeping track of that part of the integral in the course of a u-substitution will give you the right antiderivative. Do not get into the careless habit of omitting the dx (or dt or du) in an integral.

(b) Again we take $u = ax + b$, so that $dx = \frac{1}{a}du$, and our integral becomes

$$\int \sin(ax+b)\,dx = \int \sin u\,\frac{1}{a}du = \frac{1}{a}(-\cos u) + C = -\frac{1}{a}\cos(ax+b) + C.$$

You should check this formula by taking the derivative of $-\frac{1}{a}\cos(ax+b)$.

(c) This works just like part (b). The result is $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + C$.

EXAMPLE 16.4. Find:

$$\text{(a)} \int x\cos(x^2)\,dx; \qquad \text{(b)} \int \frac{\cos(\pi t)dt}{\sin^2(\pi t)}; \qquad \text{(c)} \int \frac{\sin\frac{\pi t}{10}\,dt}{\sqrt{1+\cos\frac{\pi t}{10}}}.$$

(Note that in parts (b)–(c) we have brought the dt into the numerator, instead of putting it at the far right of the integral.)

(a) Here we take u equal to the expression inside the cosine, namely, $u = x^2$. Then $du/dx = 2x$, and so $du = 2x\,dx$. Fortunately, the integral contains x as well as dx, and so, bringing the x and dx together, we can replace the product $x\,dx$ by $\frac{1}{2}du$:

$$\int x\cos(x^2)\,dx = \int(\cos u)\frac{1}{2}du = \frac{1}{2}\int\cos u\,du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin(x^2) + C.$$

Notice how, when we check by taking $\frac{d}{dx}\left(\frac{1}{2}\sin(x^2)\right)$, we get $x\cos(x^2)$ because of the chain rule.

(b) Here we can greatly simplify the denominator if we take $u = \sin(\pi t)$. Then $\frac{du}{dt} = \pi\cos(\pi t)$, and so $du = \pi\cos(\pi t)dt$, which — fortunately for us — happens to be what is in the numerator (except for the constant π, which does not cause us any problems). That is, we can replace $\cos(\pi t)dt$ by $\frac{1}{\pi}du$:

$$\int\frac{\cos(\pi t)\,dt}{\sin^2(\pi t)} = \int\frac{\frac{1}{\pi}du}{u^2} = \int\frac{1}{\pi}u^{-2}du = \frac{1}{\pi}\frac{1}{-1}u^{-1} + C = -\frac{1}{\pi u} + C = -\frac{1}{\pi\sin(\pi t)} + C.$$

(c) Here we can obtain a much simpler integrand by substituting u in place of the expression in the radical: $u = 1 + \cos\frac{\pi t}{10}$. Then $\frac{du}{dt} = 0 - (\sin\frac{\pi t}{10})(\frac{\pi}{10})$, and so $du = -\frac{\pi}{10}\sin\frac{\pi t}{10}\,dt$. So the numerator in the integral is $-\frac{10}{\pi}du$, and we obtain

$$\int\frac{-10/\pi}{\sqrt{u}}du = \int -\frac{10}{\pi}u^{-1/2}du = -\frac{10}{\pi}\frac{1}{1/2}u^{1/2} + C = -\frac{20}{\pi}\sqrt{1+\cos\frac{\pi t}{10}} + C.$$

HOMEWORK 16.

1. Find the anti-derivative of each of the following:

(a) $\frac{1}{x^4} + x\sqrt{x} + 5$; (b) $(1-t)^9$; (c) $(x^2+1)^2$;
(d) $1/\sqrt[3]{1-5t}$; (e) $\sin^3 x \cos x$; (f) $x\sqrt{100-x^2}$;
(g) $x^2/\sqrt{1-x^3}$; (h) $\sin\left(\frac{2\pi}{B}(t-C)\right) + D$; (i) $\cos(\frac{t}{10})\sqrt{1-\sin(\frac{t}{10})}$;
(j) $\cos(\pi t)\cos\big(\sin(\pi t)\big)$.

2. Suppose that the $\cos(\pi t)$ term were absent in the numerator of Example 16.4(b). Find the anti-derivative by substituting $u = \pi t$ and then using the formula for the derivative of cot.

3. Find the formula for $s(t)$ for a falling body if you know $v_0 = v(t_0)$ and $s_0 = s(t_0)$ at some time t_0 (not necessarily at time 0).

4. Find the distance function $s(t)$ if $a(t) = 1/t^4$ and both velocity and distance are zero at time $t = 1$.

5. Find the distance function $s(t)$ if the acceleration function is given by the following formula, and if the initial velocity v_0 and the initial distance s_0 at time $t = 0$ are known:

(a) $a(t) = kt$ (where k is a constant); (b) $a(t) = \sqrt{2t+1}$;
(c) $a(t) = \sin(5t)$; (d) $a(t) = \sin(0.1t)/\cos^3(0.1t)$.

CHAPTER 17

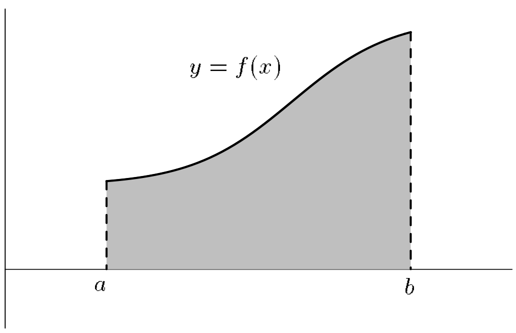
Definite Integrals: Area

A basic type of problem in calculus is to find the area under a curve. More precisely, we want to find the area bounded on the top by the graph of some function $y = f(x)$ with $f(x) \geq 0$, on the bottom by the x-axis, on the left by a vertical line $x = a$, and on the right by another vertical line $x = b$. We call this the "area under $f(x)$ between a and b." We also call it the **definite integral** of $f(x)$ from a to b.

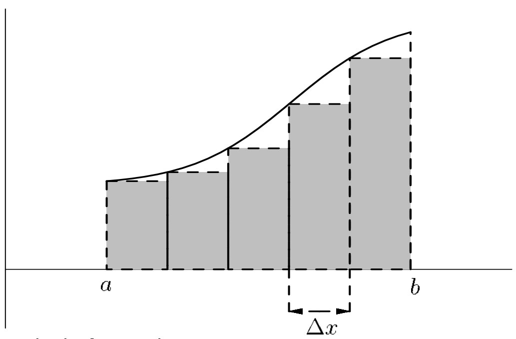
1. Riemann Sums

To understand the meaning of the definite integral it is instructive to discuss a method for obtaining an approximation to the true area area. This method is called the **Riemann sum approximation** (more precisely, the **n-th left endpoint Riemann sum approximation**, where n is some integer). The procedure is:

1. divide the interval from a to b into a certain number n of equal subintervals, each of length $\Delta x = (b - a)/n$;
2. draw a vertical line up to the curve at the endpoints of the subintervals;
3. over each subinterval form a rectangle by going up to the curve at the left endpoint and drawing a horizontal line to the right from that point on the curve;
4. take the sum of the areas of the n thin rectangles.



Area under the curve



5-th left endpoint Riemann sum

For example, if $n = 5$, so that $\Delta x = (b - a)/5$, then the heights of the 5 rectangles are:

$$f(a), \quad f(a + \Delta x), \quad f(a + 2\Delta x), \quad f(a + 3\Delta x), \text{ and} f(a + 4\Delta x)$$

(see the drawing above). All of the rectangles have the same horizontal dimension Δx. Hence, the total area, which is the 5-th left endpoint Riemann sum, is equal to

$$f(a)\,\Delta x + f(a + \Delta x)\,\Delta x + f(a + 2\Delta x)\,\Delta x + f(a + 3\Delta x)\,\Delta x + f(a + 4\Delta x)\,\Delta x\,.$$

In general, the n-th left endpoint Riemann sum, obtained from the rectangles formed over n small subintervals, is equal to

$$f(a)\,\Delta x + f(a+\Delta x)\,\Delta x + f(a+2\Delta x)\,\Delta x + \cdots + f(a+(n-1)\Delta x)\,\Delta x\,.$$

Using the summation symbol $\sum$, we can write this sum more compactly as follows:

$$(n\text{-th Riemann Sum}) = \sum_{i=0}^{n-1} f\left(a + i\,\Delta x\right)\,\Delta x\,,\ \text{where}\Delta x = \frac{(b-a)}{n}\,.$$

Here the summation notation tells us to take all integer values of i starting with $i = 0$ and ending with $i = n-1$, put each value in place of i in $f(a+i\Delta x)\,\Delta x$, and take the sum.

Example 17.1. Find the 5-th left endpoint Riemann sum for the definite integral of x^2 from 0 to 1, i.e., for the area under the parabola x^2 between 0 and 1.

We have $a = 0$, $b = 1$, $\Delta x = 0.2$, and the Riemann sum is

$$\sum_{i=0}^{4}(0.2i)^2 \cdot 0.2 = \left(0^2 + 0.2^2 + 0.4^2 + 0.6^2 + 0.8^2\right)0.2 = 0.24\,.$$

Later we will see that the actual area is a little larger, namely 1/3. Notice that rather than multiplying each term by 0.2 and then summing, it is faster to factor out the term 0.2, add the remaining terms and then multiply by 0.2.

In general, if $f(x)$ is an increasing function on the interval (a,b), then the left endpoint Riemann sum will be a little less than the true area; but if $f(x)$ is a decreasing function on (a,b), then the left endpoint Riemann sum will be a little greater than the true area.

The larger n is, the closer the Riemann sum will be to the true area. We say that the area is "the limit of the Riemann sums as n approaches ∞" and we write it in symbols this way:

$$\lim_{n\to\infty}\sum_{i=0}^{n-1} f(a+i\Delta x)\,\Delta x$$

In old books, the printed form of the letter "S" is "$\int$" and "infinitely small" values of Δx are written dx. Thus, a good symbol for the limit is

$$\int_a^b f(x)\,dx\,.$$

This number is called the "**definite integral** of $f(x)$ from a to b." The function $f(x)$ is called the **integrand**, a is called the **lower limit of integration** and b is called the **upper limit of integration**. The letter x is called the **variable of integration**.

In a definite integral it makes no difference what the variable of integration is called. In other words, the area under the function $y = f(z)$ between $z = a$ and $z = b$ means exactly the same thing as the area under the function $y = f(x)$ between $x = a$ and $x = b$. That is, both $\int_a^b f(z)dz$ and $\int_a^b f(x)dx$ are equal to the same thing because in a definite integral the variable of integration only holds a place, and does not affect the answer.

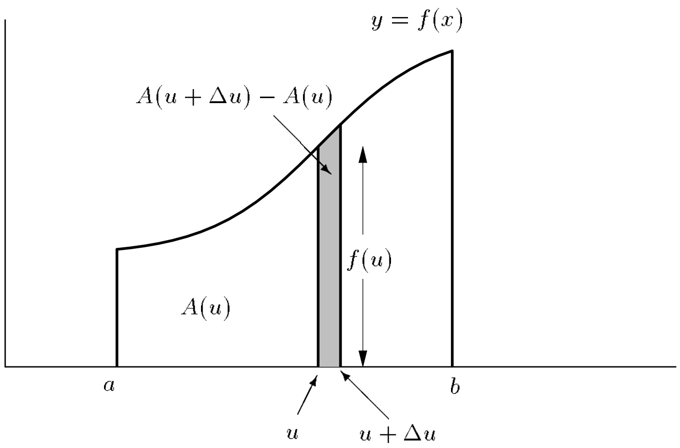
In the next section, we will discover that there is a relation between the definite integral and the **in**definite integral which you studied in the last chapter.

2. Computing the Area Under a Curve

We now describe how to find the exact area under $f(x)$ between a and b. To do this, we introduce an "area function" $A(u)$ as follows: for any u between a and b we set

$$A(u) = \text{area under } f(x) \text{ between } a \text{ and } u.$$

If we could find a formula for the function $A(u)$, then we would just substitute $u = b$



in order to obtain the area between a and b that we want. To find such a formula, we proceed indirectly. Instead of asking for $A(u)$ itself, we ask *how much the area function changes* when we go from u to $u+h$, and then we take the difference quotient $\big(A(u+h) - A(u)\big)/h$. By the definition of the area function,

$$\begin{aligned} A(u+h) - A(u) &= \text{area under } f(x) \text{ between } u \text{ and } u+h \\ &\approx \text{area of the rectangle above the interval } (u, u+h) \text{ of height } f(u) \\ &= h\, f(u). \end{aligned}$$

This approximate equality between the area under the curve and the area of the thin rectangle becomes very close to equality when h is small. Dividing through by h, we find that

$$\frac{A(u+h) - A(u)}{h} \approx f(u),$$

with the approximate equality becoming an equality in the limit as h goes to 0. That is, the limit of the difference quotient — the derivative of $A(u)$ — is equal to $f(u)$:

$$\frac{dA}{du} = f(u).$$

To see what this says, imagine that the area under the curve to the right of u is covered over by a card. The derivative of the area function $A(u)$ means the rate at which area is being uncovered as you move the card to the right. Clearly this depends on how wide the area is at the edge of the card — and this is precisely $f(u)$.

Once we know that dA/du is $f(u)$, this means that $A(u)$ is an antiderivative of $f(u)$. So suppose that we find the antiderivative $\int f(u)du = F(u) + C$, using the methods in the last section. Our function $A(u)$ must correspond to a certain value of C. Just as before, we need some initial information to determine this value. For the area function $A(u)$ the initial information is: when $u = a$ there is no area, i.e., $A(a) = 0$. Thus, $0 = A(a) = F(a) + C$, and hence $C = -F(a)$. We conclude that $A(u) = F(u) - F(a)$. This means that the total area under the curve between a and b is $A(b) = F(b) - F(a)$. In other words, *the area under $f(x)$ between a and b can be computed by finding an antiderivative of $f(x)$, substituting b and a in that antiderivative, and subtracting.* Using the definite integral notation, we conclude that

$$\text{Area under } f(x) \text{ between } a \text{ and } b = \int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a),$$

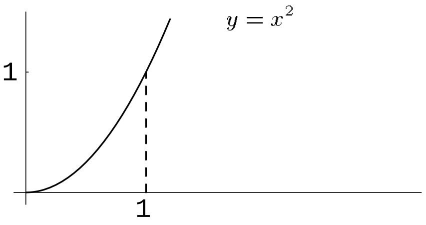
where $F(x)$ is an antiderivative of $f(x)$. Here we used the notation $F(x)\Big|_a^b$ to mean "substitute b and a into $F(x)$ and then subtract."

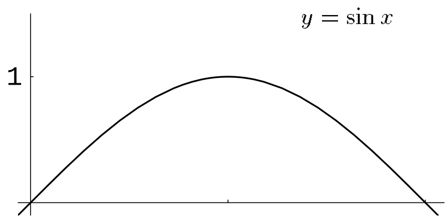
Note: You can tell whether an integral is a definite integral or an indefinite integral by whether or not there are limits of integration on the integral sign. *The answer to a definite integral problem is usually a specific value, not a function with an unknown constant C as in an indefinite integral problem. However, we will see many definite integrals with variable limits and these evaluate to functions of the limits, but without an arbitrary constant stepping in!*

EXAMPLE 17.2. Find the area under (a) $y = x^2$ between 0 and 1, and (b) one arch of the sine curve.

(a) $\int_0^1 x^2 dx = \frac{1}{3}x^3 \Big|_0^1 = \frac{1}{3}(1^3 - 0^3) = \frac{1}{3}$.

(b) An arch of the sine curve is the region under $y = \sin x$ between $x = 0$ and $x = \pi$, so its area is $\int_0^\pi \sin x\, dx = -\cos x \Big|_0^\pi = -\cos\pi - (-\cos 0) = -(-1) - (-1) = 2$.

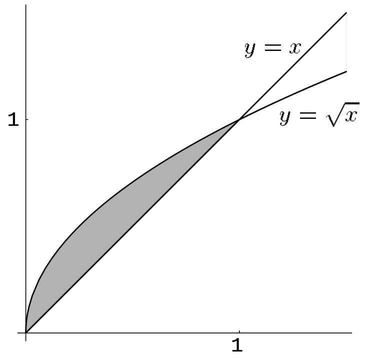




EXAMPLE 17.3. Find the area contained between the curve $y = \sqrt{x}$ and the line $y = x$.

The way to think of the area between two curves is: the area under the top curve between a and b minus the area under the bottom curve between a and b. Here a and b are the x-coordinates of the intersection points. In this case the square-root function and the line $y = x$ intersect at the origin and at the point (1,1). Between 0 and 1 the top curve is $\sqrt{x}$. So the area between them is

$$\int_0^1 \sqrt{x}\,dx - \int_0^1 x\,dx = \frac{2}{3}x^{3/2}\Big|_0^1 - \frac{1}{2}x^2\Big|_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}.$$



3. Substitution

In integrals evaluated using a u-substitution, as before you must change **EVERYTHING** from the old variable x (or t) to the new variable u. In the case of a definite integral, "everything" includes the limits of integration. You get the new limits of integration (the u-limits of integration) by asking the question: "as x goes from a to b the variable u goes from (what?) to (what?)." To answer this question, substitute a in place of x in the formula that gives u in terms of x; this gives the lower u-limit of integration. Then substitute b in place of x in the same formula to get the upper u-limit of integration. Here are some examples:

EXAMPLE 17.4. Find the area under (a) $y = \sqrt[3]{7x-6}$ between 1 and 2, and (b) $y = \sin\left(\frac{\pi x}{2}\right)\sqrt{1+\cos\left(\frac{\pi x}{2}\right)}$ between 0 and 1.

(a) In $\int_1^2 \sqrt[3]{7x-6}\,dx$ we make the substitution $u = 7x - 6$, so that $du = 7\,dx$ and $dx = \frac{1}{7}du$. This transforms the integral to

$$\int \sqrt[3]{u}\frac{1}{7}\,du = \frac{1}{7}\int u^{1/3}\,du.$$

What about the limits of integration? When x is equal to the lower limit of integration 1, we have $u = 7\cdot 1 - 6 = 1$; whereas when x is equal to the upper limit of integration 2, we have $u = 7\cdot 2 - 6 = 8$. Thus, our new limits of integration are 1 and 8. So the area is

$$\frac{1}{7}\int_1^8 u^{1/3}du = \frac{1}{7}\cdot\frac{3}{4}u^{4/3}\Big|_1^8 = \frac{3}{28}\left(8^{4/3} - 1^{4/3}\right) = \frac{3}{28}(16-1) = \frac{45}{28}.$$

(b) If we make the substitution $u = 1 + \cos\left(\frac{\pi x}{2}\right)$, so that $du = -\frac{\pi}{2}\sin\left(\frac{\pi x}{2}\right)\,dx$, we see that as x goes from 0 to 1 the variable u goes from 2 to 1. We obtain:

$$\begin{aligned}\int_0^1 \sin\left(\frac{\pi x}{2}\right)\sqrt{1+\cos\left(\frac{\pi x}{2}\right)}dx &= \int_2^1 \sqrt{u}\left(-\frac{2}{\pi}\right)\,du \\ &= -\frac{2}{\pi}\cdot\frac{2}{3}u^{3/2}\Big|_2^1 \\ &= -\frac{4}{3\pi}\left(1-2^{3/2}\right) = \frac{4}{3\pi}(2\sqrt{2}-1) = 0.776\,.\end{aligned}$$

4. Some Properties of Definite Integrals

1. Like indefinite integrals, they are linear. This means, for example, that any constant inside the integral can be pulled outside (as we did in Example 17.4(a) with the constant 1/7). Also, if you have the definite integral from a to b of $f(x)+g(x)$, it makes no difference whether you work with the sum all at once, or find the definite integral of $f(x)$ and $g(x)$ separately and then add the results. (The same observation applies to $f(x) - g(x)$.)
2. If c is a number between a and b, then the area between a and b is equal to the sum of the area between a and c and the area between c and b:
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\,.$$
Actually, when you use the information from the next item below, we can get that the above formula is valid even when c is not between a, b.
3. If the lower limit of integration is greater than the upper limit, i.e., if b is on the bottom and a is on the top (so you are traversing the area under the curve from right to left rather than from left to right), then the effect is to put in a negative sign. This is because $\int_b^a f(x)dx = F(a) - F(b) = -(F(b) - F(a))$. In Example 17.4(b), after making the u-substitution we obtained an integral of the form $\int_2^1$, which was negative. (We also had a negative constant in front of the integral, so that our final answer was positive.)
4. If the function $f(x)$ is negative, then $\int_a^b f(x)dx$ is negative (assuming that $b > a$), i.e., area below the x-axis is counted as negative area. For example, if we compute $\int_0^{2\pi}\sin x\,dx$ (see Example 17.2(b)), we obtain $-\cos x\,\Big|_0^{2\pi} = 0$ as our answer. This makes sense, because $\int_\pi^{2\pi}\sin x\,dx$ is an arch of the sine curve under the x-axis, and so has area -2. By property (2) above, we have $\int_0^{2\pi}\sin x\,dx = \int_0^{\pi}\sin x\,dx + \int_\pi^{2\pi}\sin x\,dx = 2 + (-2) = 0$, i.e., the two equal arches, one above and one below the x-axis, cancel.

5. Application to Degree-Days

The concept of "degree-days" arises when one wants to estimate heating costs. Suppose that in the winter you keep a building heated to a constant temperature of 65°F. Let $f(t)$ be the outside temperature. Then *the cost of heating the building between time $t = a$ and time $t = b$ is roughly proportional to the area between the line $y = 65$ and the temperature curve $y = f(t)$ from $t = a$ to $t = b$.* For example, if the temperature outside has the constant value $f(t) = D$, then the area between the two lines $y = 65$ and $y = D$ is simply $(65 - D)(b - a)$ (the units are units of temperature times time, i.e., degree-days). In that case all we're saying is that the heating cost is proportional to the temperature difference times the time interval. But when $f(t)$ is not constant we need to use definite integration to find this area.

EXAMPLE 17.5. Using the temperature function in Example 8.2 in the section on trig functions, find the number of degree-days in January in an average year in Seattle.

Our temperature function is $14 \sin\left(\frac{2\pi}{360}(t - 120)\right) + 51$. Subtracting this from 65 and simplifying, we obtain $65 - (14\sin(\cdots) + 51) = 14\Big(1 - \sin(\cdots)\Big)$. Our time interval is 0 to 30 (recall that we supposed that every month has 30 days). So the number of degree-days is

$$\begin{aligned} \int_0^{30} 14\left\{1 - \sin\left(\frac{2\pi}{360}(t - 120)\right)\right\} dt &= 14\left\{t - \left(\frac{360}{2\pi}\right)\left(-\cos\left(\frac{2\pi}{360}(t - 120)\right)\right)\right\}\Big|_0^{30} \\ &= 14\left\{30 + \frac{180}{\pi}\left(\cos\left(-\frac{1}{2}\pi\right) - \cos\left(-\frac{2}{3}\pi\right)\right)\right\} \\ &= 14\left\{30 + \frac{180}{\pi}(0 - (-0.5))\right\} = 821. \end{aligned}$$

Thus, based on this information, every January we can get a rough idea of whether to expect high or low heating bills by comparing the total number of degree-days measured in the particular month with the norm for January, which is 821.

6. Velocity and Distance Traveled As Area

In Chapter 16 we saw that the velocity is the antiderivative of the acceleration. Another way to say this is that *velocity is the area under the acceleration graph.* More precisely, the area under $a(t)$ between $t = t_1$ and $t = t_2$ is equal to the amount the velocity increases during this time interval. In mathematical notation:

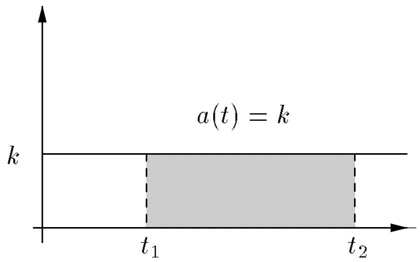
$$v(t_2) - v(t_1) = \int_{t_1}^{t_2} a(t)dt.$$

The graph at the left illustrates this in the special case when $a(t) = k$ is a constant function.

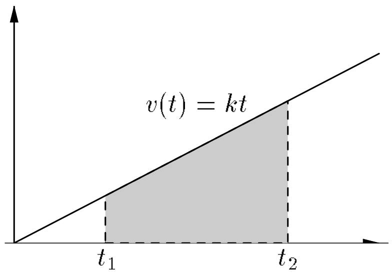
In Chapter 16 we also saw that the distance is the antiderivative of the velocity. In other words, *distance traveled is the area under the velocity graph.* More precisely, the area under $v(t)$ between $t = t_1$ and $t = t_2$ is equal to the distance traveled during this time interval:

$$s(t_2) - s(t_1) = \int_{t_1}^{t_2} v(t)dt.$$

The graph at the right illustrates this in the special case when $v(t) = kt$.



Shaded area is change in velocity during time interval.



Shaded area is distance traveled during time interval.

Note: By "distance traveled" we mean the *net* distance. In other words, the backward distance we travel when the velocity is negative (represented by area that is *beneath* the t-axis) counts with a negative sign, i.e., it cancels some of the positive distance we traveled earlier.

HOMEWORK 17.

1. In each case find the n-th left endpoint Riemann sum (for the indicated value of n), and compare with the true value of the area obtained by the anti-derivative method. In each case draw a picture showing the area in question and the rectangles in the Riemann sum, from which it should be clear why the true value is greater or less than the Riemann sum.

(a) $\int_0^2 (4-x^2)dx$, $n=4$; (b) $\int_0^2 (4-x^2)dx$, $n=8$; (c) $\int_0^4 \sqrt{x}dx$, $n=4$;
(d) $\int_0^{\pi/2} \cos x\, dx$, $n=3$; (e) $\int_0^{\pi/2} \cos x\, dx$, $n=6$.

2. Compute: (a) $\sum_{i=1}^5 i^2$; (b) $\sum_{i=0}^3 \frac{1}{1+i}$; (c) $\sum_{i=0}^3 \cos(i\frac{\pi}{6})$.

3. Find the area under
(a) x^2+x between 1 and 4; (b) $\sqrt{4-3x}$ between 0 and 1;
(c) $1/\sqrt{4-3x}$ between 0 and 1; (d) $1/(2x+1)^3$ between 0 and 2;
(e) $\cos(\pi x)$ between $-1/2$ and $1/2$.

4. Find the area contained
(a) between the curve $y=\sqrt{1-x}$ and the coordinate axes;
(b) under both $\sin x$ and $\cos x$ in the region from $x=0$ to $x=\pi/2$;
(c) between the curve $y=\cos(\frac{\pi}{2}x)$ and the curve $y=x^2-1$;
(d) between the line $y=2$ and the branch of the curve $y=\sec^2 x$ lying between the asymptotes at $\pm\pi/2$.

5. Evaluate the following definite integrals:
(a) $\int_1^3 \sqrt{4x-3}dx$; (b) $\int_{-1}^1 (x^2+1)^2dx$; (c) $\int_{-1}^1 (x^3+x)^3dx$;
(d) $\int_0^{0.5} \sin^3(\pi x)\cos(\pi x)dx$; (e) $\int_0^{\pi/4} \frac{\sin x\, dx}{\cos^5 x}$; (f) $\int_0^3 x\sqrt{9-x^2}dx$.

6. Find the normal number of degree-days in Fairbanks (a) during an entire year, (b) during the coldest 2-month period (i.e., the time period extending 30 days before and 30 days after the coldest day), and (c) during the warmest 2-month period. Assume that the inside temperature is maintained at 65°F, and use the formula for the normal mean daily temperature that is given in Problem 6 of the homework set on trig functions.

7. In each case, find formulas and draw graphs for $a(t)$ and $v(t)$, and find the net distance traveled during the given time interval. (a) The initial velocity is zero. The acceleration starts at 0, increases linearly to 4 units/sec^2 during the first two seconds, remains constant at 4 units/sec^2 for $2\le t\le 4$, and drops off linearly to 0 during the last 4-second period $4\le t\le 8$. (b) The initial velocity is zero. The accleration is the minimum of $\sin t$ and $1-t$ for $0\le t\le 1$. (Use Newton's method to find the point of intersection of these two functions, maintaining four significant figures of accuracy throughout the problem.)

CHAPTER 18

The Fundamental Theorem

The so-called "fundamental theorem of calculus" says that taking the derivative and taking the integral are inverse processes. More precisely, suppose you define a function using a definite integral as follows:

$$F(u) = \int_a^u f(x)\, dx\,,$$

where u is the variable and x is only a "dummy variable" (variable of integration). This $F(u)$ is simply the "area function" we defined before (except that we called it $A(u)$ rather than $F(u)$ in the last section). In the last section we showed that the derivative of this area function is $f(u)$. Thus, taking the derivative of F returns us to the original function f that we integrated to get F. In words, the fundamental theorem says that *if a function is defined by taking the definite integral from a fixed lower limit of integration up to a variable upper limit of integration, then its derivative is obtained by simply dropping the integral sign and taking the integrand with the dummy variable replaced by the variable in the upper limit of integration.* In symbols,

$$\frac{d}{du}F(u) = \frac{d}{du}\int_a^u f(x)dx = f(u).$$

EXAMPLE 18.1. Find d/du of the function $F(u)$ defined as

$$\text{(a)} \int_0^u x^2\, dx; \ \text{(b)} \int_1^u x^2\, dx; \ \text{(c)} \int_0^u \sqrt{1+\tan x}\, dx.$$

In parts (a)–(b), check your answer by finding a formula for $F(u)$ and computing the derivative of this formula.

(a) By the fundamental theorem, $\frac{d}{du}\int_0^u x^2 dx = u^2$. Let us check this by finding $F(u) = \int_0^u x^2 dx = \frac{1}{3}x^3 \Big|_0^u = \frac{1}{3}u^3$. Since $\frac{d}{du}\left(\frac{1}{3}u^3\right) = u^2$, the answer checks.

(b) By the fundamental theorem, $\frac{d}{du}\int_1^u x^2 dx = u^2$, the same answer as in part (a). But our function $F(u)$ is not quite the same. Namely,

$$F(u) = \int_1^u x^2\, dx = \frac{1}{3}x^3 \Big|_1^u = \frac{1}{3}u^2 - \frac{1}{3}\,.$$

Of course, this $F(u)$ still has derivative u^2. This example illustrates the fact that the derivative of a function of the form $F(u) = \int_a^u f(x)dx$ does not depend on what constant value we choose for the lower limit of integration a.

(c) By the fundamental theorem, $\frac{d}{du}\int_0^u \sqrt{1+\tan x}\, dx = \sqrt{1+\tan u}$. Here we cannot find a formula for $F(u)$, since we do not know how to find the anti-derivative of a complicated function like $\sqrt{1+\tan x}$. But thanks to the fundamental theorem, if all we want is the *derivative* of $F(u)$ (and not $F(u)$ itself), we never have to worry about anti-derivatives. The derivative of $F(u)$ can be determined right away without any work.

1. The Fundamental Theorem together with the Chain Rule

Suppose that u is a function of some other variable, like time, and we want to find $\frac{d}{dt}F(u(t)) = \frac{d}{dt}\int_a^{u(t)} f(x)dx$. This can be done using the chain rule:

$$\frac{dF}{dt} = \frac{dF}{du}\cdot\frac{du}{dt} = f(u(t))u'(t).$$

EXAMPLE 18.2. Suppose that in Example 1(c) we have $u = t^2$, and we want to find the time derivative of the integral. Then

$$\frac{d}{dt}\int_0^{t^2}\sqrt{1+\tan x}\,dx = \left(\frac{d}{du}\int_0^{u}\sqrt{1+\tan x}\,dx\right)\frac{du}{dt} = \sqrt{1+\tan u}\,2t = 2t\sqrt{1+\tan(t^2)}.$$

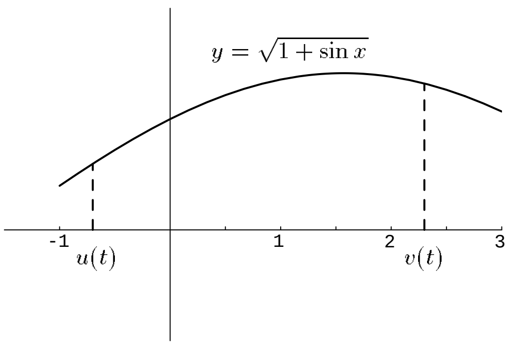
If the function $F(u(t))$ is defined by an integral in which $u(t)$ occurs in the *lower* limit of integration, then simply reverse the upper and lower limits, and put in a minus sign, i.e., $\int_u^a f(x)dx = -\int_a^u f(x)dx$ (see property (3) of definite integrals). What if a function appears in *both* limits of integration, i.e., what if you want to take d/dt of $\int_{u(t)}^{v(t)} f(x)dx$, where u and v are each functions of t? Then choose a constant c and use properties (2) and (3) of definite integrals with a replaced by u and b replaced by v:

$$\int_u^v f(x)\,dx = \int_c^v f(x)\,dx + \int_u^c f(x)\,dx = \int_c^v f(x)\,dx - \int_c^u f(x)\,dx\,.$$

We already know how to take the derivative of each of the two terms on the right. Thus,

$$\frac{d}{dt}\int_{u(t)}^{v(t)} f(x)dx = f(v(t))\frac{dv}{dt} - f(u(t))\frac{du}{dt}.$$

EXAMPLE 18.3. Suppose that the curve $\sqrt{1+\sin x}$ forms the top of a shape that is bounded below by the x axis and on the left and right by vertical lines whose position changes with time. Suppose that the vertical line on the left is moving steadily to the right at speed 0.1, starting at -1 at time $t = 0$. Meanwhile, the vertical line on the right is oscillating between 1 and 3 according to the formula $2+\sin(0.25\pi(t-6))$. Find a formula in terms of t for the rate of change of the area of the shape between the two vertical lines.



We have

$$
\begin{aligned}
&\frac{d}{dt}\int_{-1+0.1t}^{2+\sin(0.25\pi(t-6))}\sqrt{1+\sin x}\,dx \\
&\quad= \sqrt{1+\sin(2+\sin(0.25\pi(t-6)))}\left(\frac{d}{dt}(2+\sin(0.25\pi(t-6))\right) \\
&\qquad - \sqrt{1+\sin(-1+0.1t)}\left(\frac{d}{dt}(-1+0.1t)\right) \\
&\quad= 0.25\pi\cos(0.25\pi(t-6))\sqrt{1+\sin(2+\sin(0.25\pi(t-6)))} \\
&\qquad - 0.1\sqrt{1+\sin(-1+0.1t)}.
\end{aligned}
$$

HOMEWORK 18.

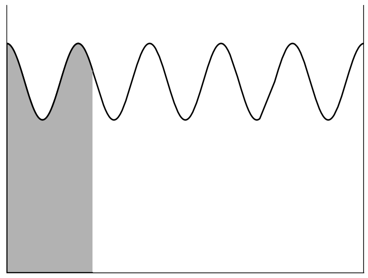
1. Find d/du of the following functions of u:

$$\text{(a)} \int_0^u \sin^2 x\,dx; \quad \text{(b)} \int_{-1}^u \sqrt{1+t+t^2}\,dt; \quad \text{(c)} \int_u^\pi \sqrt{1+\sin^2 y}\,dy;$$
$$\text{(d)} \int_{-u}^u \sin^2 x\,dx; \quad \text{(e)} \int_{-u}^u \sin^3 x\,dx.$$

2. Find d/dt of the following functions of t:

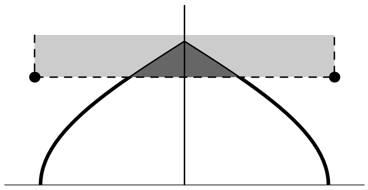
$$\text{(a)} \int_1^{a+bt} \frac{dx}{x}; \qquad \text{(b)} \int_0^{-\frac{1}{2}gt^2+v_0t+s_0} \sin^2 y\,dy;$$
$$\text{(c)} \int_{t^2}^{t^3} \cos(x^2)dx; \quad \text{(d)} \int_{a+bt}^{A\sin(\alpha(t-C))} \sqrt{1+u^2}du.$$

3. The stage of a puppet show has its upper edge defined by $y = 20 + 10\cos^2(0.1\pi x)$ (the units are cm). Its left side is the y-axis and its lower edge is the x-axis (see the drawing). After the show the curtain closes from left to right at 2 cm/sec. (a) Write a definite integral for the curtain's area as perceived by the audience after t seconds. (b) What is the rate of change of the curtain's area at time t?



4. The same as Problem 3, except that now the upper edge is defined by $y = 40/\left(1+(0.01x-1)^2\right)$.

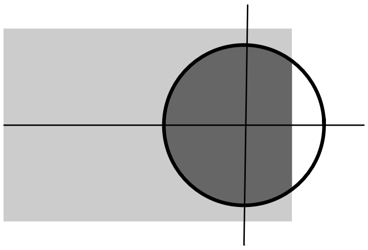
5. The opening to a stage has its highest point 48 feet above the stage, it is symmetrical about the center line (which we'll take as the y-axis), and it is bounded by the flat stage and the curve $x = \pm 48\cos(\pi y/96)$ (see the drawing). A curtain with flat bottom is lowered by attaching two heavy weights, one at each end, and letting them fall (with acceleration 32 ft/sec^2), bringing down the curtain.



(a) Write a definite integral for the amount of the stage blocked by the curtain after t seconds. (b) At time t, what is the rate of change of the area blocked by the curtain?

6. A flat vertical plate is moving from left to right (back and forth) across a water pipe 1 cm in radius.

(a) Write a definite integral for the area blocked off by the plate when it is u cm to the right of center (the area shaded in the drawing). (b) Suppose that u is a sinusoidal function of time. If u starts at -1 when $t = 0$ and sweeps back and forth once every 2 seconds, find a formula for $u(t)$. (c) Find a formula for the rate of increase of the area blocked (in cm^2/sec) at time t.



Lab Projects for Ma113

Lab projects are generally stated as a topic of investigation rather then as a specific exercise. You are required to perform a detailed analysis of the problem. This will include: setting up the problem, making a labeled diagram of the problem with information acquired, performing the necessary calculations and visually displaying the results (usually as graphs of functions). You must then organize this information into a full written report.

Lab projects are done by collaborative groups of three or four students who are expected to organize themselves, do the work, and prepare the report as a group. Each person in the group must sign the final report and be aware that in doing so he or she is guaranteeing that everyone who signed has done his or her fair share of the work. This does not mean that everyone did exactly the same number of tasks, spent the same amount of time, etc. It may well be the case that one member of the group provides the majority of the ideas, another provides the organization and direction, and another actually prepares the "write-up". It does mean that each person who signs the final report is personally satisfied that every author contributed in fair measure *and fully understands the report.*

The reports will be graded with each student in the group receiving the group grade. The grade may be determined either by the written report alone or in equal measure by an oral presentation by a randomly selected member of the group. Report grading will be on correctness, completeness, correct English, demonstration of understanding of the problem and solution, and style. Bonus points may be given for originality.

The following are general guidelines and expectations:

The Lab Session:

- Get started immediately. You will not be able to complete the project all in the last minute. Do not waste this valuable time.
- Read over the laboratory material carefully before you begin discussing or completing any part of it. This should be done before the lab sessions designated for this activity. Give some advanced thought to what you are going to do and the anticipated form of the report you are going to prepare.

- Initially you may have some trouble getting started. Don't panic, a discussion with your group partners will usually generate some ideas. Your TA can also give you a 'push' in the right direction. There will likely be a general discussion on the problem in lecture. Your TA may also be able to make suggestions as to how to organize your efforts.
- The generality of projects like these tend to make them quite different from the standard "exercises" of the homework. You will need to make assumptions to simplify the problem. Be sure to discuss such assumptions in detail in your final write-up.

- Plan on spending some time outside of lab on the report. In particular, plan on writing the final report outside of class.
- Your TA will be happy to go over complete drafts of your final report with you if they are ready at least two days before the submission deadline. He or she will then be able to point out areas of concern and make suggestions on improving the final submission.

The Lab Report: The report should be readable by anyone who can understand the mathematics with no reference to any other source It must at least have the following identifiable features:

1. **A Cover Page** signed by each member of the team which produced the report. A signature on a report means that each person signing attests to the fact that each contributed approximately equally to the work. Each member of the team is responsible for any errors; each member of the team may be asked to explain any detail of a report.
2. A **complete statement of the problem**
3. An **analysis** of the problem, complete with carefully drawn sketch which contains a complete explanation of the solution, derives and explains all equations and sets up all calculations that need to be done. This may be done as part of a Maple worksheet and should contain carefully annotated graphs of the pertinent functions. This is the most important part of the report.
4. **Details of the calculations** (this can be an annotated Maple worksheet. It must be clear to the reader what each calculation is intended to do. This can be done with comments.

   The report should be free of any extraneous calculations. Do not include any intermediate calculations unless you need to refer to them or you feel that their presence makes the report easier to read.

   Most of the graphs will be in this section. It should be clear to the reader why one doing a certain graph before it is actually done. This can be done with a comment. Features of importance on a graph should be labeled and interpreted either on the graph itself or immediately following it. For instance the axes of all graphs should be labeled and a caption provided for each graph which explains its purpose. This labeling can be done by hand on computer-generated drawings. The function(s) being graphed should be in the caption or penciled on the graph.

5. **Conclusions and interpretations** . At the very least this should briefly explain how what has been done answers the original question or solves the original problem. If appropriate, it should refer back to graphs and relate features of the graphs to numerical calculations. Finally, there should be a summary paragraph or sentence describing what has been accomplished and (possibly) suggesting directions for further study. problem(s).

A good report will not be very long - probably three to five typed pages, excluding graphs.

The report should be free of any extraneous calculations (e.g. material that should have been cut out of a worksheet.)

Finally, there should be a summary paragraph or sentence describing what has been accomplished and (possibly) suggesting directions for further study. problem(s).

Laboratory Calculus I: Lab Project

Bouncing Ball

Date: ______________________

Team Members:

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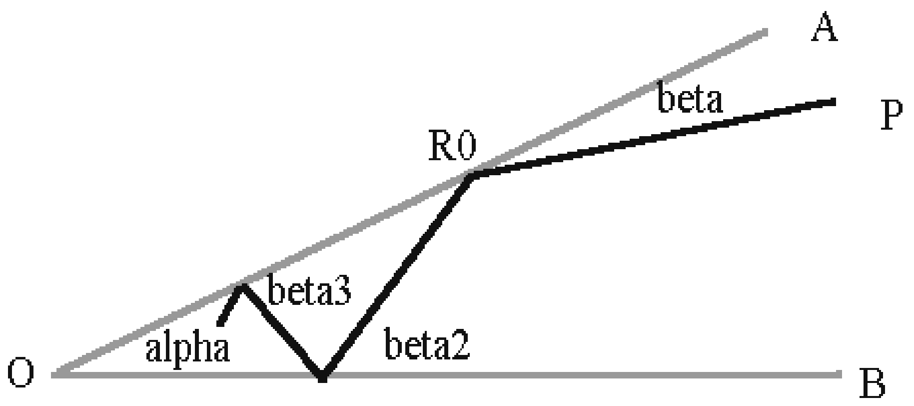

Introduction: The famous magician Michael Moshen has an act in which he stands in a large triangular sculpture and bounces balls off the sides, keeping them in motion. This is a look at the elementary mathematics of what he is doing.

The Context to be Investigated: Two plane surfaces, one parallel to the ground, meet at an angle $\theta = 30$ degrees. A ball is thrown against the upper plane at an angle $\beta = 5$ degrees with a velocity of 20 feet per second. It hits 10 feet from the corner and five feet above the ground. The ball bounces elastically, with no spin or air resistance but is acted upon by gravity.

Develop a mathematical model for the path the ball follows with and without the influence of gravity. First assume that there is no gravity and calculate the path of the ball as it bounces between the planes. Vary the initial velocity, angle of entry, and angle made by the two planes. Can the ball be made to bounce any number of times?

Repeat the investigation with the influence of gravity. Compare the paths of the ball with and without gravity for different initial velocities, angles of entry, angles between the planes, etc.

Finally, repeat the investigation for a ball that is not perfectly elastic. Suppose the ball always rebounds from a surface with the component of its velocity normal to the surface having only 75% of its magnitude prior to impact.

Laboratory Calculus I: Project

Forces on a Piston

Date: ______________________

Team Members:

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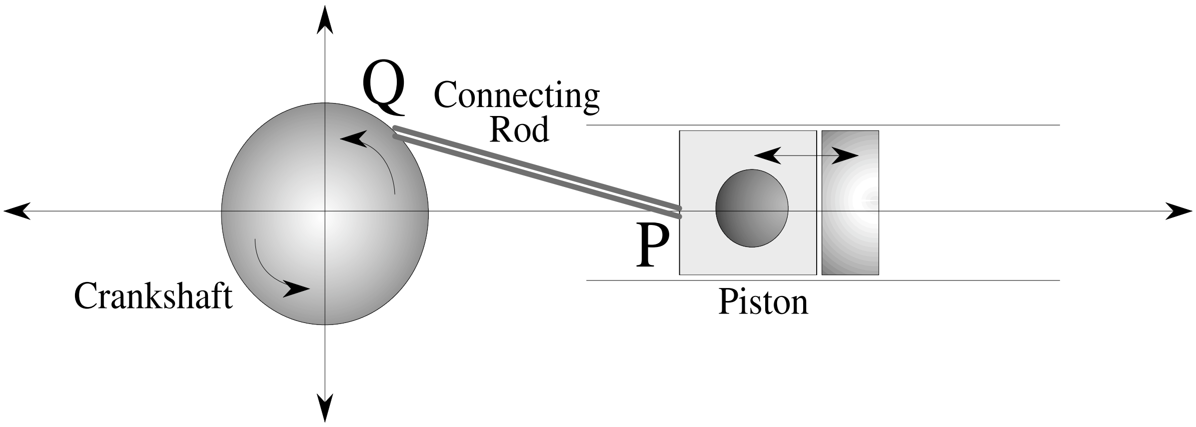
The problem: In an internal combustion engine, each piston is housed in a cylinder and is attached to the crankshaft by a connecting rod as indicated in the diagram below. The piston moves back and forth in the cylinder, and, in response the crankshaft rotates. When the crankshaft is turning, even at a constant angular velocity the (linear) velocity and speed of the piston are constantly changing. Investigate the effect of the length of the connecting rod on the velocity and acceleration of the piston.



1. Suppose the crank is 1.5 inches in radius and the connecting rod is 6 inches long and the crank is turning at a constant 1 revolution per second. Assume the piston is moving on the x-axis and express the position of the point at which the rod connects to the piston (the wrist pin) as a function of t (in seconds).

   Hint: Let r denote the crank radius, l the length of the connecting rod, $\theta(t)$ the crank angle at time t and assume $\theta(0) = 0$ and that the rotation is counterclockwise. We can place the center of the crank at the origin. Then if $x(t)$ is the position of the wrist pin at time t, the law of cosines says that

   $$l(t)^2 = r^2 + x(t)^2 - 1rx(t)\cos(\theta(t)).$$

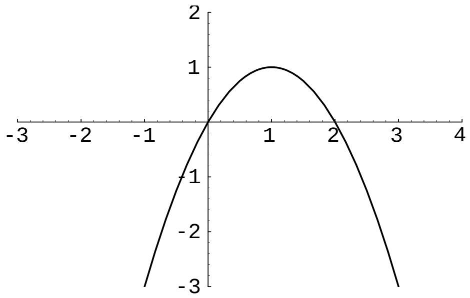
   Solve for $x(t)$ and choose one of the solutions (e.g. the one for which $x(t) = r + l$ when $t = 0$).
2. Calculate expressions for the position, velocity, and acceleration of the wrist pin as a function of time and plot all three on the same graph for t from 0 to 2 seconds. At what times are the velocity and acceleration maximal and minimal? What are the maximum velocity and acceleration?
3. Repeat the above calculations for various lengths of the connecting rod from 2 to 8 and observe the effect of these changes on the velocity and acceleration of the wrist pin. Note particularly the change in shape of the acceleration curve and formulate a conjecture as to length of the connecting rod for which marks

the transition from one shape to the other. Develop additional evidence for your conjecture through additional experiments.

Practice First Midterms

Practice First Midterm #1 (50 points in all, time = 1 hour)

1. (12 points) The graph of $y = f(x) = 2x - x^2$ is given at the right. Find the domain of each of the following functions. You do **not** need to use any algebra in this problem.



(a) $\sqrt{f(x)} = \sqrt{2x - x^2}$
(b) $\sqrt{f(-x)} = \sqrt{-2x - x^2}$ (c) $\frac{1}{f(x-2)}$
(d) $\frac{1}{\sqrt{f(x+1)}}$

2. (12 points) At time $t = 0$ an object starts from 0 moving to the right at 2 m/sec. Its velocity changes, as shown in the graph at the right. This graph shows the velocity with which the object is moving to the right as a function of time. Give the letters of **ALL** labeled points where

(a) the object is actually moving rightward;

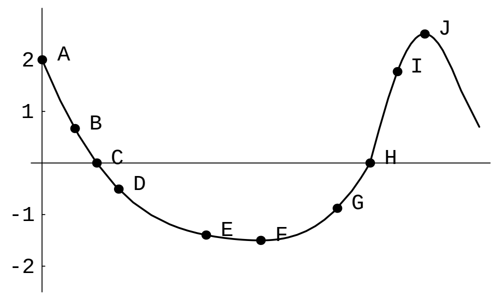
(b) the object is actually moving leftward;

(c) the object has positive (rightward) acceleration;

(d) the object has zero acceleration;

(e) the object's speed heading to the right is maximum;

(f) the object's speed heading to the left is maximum.



3. (13 points) A box with square top and bottom and rectangular sides has surface made up of two $s \times s$ sides and four $s \times t$ sides. Thus, if your box has dimensions 5 ft $\times$ 5 ft $\times$ 4 ft, then the total surface area A is 130 sq ft. If you want to increase s by 1 inch to 5 ft 1 in, how should t change so that the surface area stays the same? Use the tangent line approximation, and give your answer in feet.

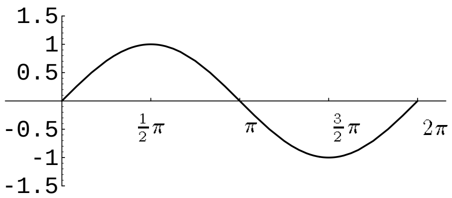
4. (13 points) A cubic container with side 9 cm holds $9^3 = 729$ cubic cm. Suppose you have a bottle containing 750 ml of liquor. (A milliliter (ml) is the same thing as a cubic cm.) Use the tangent line approximation to find the side of a cubic container needed to hold the 750 ml of liquor.

Practice First Midterm #2 (50 points in all, time = 1 hour)

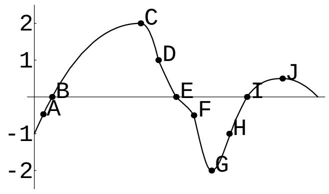
1. (8 points) Sketch the graph of the functions

$$\text{(a) } y = (x+2)^2 - 4, \qquad \text{(b) } y = -\sqrt{25 - (5x)^2}$$

2. (6 points) The graph of the function $g(x)$ on the interval $[0, 2\pi]$ is pictured below to the left. Find the domain of $y = f(x) = 1/\sqrt{g(x + (\pi/2))}$. (This function $g(x)$ is actually the sine function restricted to the interval $[0, 2\pi]$.)



Graph for Problem 2



Graph for Problem 3

3. (12 points) At time $t = 0$ an object starts from 0 moving to the left at 1 m/sec. Its velocity changes, as shown in the graph above at the right. This graph shows the velocity with which the object is moving to the right as a function of time. Give the letters of **ALL** labeled points where

(a) the object is actually moving rightward;
(b) the object is actually moving leftward;
(c) the object has positive (rightward) acceleration;
(d) the object has zero acceleration;
(e) the object's speed heading to the right is maximum;
(f) the object's speed heading to the left is maximum.

4. (12 points) From a point **exactly** 75 m above the ground you throw a stone upward at v_0 m/sec, and you time how long it takes to hit the ground. If we neglect air resistance, the height of the stone at time t is given by the falling body formula $s = -\frac{1}{2}gt^2 + v_0 t + s_0$, where $g = 9.8$ m/sec^2.

(a) If you time 5 sec for the stone to reach the ground, at what upward speed did you throw the stone?

(b) If your stopwatch is accurate only to ± 0.1 sec, how accurate is your answer to part (a)? Use the tangent line approximation.

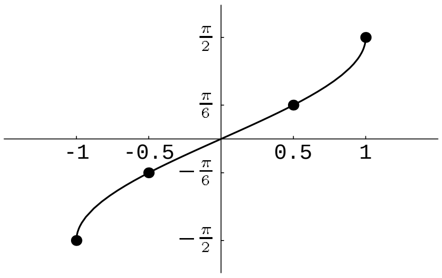
5. (12 points) You have a 1000-square-foot apartment which you're renting for the summer to someone from Europe, so you have to convert its area to square meters in order to explain its size. You know that there are about 10 square feet in one square

meter. More precisely, 1 square meter = 10.77 square feet. Use the tangent line approximation to find the area of your apartment in square meters. You do **NOT** need a calculator to do this problem correctly.

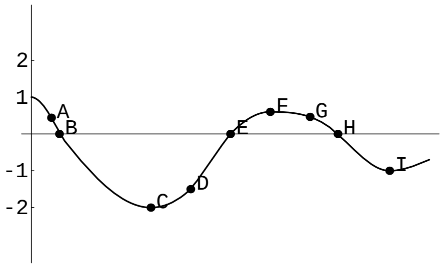
Practice First Midterm #3 (50 points in all, time = 1 hour)

1. (8 points) The graph of $y = f(x)$ is pictured below to the left. It is called the "arcsine" function, and its domain is the interval $[-1, 1]$. Find the domain of

$$y = \frac{1}{\sqrt{\frac{\pi}{6} + f(x-1)}}.$$



Graph for Problem 1



Graph for Problem 2

2. (12 points) At time $t = 0$ an object starts from 0 moving to the right at 1 m/sec. Its velocity changes, as shown in the graph above at the right. This graph shows the velocity with which the object is moving to the right as a function of time. Give the letters of **ALL** labeled points where

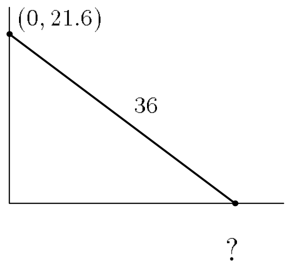
(a) the object is actually moving rightward;
(b) the object is actually moving leftward;
(c) the object has positive (rightward) acceleration;
(d) the object has zero acceleration;
(e) the object's speed heading to the left is maximum.

3. (15 points) In this problem use the formula $s = \frac{1}{2}gt^2$ for the distance an object falls in t seconds (with no air resistance). Upon arrival on a new airless planet, you decide to measure its gravitational acceleration g by dropping a stone from a height of **exactly** 11.25 meters and measuring the time t it takes to fall to the ground.

(a) If you time 3.00 sec for the stone to reach the ground, what is g?

(b) If you do not know the time precisely, but only know that it is between 3.00 and 3.10 sec, what is the range you know g to be in? Use the tangent line approximation.

4. (15 points) You lean a yardstick (exactly 36 inches long) against a wall, and measure the height up the wall to where the yardstick touches to be 21.6 inches $\pm$ 0.2 inches. Find the distance the other end of the yardstick is from the bottom of the wall (see the diagram to the right). Include the error, which you find using the tangent line approximation.



Practice First Midterm #4 (50 points in all — time = 1 hour)

1. (10 points) Let $f(x) = \sqrt{625 - x^2}$. Find the domain of each of the following functions:

$$\text{(a)} \ \frac{x}{5f(x+10)}, \qquad \text{(b)} \ \sqrt{f(x) - 24}.$$

2. (15 points) The downward velocity of a parachutist measured in m/sec is given by the graph at the bottom of the page on the left.

(a) Sketch the graph of downward acceleration.

(b) At what instant is the downward velocity the greatest? What (in practical terms) is happening at that instant? At what instant does the acceleration have the greatest positive value? What is happening at that instant? At what instant does the acceleration have the greatest negative value?

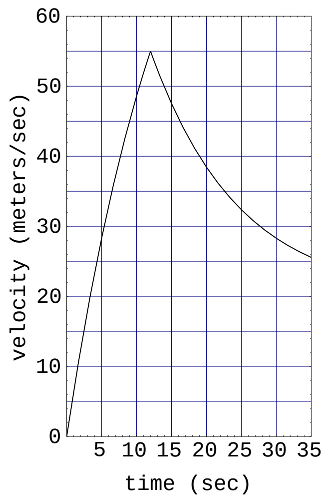
3. (10 points) Sketch the circle $y = \pm\sqrt{100 - x^2}$ and the ellipse $y = \pm 2\sqrt{100 - x^2}$. The point $(8, 12)$ is on the latter ellipse. Use the tangent line approximation to find the y-coordinate of the point on the ellipse in the first quadrant whose x-coordinate is 7.7. You *must* use the tangent line approximation in this problem, showing your work clearly.

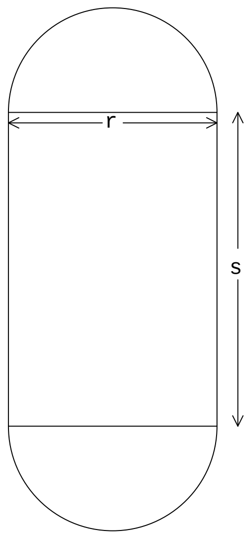
4. (15 points) A game board is to be constructed in the shape of a rectangle of dimensions $r \times s$ with a half-circle attached to each of the two sides of length r (see the bottom of this page at the right).

(a) Find a formula for the area of the board (rectangle plus two half-circles) in terms of r and s.

(b) You first construct a board having dimensions $r = 20$ cm, $s = 30$ cm. What is the area of the board?

(c) Now you want to increase r to 21 cm. What must you do to s so that the total area of the board remains the same? Use the tangent line approximation. Be sure to show clearly what the function is to which you're applying the tangent line approximation formula.

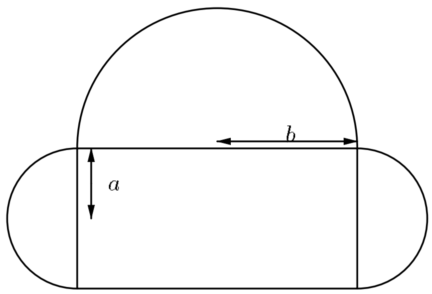




Practice Second Midterms

Practice Second Midterm #1 (50 points in all, time = 1 hour)

1. (13 points) A rectangle having dimensions $2a$ and $2b$ has semicircles on three sides, as shown in the drawing at the right. Suppose that at first $a = 10$ and $b = 20$, but then you want to increase a to 10.100 and decrease b in such a way that the total area of the shape (the rectangle plus the three semicircles) remains constant. Use the tangent line approximation to determine your new value for b.



2. (12 points) Let $y = f(x) = x - \big(1/(2x^2)\big)$.

(a) Find the x- and y-coordinates of any upper and lower turning points, and **use the second derivative test** to determine if they are max or min.

(b) Find all intervals of values of x where the function is **increasing**.

(c) Find all vertical asymptotes (if there are any).

(d) Find all points of inflection (if there are any), and determine the upward or downward concavity of each part of the curve.

(e) Sketch the curve. Show in your sketch what $f(x)$ looks like as x approaches $+\infty$ and $-\infty$, and as x approaches the vertical asymptote (if there is one) from the left and from the right.

3. (10 points) Suppose that in Wanganui, New Zealand the mean daily temperature (averaged over many years) is a sinusoidal function of time. Let t be the time **in months** from the beginning of the year, and let y be the temperature in °F. Suppose that the hottest temperature is 75°F in mid-January (that is, $t = 0.5$), and the coldest is 35°F in mid-July.

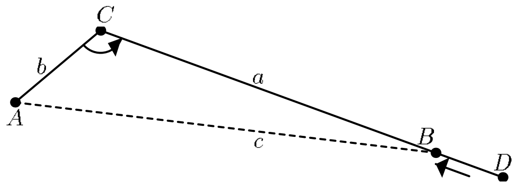
(a) Graph the temperature function $y = f(t)$.

(b) Give its equation, in the form $y = A \sin\big(\frac{2\pi}{B}(t - C)\big) + D$.

(c) Suppose that the typical Wanganuian homeowner turns on the heat when the mean daily temperature drops below 65°F. In an average year, find the time interval during the year when she has the heat turned on.

4. (15 points) According to the law of cosines for the triangle ABC below, we have: $c^2 = a^2 + b^2 - 2ab\cos C$. Suppose that $C = 120°$, $b = 50$ meters, and you are running

from D to C at 5.70 m/sec. How fast is your distance c from A decreasing when you are at a distance $a = 160$ m from C?

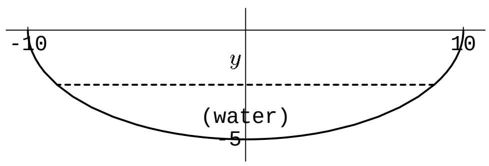


Practice Second Midterm #2 (50 points in all, time = 1 hour)

1. (13 points) Let $y = f(x) = x^2 - \frac{1}{6}x^4$.
(a) Find the x- and y-coordinates of any upper and lower turning points.
(b) Find all intervals of values of x where the function is **increasing**.
(c) Find all points of inflection.
(d) For what values of x is the function concave **upward**?
(e) Sketch the curve. Show in your sketch what $f(x)$ does as x approaches $+\infty$ and $-\infty$.

2. (12 points) A bicycle has a wheel whose **circumference** is exactly 2 meters. It is traveling at 5 m/sec from left to right. At time $t = 0$ it picks up a pebble in the tread of the wheel. Take $(0, 0)$ to be the point where the pebble attaches itself to the tread. In this problem use the parametric equations of the cycloid: $x = r\theta - r\sin\theta$, $y = r - r\cos\theta$.
(a) What is the angular velocity $d\theta/dt$ of the wheel in rad/sec?
(b) Find formulas for the vertical velocity and the horizontal velocity of the pebble in terms of θ.
(c) Find the slope of the direction of motion of the pebble when $t = 1/3$ sec. Draw a picture of the wheel showing the position of the pebble when $t = 1/3$ sec, and draw an arrow showing the direction of motion of the pebble when $t = 1/3$ sec.

3. (12 points) (a) Write the implicit equation of the ellipse whose bottom half is pictured below to the right, and solve for x in terms of y.
(b) Suppose that the drawing is the side view of a container which has a circular top (and circular cross-section at any level). That is, the container is obtained by rotating the bottom half of the ellipse around the y-axis. Suppose the container is being filled with water. Let $A = f(y)$ be the **surface area** of the top of the water when it reaches the level y on the y-axis. The units are **meters**. Write a formula for A in term of y.

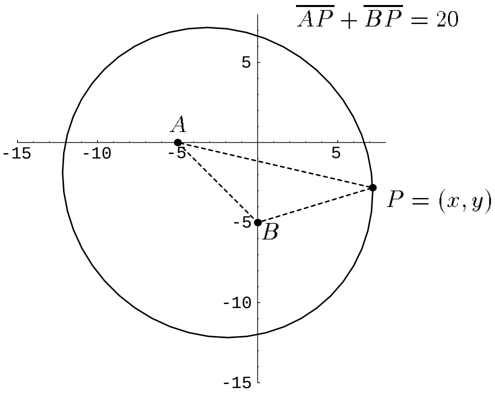


(c) If the water is 250 cm = 2.5 m deep at its deepest point and is rising at 1 cm/sec, how fast is the surface area of the top of the water increasing?

4. (13 points) The ellipse pictured to the right has foci at the two points $(-5, 0)$ and $(0, -5)$. The distance of any point (x, y) on the ellipse from $(-5, 0)$ plus its distance from $(0, -5)$ has the constant value 20.

(a) Write an equation for this ellipse in the form $F(x, y) = \text{Const}$.

(b) Notice that the point $(0, -12)$ is on the ellipse. Find the slope of the tangent line to the ellipse at $(0, -12)$.



Practice Second Midterm #3 (50 points in all, time = 1 hour)

1. (12 points) Let $y = f(x) = 0.25\,x^4 - 2\,x^3 + 4.5\,x^2$.

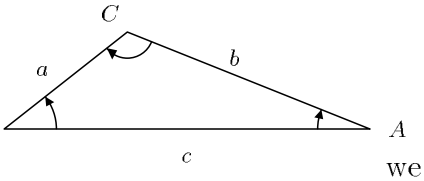
(a) Find the x- and y-coordinates of any upper and lower turning points.

(b) Find all points of inflection.

(c) Sketch the curve, labeling all of the points found in parts (a) and (b) ("max," "min," "inflec").

(d) Give the exact interval(s) where the curve is concave downward.

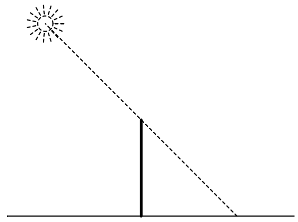
2. (12 points) In the triangle shown below, the sides a and b are pieces of metal hinged at C. Here a is **exactly** $1.50\,m$ long and b is **exactly** $2.50\,m$ long. We measure the side c to be $3.50\,m$, but our measurement of c has an error of up to a centimeter, i.e., $\pm 0.01\,m$. Using the law of cosines,



$$c^2 = a^2 + b^2 - 2ab\cos C,$$

we find that the angle C is $120°$. Use the tangent line approximation and implicit differentiation to determine the error **in degrees** (not radians) in our value for the angle C.

3. (12 points) You are 5 ft tall, and the sun is casting a 5 ft shadow. (See the diagram at the right.) If the sun is descending toward the horizontal at the rate of 1/4 degree per minute, how fast is your shadow lengthening? (First express the length of your shadow as a function of the sun's angle above the horizontal.)



4. (14 points) An object oscillates horizontally at the end of a spring. Its sinusoidal horizontal motion goes from a minimum position of $x = 0$ to a maximum position of $x = 4\,\text{m}$. It goes back and forth once every 2 seconds, starting at $x = 0$ at time $t = 0$. Meanwhile, at time $t = 0$ the whole spring system is dropped from a height of 19.6 meters. Neglect air resistance, and take $g = 9.8\ \text{m/sec}^2$.

(a) Write a formula for $y(t)$.

(b) Draw a graph of x as a function of t, and find a formula for $x(t)$.

(c) At the instant $t = 1.25$ sec, find the horizontal and vertical components of velocity, and the direction (slope) of the path.

Practice Second Midterm #4 (50 points in all, time = 1 hour)

1. (15 points) Let $y = f(x) = \frac{1}{120}\left(2x^6 - 9x^4 - 108x^2\right)$.

(a) Find the x- and y-coordinates of any upper and lower turning points.

(b) Find all intervals of values of x where the function is **increasing**.

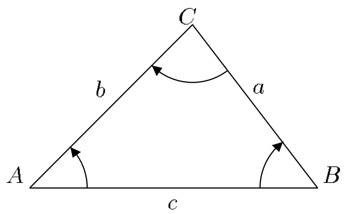
(c) Find all points of inflection.

(d) For what values of x is the function concave **downward**?

(e) Sketch the curve. Show in your sketch what $f(x)$ does as x approaches $+\infty$ and $-\infty$.

2. (10 points) A circular oil slick of uniform thickness is caused by a spill of 1 cubic meter of oil. The thickness of the oil is **decreasing** at the rate of 0.1 cm/hr as the oil spreads (note: 1 cm = 0.01 m). At what rate is the radius of the slick increasing when it is 8 meters? (Note: the volume of a circular oil slick is equal to $\pi \times (\text{radius})^2 \times \text{thickness}$.)

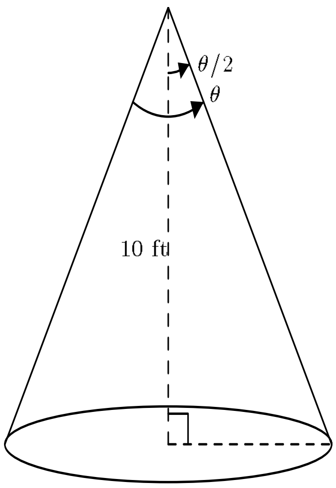
3. (10 points) In the triangle pictured at the right, let A, B, C be the angles at the three vertices, and let a, b, c be the sides opposite those angles. According to the "law of sines," you always have:



$$\frac{b}{a} = \frac{\sin B}{\sin A}.$$

Suppose that a and b are pieces of metal which are hinged at C. At first the angle A is 45° and the angle B is 60°. You then widen A to 46°, without changing the sides a and b. What happens to the angle B? Use the tangent line approximation.

4. (15 points) A conical pile of sand is exactly 10 ft high. Suppose that you have an instrument to measure the vertex angle θ (see the diagram at the right).



(a) Express the volume V of sand in terms of θ.

(b) Suppose you measure θ to be 60° with an error of $\pm 1°$. Find V, and use the tangent line approximation to determine the error in your value for V. (Warning: use radians for angles.)

Practice Final Exams

Practice Final Exam #1.

(100 points in all; time = 3 hours, checking your work carefully)

1. (15 points) Evaluate: (a) $\int_0^3 \sqrt[3]{1-3x}\,dx$ (b) $\int \sin\left(\frac{\pi}{10}(t-1)\right)\cos\left(\frac{\pi}{10}(t-1)\right)\,dt$.

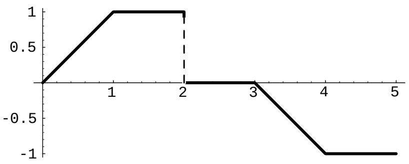
2. (15 points) A light is at the top of a pole 14 meters high. A ball is dropped from the same height from a point 7 meters from the light. How fast is the shadow of the ball moving along the ground 0.8 sec later? (Assume that the ball falls a distance $s = 4.9t^2$ meters in t seconds.)

3. (15 points) At time $t = 0$ an object is at rest. It then accelerates according to the graph at the right, where the acceleration $a(t)$ is measured in m/sec^2.

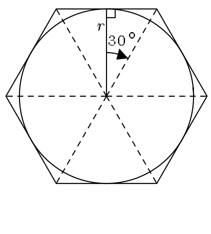
(a) Sketch the graph of the velocity function $v(t)$ during these 5 seconds.

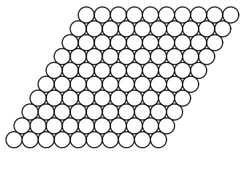
(b) Find a formula for $v(t)$ for t between 0 and 1, for t between 1 and 2, and for t between 2 and 3.

(c) How far does the object travel during the first 3 seconds?



4. (15 points) Cylindrical cans with circular tops and bottoms are to be manufactured to contain a given volume. There is no waste involved in cutting the tin that goes into the vertical side of the can. But the two circular end pieces are cut from a sheet, as shown at the bottom of the page. If the circles for the tops and bottoms of the cans are fit as snugly as possible, each uses up a hexagonal area of tin, as shown in the picture above. First find the area of one of the hexagons in terms of r. (Hint: It consists of 6 equal equilateral triangles.) Now find the ratio of height to radius for the most economical cans. In other words, minimize the total area of tin needed to make a can.





5. (20 points) A 4-centimeter rod is attached at one end A to a point on a wheel of radius 2 cm. The other end B is free to move back and forth along a horizontal bar that goes through the center of the wheel. At time $t = 0$ the rod is situated as in

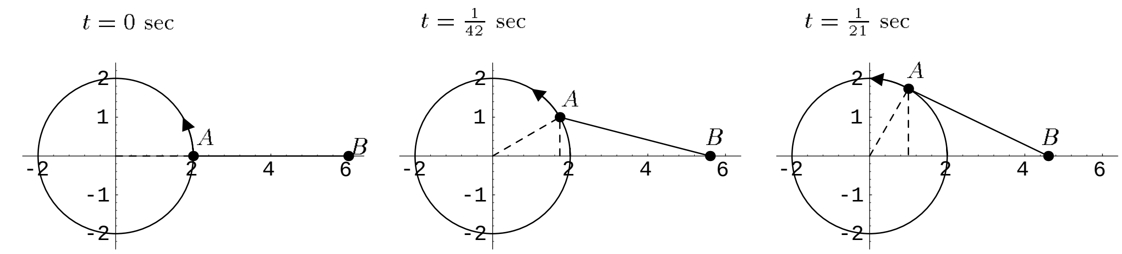
the diagram at the left below. The wheel rotates counterclockwise at 3 1/2 rev/sec. Thus, when $t = 1/21$ sec, the rod is situated as in the diagram at the right below.

(a) How far is the right end of the rod (the point B) from the center of the wheel at time $t = 1/21$ sec?

(b) Express the position of the right end of the rod as a function of t.

(c) Express the speed of the right end of the rod as a function of t.

(d) Use parts (a), (c) and the tangent line approximation to find the distance from B to the center of the wheel at time $t = 1/20$ sec.

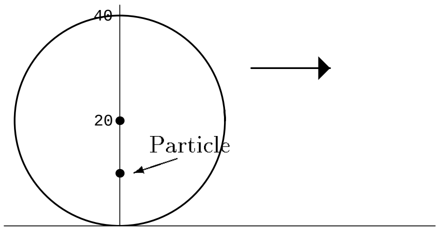


6. (20 points) A wheel 20 cm in radius rolls along a flat surface without slipping at a speed of 5 cm/sec. A particle is caught exactly half way up one of the spokes. At time $t = 0$ the particle is at its lowest point (a height of 10 cm above the surface). The drawing below shows the wheel at time $t = 0$.

(a) Find an expression for the horizontal position of the particle (its x-coordinate) at time t.

(b) What is the x-coordinate of the particle when $t = 3$ sec? In this part and the next, carry out your computations to five places beyond the decimal point.

(c) Suppose that you want to find the time when the particle has moved a horizontal distance of 10 cm, i.e., when its x-coordinate is 10. Write a function $f(t)$ that equals zero at this instant. Then use Newton's method with $t_0 = 3$ to find successive approximations t_1, t_2. Also compute the horizontal position of the particle at time t_2. Please show your work clearly.



Practice Final Exam #2.

(100 points; time = 3 hours, checking your work carefully)

1. (15 points) Evaluate: (a) $\int \frac{dt}{(1-2t)^3}$ (b) $\int_0^{\pi/6} \sqrt{\cos(3x)}\sin(3x)\,dx$.

2. (10 points) A wheel 9 inches in radius is rolling along a pavement without slipping. Its angular velocity at time t seconds is $60 + 20\sin(\pi t)$ rev/sec.

(a) How many revolutions does the wheel go through between $t = 0$ and $t = \frac{1}{3}$ sec?

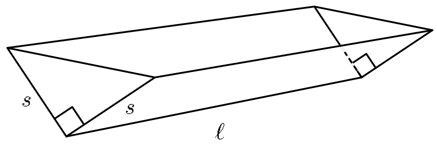
(b) How many radians does the wheel rotate during this time?

(c) How far does the wheel roll during this time? (Give your answer in feet.)

3. (10 points) A girl flies a kite at a height of 240 ft, the wind carrying the kite horizontally away from her at a rate of 20 ft/sec. How fast must she let out the string when the kite is 400 ft away from her?

4. (10 points) You are making a rectangular box with square bottom. If the bottom has side 2 ft and the height is 4 ft, then the total area of all six sides is 40 ft^2. If you decrease the height to 3.9 ft, what should the dimensions of the square bottom be in order for the surface area to remain the same? Use implicit differentiation and the tangent line approximation.

5. (10 points) You are building a trough whose sides are $45° - 45° - 90°$ triangles (see the diagram at the right). Those triangular sides are three times more expensive (in cost per unit area) than the two rectangular pieces that run the length of the trough. You want the trough to hold 6 ft^3. What dimensions s and ℓ will minimize the cost?



6. (25 points) The purpose of this problem is to sketch the function $y = f(x) = x^2 + \sin(2x)$ between $x = -\pi/2$ and $x = \pi/2$.

(a) On the same graph, sketch both the curve $y = x^2$ and the curve $y = \sin(2x)$ between $x = -\pi/2$ and $x = \pi/2$.

(b) What equation must be satisfied in order for $y = f(x) = x^2 + \sin(2x)$ to have a maximum or minimum at x?

(c) Find the x- and y-coordinates of all points of inflection of $y = f(x) = x^2 + \sin(2x)$ between $x = -\pi/2$ and $x = \pi/2$.

(d) Give a value of x near a minimum of $y = f(x) = x^2 + \sin(2x)$. Do this by finding a minimum of $\sin(2x)$. Call this value x_0.

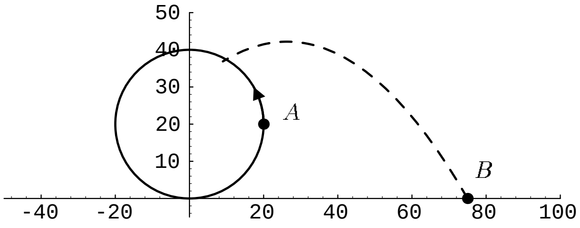
(e) Use Newton's method with two iterations to get a more accurate value x_2 for the x-coordinate of the minimum point of $y = f(x) = x^2 + \sin(2x)$.

(f) Do you think the function $y = f(x) = x^2 + \sin(2x)$ has any other maxima or minima besides the minimum you found in part (e)? (You are not expected to do any computations or take a long time on this part.)

(g) Sketch the graph of $y = f(x) = x^2 + \sin(2x)$.

(Continue to Problem 7 on the next page)

7. (20 points) You are standing at B, a distance of 75 ft from the bottom of a ferris wheel which is 20 ft in radius. Your arm is at the same level as the bottom of the ferris wheel. At time $t = 0$ your friend is at the point A rotating counterclockwise at the rate of 1 revolution every 12 seconds. At that instant you throw a ball to her. You throw the ball at 60 ft/sec at an angle of 60° above the horizontal. Take $g = 32$ ft/sec^2, and neglect air resistance.



(a) What is your friend's angular velocity in rad/sec?

(b) Using the x- and y-coordinates in the diagram, write parametric equations for your friend's position.

(c) Find the horizontal and vertical components of the velocity of the ball at time $t = 0$. Notice that the horizontal motion is in the *negative* x-direction. Round off your answers to the nearest ft/sec.

(d) Write parametric equations for the x- and y-coordinates of the position of the ball after t sec.

(e) At time $t = 2$ sec, how far is the ball from your friend?

(f) Express the distance from the ball to your friend at time t in terms of t.

(g) Suppose you want to find the closest distance the ball gets to your friend. Find a function of t that must be equal to zero at the moment of this minimum distance. This is the function that would be put into the Newton's method formula (with initial guess $t_0 = 2$ sec) to find the time when the ball is closest to your friend.

Practice Final Exam #3.

(100 points in all, time = 3 hours, checking your work carefully)

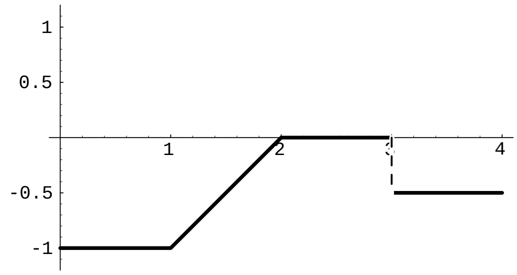
1. (15 points) Evaluate: (a) $\displaystyle\int \frac{dz}{\sqrt{1-7z}}$ (b) $\displaystyle\int_0^\pi \sin(x/4)\cos^2(x/4)\,dx$.

2. (15 points) At time $t = 0$ an object is moving to the right at 2 m/sec. It then decelerates according to the graph below, which shows rightward acceleration as a function of t. Here $a(t)$ is measured in m/sec^2.

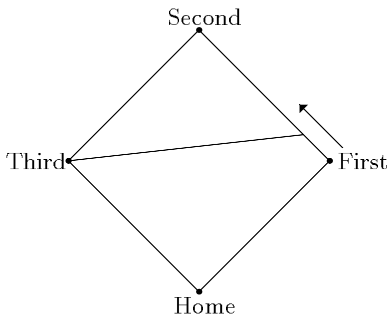
(a) Draw the graph of the velocity function $v(t)$ during these 4 seconds.

(b) Find formulas for $v(t)$ for t in the intervals (0,1), (1,2), (2,3) and (3,4).

(c) How far does the object travel between $t = 0$ and $t = 4$?



3. (10 points) A baseball diamond is a square 90 ft on a side. A player runs from first base to second base at 16 ft/sec. At what rate is the player's distance from third base decreasing when the player is 20 ft from first base?



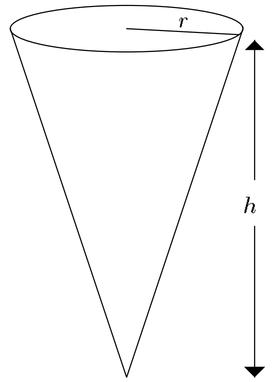
4. (15 points) Two legs of a right triangle are 7 and 24 (so the hypotenuse is 25). If you want to shorten the shorter leg to 6.980 and keep the perimeter of the right triangle (the sum of the three sides) constant, how long should you make the longer leg? Use the tangent line approximation and implicit differentiation.

5. (20 points) In this problem you want to find the best ratio of height to radius $x = h/r$ so that a cone whose side has fixed area holds maximum volume. Use the formulas $\frac{1}{3}\pi r^2 h$ for the volume of a cone and $\pi r\sqrt{r^2+h^2}$ for the side of a cone.

(a) Express both the volume and area of side in terms of r and x (without h).

(b) Suppose the side area must be a constant C. Solve for r in terms of x. (The constants π and C should also appear in your formula.)

(c) Find the ratio of height to radius for which the cone of side area C holds the most volume.



(Continue to Problems 6 and 7 on the next page)

6. (10 points) Use Newton's method with two iterations to find a number that is equal to the sum of its square root and its cube root. 3 is a good initial guess, since $\sqrt{3} + \sqrt[3]{3} = 3.1743$.

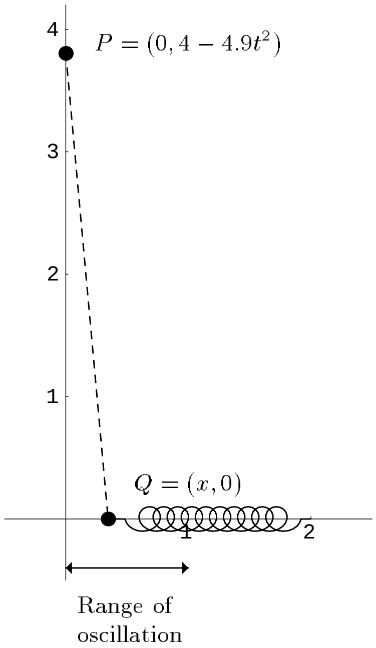
7. (15 points) At time $t = 0$ an object P is dropped from a height of 4 m on the y-axis. At the same instant an object Q at the origin is released so that it can oscillate back and forth on a horizontal spring. The point Q moves sinusoidally, returning to the origin each 1 sec. When the spring is most compressed, Q is at 1 m on the x-axis. Your purpose in this problem is to find the closest distance between P and Q.

(a) Write a formula for Q's position as a function of time in the form

$$x = A \sin\left(\frac{2\pi}{B}(t - C)\right) + D\,.$$

(b) Write a formula for the **square** of the distance from P to Q as a function of time.

(c) Find a function of t that is zero at the instant when P and Q are closest. This is the function that would be put into Newton's method to find the time when P and Q are closest.



Practice Final Exam #4.

(100 points in all, time = 3 hours, checking your work carefully)

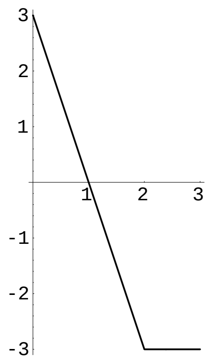
1. (20 points) Evaluate (in part (b) express your answer in terms of B):

$$(a)\ \int_0^3 \frac{dt}{\sqrt[4]{1+5t}}\quad (b)\ \int_0^{B/6} \cos\left(\frac{2\pi x}{B}\right)\sin\left(\frac{2\pi x}{B}\right)\,dx.$$

2. (15 points) At time $t = 0$ an object is moving to the right at 3 m/sec. It then accelerates according to the graph at the right, which shows rightward acceleration as a function of t (measured in m/sec^2).

(a) Find a formula for $v(t)$ when t is between 0 and 2 and when t is between 2 and 3.

(b) How far does the object travel in the 3 seconds?

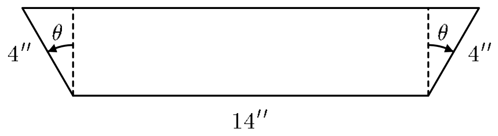


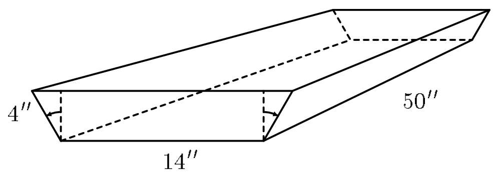
3. (15 points) The trough shown at the right is to be constructed from a piece of metal 50 inches long and 22 inches wide. The trough is to be made by turning up strips of width 4 inches to make equal angles θ with the vertical. **See the diagram at the right.**

(a) Express the volume of the trough in terms of the angle θ.

(b) Let $X = \sin\theta$. Find an expression of the form $aX^2 + bX + c$ which must equal 0 if θ is chosen so as to maximize the volume of the trough.

(c) Find the maximum possible volume of the trough. (In this problem you will need to use the quadratic formula $(-b \pm \sqrt{b^2 - 4ac})/2a$ to solve for $X = \sin\theta$.)

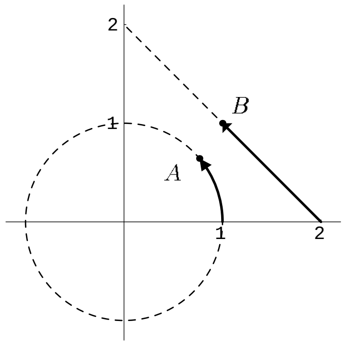




4. (10 points) Use Newton's method to find the side s of a cube for which the sum of its volume and its surface area is 1500. Use enough iterations so as to be sure that your answer is accurate to four significant figures.

(Continue to Problems 5 and 6 on the next page)

5. (15 points) The point A revolves counterclockwise around the unit circle at the rate of 1 revolution every 9 seconds. At $t = 0$ it starts at the point $(1, 0)$. The point B starts at the point (2,0) and travels in a straight line at constant velocity, reaching the point (0,2) after 2 seconds.



(a) Find parametric equations for both points A and B.

(b) Find a function which must be equal to zero at the instant when A and B are closest. This is the function that you could put into Newton's method to find the instant when A and B are closest.

6. (25 points) At time $t = 0$ a ball is thrown out of a 50-meter high window at a velocity of 10 m/sec at an angle of α radians above the horizontal. Take the ground to be the x-axis and the side of the building from which the ball was thrown to be the y-axis. Neglect air resistance, and take $g = 9.8$ m/sec^2.

(a) Find parametric equations for the x- and y-coordinates of the ball at time t. Some trig functions of α should appear in your equations.

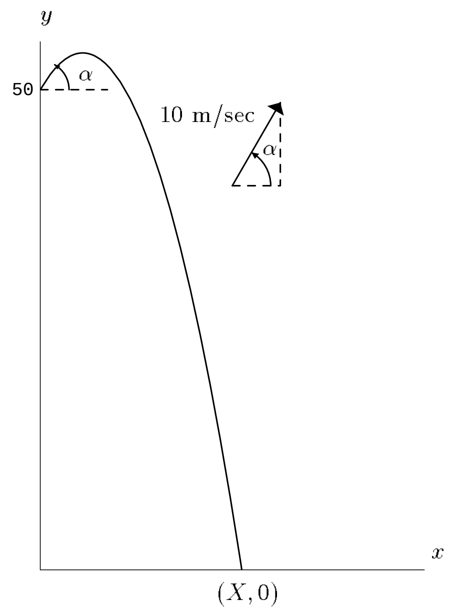
(b) Let T be the time when the ball hits the ground. Express T as a function of α (you will need the quadratic formula

$$(-b \pm \sqrt{b^2 - 4ac})/2a,$$

where the sign in front of the square root should be chosen so as to give positive T).

(c) Let X be the distance of the ball from the building when it hits the ground. Express X as a function of α.

(d) Suppose that α is measured to be $60°$ with an error of $\pm e$ degrees: $\alpha = 60° \pm e°$. Use the tangent line approximation to determine how sensitive X is to the error in α. In other words, find the constant of proportionality which relates the error in X (in meters) to the error e in α (in degrees).



Answers

1. Answers to Homework Problems

Answers to Selected Problems in Chapter 1. 1: m, b and distance are: (a) 1.5, -3, $\sqrt{13}$; (b) -3, 2, $\sqrt{10}$; (c) 1, 0, $2\sqrt{2}$; (d) 0, 3, 6; (e) 1, 1, $5\sqrt{2}$; (f) -3, 0.02, $0.02\sqrt{10}$. 2: y- and x-intercepts are: (a) 2, -1; (b) 6, 6; (c) 1/2, -1; (d) 3/2, none; (e) -2, -3. 3: (a) $x^2+y^2=9$, (b) $(x-5)^2+(y-6)^2=9$, (c) $(x+5)^2+(y+6)^2=9$, (d) $x^2+(y-3)^2=9$ (or $x^2+y^2-6y=0$), (e) $x^2+y^2+6y=0$, (f) $x^2-6x+y^2=0$. 4: radius and center are: (a) 5, $(0,-5)$; (b) 7, $(5,0)$; (c) 5, $(3,4)$.
5: $y=1.8x+32$. 6: $y=0.15x+10$. 7: $y=0.08x-3.2$. 8(b); $y=3x+120$. 9: (a) $y=0$ for $0\le x\le 100$, $y=0.1x-10$ for $100\le x\le 1000$, $y=x-910$ for $x\ge 1000$; (b) he's introducing a "negative income tax" as a rather cheap way of getting good PR — so that, for example, a penniless person receives 10 gold coins from the King (since the y-intercept is -10). 10:(a) $P=-0.0001x+2$; (b) $x=-10000P+20000$. 11: $y=0.15x$ for $0\le x\le 19450$, $y=0.28x-2528.5$ for $19450\le x\le 47050$, $y=0.33x-4881$ for $47050\le x\le 97620$.

Answers to Selected Problems in Chapter 2. 1: $[1.5,\infty)$. 2: $x\ne -1$. 3: $x\ne\pm1$. 4: $(-\infty,0)$. 5: all x. 6: $[0,\infty)$. 7: $[h-r,h+r]$. 8: x negative or in $[1,\infty)$. 9: $\left(-\frac{1}{3},\frac{1}{3}\right)$. 10: x nonnegative but $x\ne 1$. 11: x nonnegative but $x\ne1$.

Answers to Selected Problems in Chapter 3. 1: (a) -5, (b) -2.47106, (c) -2.40679, (d) -2.40068, (e) -2.4. 2: (a) $-4/3$, (b) $-24/7$, (c) $7/24$, (d) $3/4$. 3: (a) 5, (b) 4.1, (c) 4.01, (d) 4.001; limit=4. 4: (a) -10.29, (b) -9.849, (c) -9.8049; $-9.8-4.9\Delta t$; -9.8 m/sec. 5: (a) -0.107527, (b) -0.110742, (c) -0.111074; $-1/\big(3(3+\Delta x)\big)$; $f'(3)=-1/9$. 6: $(2+\Delta x)+1/(1+\Delta x)$; $f'(1)=3$. 7: (a) 3.31, (b) 3.003001, (c) 3.0000300001; $3+3\Delta x+(\Delta x)^2$; $f'(1)=3$.

Answers to Selected Problems in Chapter 4. *(first two problems)*
1: $f'(x)=-x/\sqrt{169-x^2}$. 2: (a) $f'(t)=-9.8t$, (b) $f'(x)=2x+(1/x^2)$, (c) $f'(x)=2ax+b$, (d) $f'(x)=3x^2$.

Answers to Selected Problems in Chapter 5. 1:(a) 4.4, (b) 0.9, (c) 1.01, (d) 0.0099, (e) 2.0025, (f) 5.1, (g) 4.9, (h) 2.001. 2: 3.94 3: 0.9433. 4: 15 ft 2 in. 5: 208.9 ft. 6: $25\pi\pm4\pi(\Delta x)$. 7: ±30, $\pm1\%$ and $\pm3\%$.
8: (a) 5 sec, (b) ±0.025 sec, $\pm0.5\%$, (c) $\pm\frac{1}{2}p\%$. 9: (a) $r=CA^{1/2}$, where $C=1/\sqrt{\pi}$, (b) $\pm0.5\%$, (c) $\pm\frac{1}{2}p\%$. 10: 8 ± 0.15. 11: $\pm3p\%$. 12: (a)$\pm\frac{1}{2}p\%$, (b) $\pm\frac{1}{3}p\%$. 13: (a) $\pm p\%$, (b) $\pm3p\%$, (c) $\pm2p\%$. 14: $\pm\frac{2}{3}p\%$. 15: (a) $\pm2p\%$, (b) $\pm p\%$, (c) $\pm p\%$. 16: 1 83/84. 17: $\Delta x=-1/12$ cm.

Answers to Selected Problems in Chapter 6. 1: (a) $4x^3-9x^2+x+7$, (b) $3x^2-4x+2/\sqrt{x}$, (c)-(d) $6x(x^4+2x^2+1)$,
(e) $\sqrt{169-x^2}-x^2/\sqrt{169-x^2}$, (f) $(2x-4)\sqrt{25-x^2}-x(x^2-4x+5)/\sqrt{25-x^2}$,
(g) $(-x^2+2x+2)/(1-x)^2$, (h) $-25/\left(x^2\sqrt{25-x^2}\right)$, (i)-(j) $-\sqrt{x}/\sqrt{169-x^2}-\frac{1}{2}\sqrt{169-x^2}/x^{3/2}$,
(k) $\frac{1}{2}\left(3x^2-2x+(1/x^2)\right)/\sqrt{x^3-x^2-(1/x)}$, (l) $300x/(100-x^2)^{5/2}$,
(m) $(\frac{1}{3}+x^2)(x+x^3)^{-2/3}$. (n) $\left((x^2+1)^2+\sqrt{1+(x^2+1)^2}\right)^{-1/2}x(x^2+1)\left(2+\left(1+(x^2+1)^2\right)^{-1/2}\right)$.

2: (a) $-9.8t+20$, -9.8; (b) a, 0; (c) $\frac{1}{2}a(at+b)^{-1/2}, -\frac{1}{4}a^2(at+b)^{-3/2}$; (d) $\frac{1}{2}(2at+b)(at^2+bt+c)^{-1/2}$, $a(at^2+bt+c)^{-1/2} - \frac{1}{4}(2at+b)^2(at^2+bt+c)^{-3/2}$, which simplifies to $\frac{1}{4}(4ac-b^2)(at^2+bt+c)^{-3/2}$.
3: $v(t) = -5+2t$, $a(t) = 2$; the bicycle is moving leftward initially at 5 m/sec, but its speed is decreasing; after 2.5 sec it has slowed to 0 m/sec, at which point it reverses direction and starts moving to the right.
4: $2 + \frac{n}{4}\Delta x$. 5: $\pm 0.96\Delta x$. 6: $\frac{500}{3}\pi \pm 40\pi(\Delta x)$. 7: (a) $f(t) = \sqrt{100+(78-4.9t^2)^2}$; (b) $-9.8t(78-4.9t^2)/\sqrt{100+(78-4.9t^2)^2}$.

Answers to Selected Problems in Chapter 7. 1: (a) 1.992, (b) 5.0129, (c) 4.00133. 2: (a) $y' = x/y$, (b) $y' = -(2x+y)/(x+2y)$, (c) $y' = (3x^2+y^2-2xy)/(3y^2+x^2-2xy)$. 3: $y' = 1$. 4: $y = -\frac{1}{13}x + \frac{14}{13}$, 0.9923. 5: B is 4 times as strong as A. 6: $y' = 3/19$. 7: (a) $dT/dh = -38.64°$F/hour, (b) $h = -0.0259T + 5.493$. 8: (a) $\sqrt{100+(10+0.3t)^2} + \sqrt{(x-10)^2+(10-0.3t)^2} =$const, (b) 12.492 sec.

Answers to Selected Problems in Chapter 8. 1: (a) $\pi/12$, (b) $2\pi/3$, (c) $\pi/2$, (d) 20π. 2: (a) 25 rad/sec (b) 6.944 rad/sec. 3: (a) $30\sin\theta$, (b) $\tan\theta$, (c) $60(1-\cos\theta)$. 4: $\sin(\alpha+\beta) = \overline{BC} = \overline{DF} = \overline{DE} + \overline{EF} = \overline{AE}\sin\alpha + \overline{BE}\cos\alpha = \cos\beta\sin\alpha + \sin\beta\cos\alpha$. 6: warmest: 62° on July 12; coolest: $-12°$ on January 10. 7: $F = 50\sin\left(\frac{\pi}{2}t\right) + 150$, $R = 2500\sin\left(\frac{\pi}{2}(t-3)\right) + 7500$. 8: (a) $20\sin\left(\frac{2\pi}{360}(t-109)\right) + 40$; (b) $2\sin\left(\frac{2\pi}{360}(t-81)\right) + 7$; (c) $5\sin\left(\frac{2\pi}{24}(t-11)\right) + 20$; (d) $10\sin(5\pi t)$; (e) $0.4\sin\left(25(t-\frac{\pi}{50})\right) + 0.4$ (or, equivalently, $0.4(1-\cos(25t))$).

Answers to Selected Problems in Chapter 9. 1: 0.96509 (exact value=0.96569). 2: (a) $-\csc^2 x$, (b) $-\cot x\csc x$. 3: (a) 0.46977, (b) 0.484885, (c) 0.93018. 4: (a) $\frac{\pi}{9}\cos\left(\frac{2\pi}{360}(t-109)\right)$; (b) $\frac{\pi}{90}\cos\left(\frac{2\pi}{360}(t-81)\right)$; (c) $\frac{5\pi}{12}\cos\left(\frac{2\pi}{24}(t-11)\right)$; (d) $50\pi\cos(5\pi t)$; (e) $10\sin(25t)$.
5: (a) $\cos^2 x - \sin^2 x$, (b) $-\sin(x)\cos(\cos x)$, (c) $\left(x\sec^2 x + \tan x\right) / \left(2\sqrt{x\tan x}\right)$, (d) $\Big((1+\sin x)\sec^2 x - \sin x\Big)/(1+\sin x)^2$. 6: 0.7632. 7: 1.241. 8: (b) $0.5 = 2\sin(\pi t)\sin(\pi x/3)$; (c) $x = 0.5$; (d) $dx/dt = -3$; (e) $x = 0.5 + (0.2 - \frac{1}{6})(-3) = 0.4$.

Answers to Selected Problems in Chapter 10. 1: (a) $x = 2+3t$, $y = 3-2t$; (b) $x = 1+6t$, $y = 2+2.5t$; (c) $x = 5\sqrt{3}t$, $y = 5t$. 2: (a) 25 rad/sec; (b) $\frac{25}{2\pi}$ rev/sec; (c) 4π m/sec; (d) 21.6 km/hr. 3: (a) 34.3 mph; (b) 55 rad/sec, 8.75 rev/sec. 4: (a) $20t$; (b) $15\pi t$; (c) $7.5t^2$. 5: (a) $x = 3\cos 0.5t$, $y = 3\sin 0.5t$; (b) $x = 3\sin 4\pi t$, $y = 3\cos 4\pi t$; (c) $x = -3\sin 0.4\pi t$, $y = -3\cos 0.4\pi t$; (d) $x = 3\sin 20\pi t$, $y = -3\cos 20\pi t$. 6: (a) $x = 10t$, $y = -4.9t^2 + 24.5t + 1$; (b) 2.5 sec; (c) 5.04 sec. 7: (a) $x = 49.5t$, $y = -16t^2 - 49.5t + 163$; (b) $(-32t - 49.5)/49.5$; (c) 52.9°, 58.7°; (d) 124 ft/sec at 66.4° below horizontal (hits ground at $t = 2$). 8: (a) $x = (v\cos\alpha)t$, $y = \frac{1}{2}gt^2 + (v\sin\alpha)t$; (b) $2v\sin\alpha/(-g)$; (c) $2v^2\sin\alpha\,\cos\alpha/(-g)$.

Answers to Selected Problems in Chapter 11. 1: (b) $\sin\theta/(1-\cos\theta)$. 2: (a) $x = 2t$; (b) $y = 3\sin(4\pi(t-\frac{3}{8}))$; (c) $y = 3\sin(2\pi(x-\frac{3}{4}))$. 3: $y = 3\sin(4\pi(\sqrt{x}-\frac{3}{8}))$. 4: $x = 2\sin(\pi t)$, $y = 100 - 16t^2 + 2\cos(\pi t)$. 5: $x = 6t - 0.5\sin(8t)$, $y = 0.75 - 0.5\cos(8t)$. 6: $x = 10t + 0.25\sin(40t)$, $y = 0.25 - 4.9t^2 + 0.25\cos(40t)$.

Answers to Selected Problems in Chapter 12. 1: 0.0199 cm/sec. 2: 0.25 m/sec. 3: 80π mi/min (=4.2 mi/sec). 4: 6.71 ft/sec. 5: 2.12 cm/sec. 6: 0.65 ft/sec 7: 7.91 m/sec 8: 1.172 m/min 9: 0.5454 m/min 10: 6 ft/sec, 2.5 ft/sec 11: 69.4 mph 12: 0.1234 ft^3/sec. 13: (a) $(a\dot{a} + b\dot{b} + 0.5a\dot{b} + 0.5b\dot{a})/\sqrt{a^2+b^2+ab}$; (b) $(a\dot{a} + b\dot{b} - a\dot{b}\cos\theta - b\dot{a}\cos\theta + ab\dot{\theta}\sin\theta)/\sqrt{a^2+b^2-2ab\cos\theta}$. 14: 81.6 m/sec 15: (a) $y = a\cos(\theta) - a\sin(\theta)\cot(\beta+\theta)$ (or else, using the law of sines, an equivalent answer is: $y = a\sin\beta/\sin(\theta+\beta)$); (b) $dy/dt = \Big(-a\sin(\theta)(1-\csc^2(\beta+\theta)) - a\cos(\theta)\cot(\beta+\theta)\Big)\,d\theta/dt$ (or else: $dy/dt = -a\Big(\sin\beta\cos(\theta+\beta)/\sin^2(\theta+\beta)\Big)\,d\theta/dt$); (c) 3.79 cm/sec. 16: (a) $a(d\theta/dt)$; (b) closing at a little over 15 rad/sec; (c) no violation of relativity, because no part of the scissors is moving faster than 300 km/sec. 17: (a) -50 m/sec; (b) 68 m/sec; (c)... 18: (a) $(a\dot{a} + b\dot{b})/\sqrt{a^2+b^2+h^2}$; (b) $(a\dot{a} + b\dot{b} + h\dot{h})/\sqrt{a^2+b^2+h^2}$.

Answers to Selected Problems in Chapter 13. 1: Both $f'(x) = 0$ and $f''(x) = 0$ when $x = 0$. If n is even, the origin is a minimum, and the curve is always concave up. If n is odd, the origin is a point of inflection, and the curve is concave up for $x > 0$ and concave down for $x < 0$.
2: (a) min at $(0.5, -0.25)$, always concave up, no inflection;

(b) max at (1,4), min at $(-1, 0)$, inflection at $(0, 2)$, conc up for $x < 0$, conc down for $x > 0$;

(c) max at (2,20), min at (4,16), inflection at (3,18), conc up for $x > 3$, conc down for $x < 3$;

(d) max at (0,3), min at $(\pm 1, 2)$, inflections at $(\pm 0.577, 2.444)$, conc down for $-0.577 < x < 0.577$, otherwise conc up, symmetric with respect to y-axis;

(e) min at $(1, -1)$, inflections at (0,0) and $(2/3, -0.5926)$, conc down for $0 < x < 2/3$, otherwise conc up;

(f) no max, min or inflections, y-axis is asymptote, conc up for $x < 0$, conc down for $x > 0$, the line $y = x$ is a slanted asymptote;

(g) no max or min, inflections at $(\pm 1, 2)$, y-axis is asymptote, conc up for $x < -1$ and $x > 1$, otherwise conc down, symmetric with respect to y-axis;

(h) min when $x = 7\pi/12$ (and $x = 7\pi/12 \pm$any multiple of π), max when $x = 11\pi/12$ (again can add or subtract any multiple of π), inflections when $x = \pi/4$, $3\pi/4$ ($\pm$ any multiple of π), conc up for $\pi/4 < x < 3\pi/4$, general appearance is oscillation around the slanted line $y = -x$;

(i) max at all points $(\frac{\pi}{4} \pm 2k\pi, \sqrt{2})$ (where k is any integer), min at $(\frac{5\pi}{4} \pm 2k\pi, -\sqrt{2})$, inflections at $(\frac{3\pi}{4} \pm k\pi, 0)$, you get a sinusoidal function;

(j) asymptotes at odd multiples of π, max when $x = -\frac{\pi}{2} \pm 2k\pi$, min when $x = \frac{\pi}{2} \pm 2k\pi$, inflections at even multiples of π, in each section between asymptotes the left half is conc down, the right half is conc up.
5: (a) max at (0,0), min at $(\pm\sqrt{10/3}, -500/27)$; (b) $x > \sqrt{2}$ and $x < -\sqrt{2}$; (c) $(\pm\sqrt{2}, -12)$; (d) use 2nd derivative test.

Answers to Selected Problems in Chapter 14. 1: (a) max value $f(2) = 5$; (b) min value $f(0) = 1$. 2: $P/4$ by $P/4$. 3: $\sqrt[3]{2V}$ by $\sqrt[3]{2V}$ by $\frac{1}{2}\sqrt[3]{2V}$; ratio=1/2. 4: $l^2/8$. 5: $x = 2000$, $P = \$6$, profit=\$5000. 6: $r\sqrt{2}$ by $r\sqrt{2}/2$, area=r^2. 7: $h/r = 2$. 8: $h/r = 8/\pi$. 9: 4/27. 10: Take $x = a$, where $f(x)$ has a corner (if $w > v$, then $f'(x)$ is never zero). 11: $h/r = 2$; $h/r = 3.5$. 12: (a) $a/6$ on a side; (b) $\left(a + b - \sqrt{a^2 + b^2 - ab}\right)/6$ on a side. 13: $0.5 - \frac{\pi}{4}k$ provided that $k < 2/\pi$; otherwise, the optimal window consists of only the semicircular piece. 14: $\theta = 30°$, max area= $l^2\sqrt{3}/12 \approx 0.144\, l^2$, bigger than in Problem 3. 15: 0.00139 hours ($\approx$ 5 sec), misses by 0.5726 mi. 16: ratio=a/b. 17: $\frac{2}{3}r\sqrt{3}$ by $\frac{2}{3}r\sqrt{6}$. 18: $1/\sqrt{3}$ ($\approx$58%). 19: 18 in $\times$ 18 in $\times$ 36 in. 20: $r = 2.71$, $h = 10.84$. 21: $h/r = \sqrt{2}$. 22: $h = 1.1798R$, $r = 0.9837R$, 28.54% 23: $c/(1+\sqrt[3]{b/a})$ from A. 24: 1/2. 25: $x = 2500$, $P = \$5$, profit=\$7000. 26: Snell's law: $\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$.

Answers to Selected Problems in Chapter 15. 1: $x_1 = 1.4765432$, $x_2 = 1.4757740$, $x_3 = 1.4757732$. 2: $x_1 = 2.25$ (exactly), $x_3 = x_2 = 2.19$ to three significant figures ($x_2 = 2.19369\ldots$, $x_3 = 2.19192\ldots$). 3: $\theta_1 = 1.9359512$, $\theta_2 = 1.9345639$, $\theta_3 = 1.9345632$. 4: Hint: Set the slope of the line joining $(-\pi/2, -1)$ to $(x, \sin x)$ equal to the derivative of $\sin x$, then multiply through by the denominator to get the $f(x)$ in Newton's method: $f(x) = (\frac{\pi}{2} + x)\cos x - \sin x - 1$; then $x_0 = 0.7$, $x_1 = 0.7633$, $x_3 = x_2 = 0.7603$. 5: $t = 0.5062$ sec, height=6.9938 ft. 6: take $t_0 = 0.7$, then $t_1 = 0.68674, t_2 = 0.68726$. 7: $x = 40\sin(\pi t/10) - 10\sin(\pi t/3)$, $y = 50 - 40\cos(\pi t/10) - 10\cos(\pi t/3)$, 9.2655 sec, 98.56 ft. 8: $l = \left(9/\sin(\frac{\pi}{3} - \theta)\right) + \left(7/\sin\theta\right)$, 0.49254 rad ($= 28.2°$), 31.89 ft

Answers to Selected Problems in Chapter 16. 1: (a) $-\frac{1}{3x^3} + \frac{2}{5}x^{5/2} + 5x + C$, (b) $-0.1(1-t)^{10} + C$, (c) $\frac{1}{5}x^5 + \frac{2}{3}x^3 + x + C$, (d) $-0.3(1 - 5t)^{2/3} + C$, (e) $\frac{1}{4}\sin^4 x + C$, (f) $-\frac{1}{3}(100 - x^2)^{3/2} + C$, (g) $-\frac{2}{3}\sqrt{1 - x^3} + C$, (h) $-\frac{B}{2\pi}\cos\left(\frac{2\pi}{B}(t - C)\right) + Dt + C'$, (i) $-\frac{20}{3}\left(1 - \sin\frac{t}{10}\right)^{3/2} + C$, (j) $\frac{1}{\pi}\sin\left(\sin(\pi t)\right) + C$. 2: $-\frac{1}{\pi}\cot(\pi t) + C$. 3: $-\frac{1}{2}g(t - t_0)^2 + v_0(t - t_0) + s_0$. 4: $\frac{1}{3}t - \frac{1}{2} + 1/(6t^2)$. 5: (a) $\frac{1}{6}kt^3 + v_0 t + s_0$, (b) $\frac{1}{15}(2t+1)^{5/2} + (v_0 - \frac{1}{3})t + s_0 - \frac{1}{15}$, (c) $-\frac{1}{25}\sin(5t) + (v_0 + \frac{1}{5})t + s_0$, (d) $50\tan(0.1t) + (v_0 - 5)t + s_0$.

Answers to Selected Problems in Chapter 17. 1: (a) 6.25 vs. 16/3=5.333, (b) 5.8125 vs. 5.333, (c) 4.146 vs. 5.333, (d) 1.239 vs. 1, (e) 1.125 vs. 1. 2: (a) 55, (b) 25/12, (c) $\frac{3}{2}+\frac{\sqrt{3}}{2}$. 3: (a) 28.5, (b) 14/9, (c) 2/3, (d) 0.24, (e) $2/\pi$. 4: (a) 2/3, (b) $2-\sqrt{2}$, (c) $\frac{4}{\pi}+\frac{4}{3}$, (d) $\pi-2$. 5: (a) 13/3, (b) 56/15, (c) 0, (d) $1/4\pi$, (e) 3/4, (f) 9. 6: (a) 14600 degree-days, (b) 4523 degree-days, (c) 277 degree-days. 7: (a) $a(t)=\begin{cases}2t, & 0\le t\le 2\\ 4, & 2\le t\le 4\\ 8-t, & 4\le t\le 8\end{cases}$; $v(t)=\begin{cases}t^2, & 0\le t\le 2\\ 4t-4, & 2\le t\le 4\\ 8t-\frac{1}{2}t^2-12, & 4\le t\le 8\end{cases}$; $s(8)-s(0)=88$. (b) $a(t)=\begin{cases}\sin t, & 0\le t\le 0.5110\\ 1-t, & 0.5110\le t\le 1\end{cases}$; $v(t)=\begin{cases}1-\cos t, & 0\le t\le 0.5110\\ t-\frac{1}{2}t^2-0.2527, & 0.5110\le t\le 1\end{cases}$; $s(1)-s(0)=0.1234$.

Answers to Selected Problems in Chapter 18. 1: (a) $\sin^2 u$, (b) $\sqrt{1+u+u^2}$, (c) $-\sqrt{1+\sin^2 u}$, (d) $2\sin^2 u$; (e) 0. 2: (a) $b/(a+bt)$, (b) $(-gt+v_0)\sin^2\left(-\frac{1}{2}gt^2+v_0t+s_0\right)$, (c) $3t^2\cos(t^6)-2t\cos(t^4)$, (d) $A\alpha\cos(\alpha(t-C))\sqrt{1+\left(A\sin(\alpha(t-C))\right)^2}-b\sqrt{1+(a+bt)^2}$. 3: (a) $\int_0^{2t}\left(20+10\cos^2(0.1\pi x)\right)dx$, (b) $2\left(20+10\cos^2(0.2\pi t)\right)$. 4: (a) $\int_0^{2t}\frac{40\,dx}{1+(0.01x-1)^2}$, (b) $\frac{80}{1+(0.02t-1)^2}$. 5: (a) $2\int_{48-16t^2}^{48}48\cos\frac{\pi y}{96}\,dy$, (b) $3072t\cos\left(\frac{\pi}{6}(3-t^2)\right)$. 6: (a) $2\int_{-1}^{u}\sqrt{1-x^2}\,dx$, (b) $u(t)=\sin\left(\pi(t-\frac{1}{2})\right)$, (c) $2\pi\cos\left(\pi(t-\frac{1}{2})\right)\sqrt{1-\sin^2\left(\pi(t-\frac{1}{2})\right)}$.

2. Answers to Practice Midterms

Answers to Practice First Midterm #1

1. (a) $0\le x\le 2$; (b) $-2\le x\le 0$; (c) all x except 2 and 4; (d) $-1<x<1$.
2. (a) ABIJ; (b) DEFG; (c) GHI; (d) FJ; (e) J; (f) F.
3. $t=f(s)=\frac{130}{4s}-\frac{1}{2}s$, $f(5\frac{1}{12})\approx f(5)+\frac{1}{12}f'(5)=4+\frac{1}{12}(-1.8)=3.85$ ft. 4. $9\frac{7}{81}=9.0864$ cm.

Answers to Practice First Midterm #2

1. (a) like the x^2-parabola, but shifted to have its vertex at $(-2,-4)$; (b) bottom half of ellipse, passes through ±1 on the x-axis and -5 on the y-axis. 2. $-\pi/2<x<\pi/2$.
3. (a) CDJ; (b) AFGH; (c) ABHI; (d) CGJ; (e) C; (f) G. 4. (a) $v_0=9.5$ m/sec; (b) $v_0=f(t)=4.9t-\frac{75}{t}$, $f(5\pm0.1)\approx 9.5\pm0.1\cdot7.9=9.5\pm0.79$ m/sec.
5. $y=f(x)=1000/x$, $f(10.77)\approx f(10)+(0.77)f'(10)=100+(0.77)(-10)=92.3$ m^2.

Answers to Practice First Midterm #3

1. $0.5<x\le 2$. 2. (a) AFG; (b) CDI; (c) DE; (d) CFI; (e) C.
3. $g=f(t)=\frac{22.5}{t^2}$; (a) $f(3)=2.50$; (b) $f(3.1)\approx f(3)+0.1\cdot f'(3)=2.33$, so g is between 2.33 and 2.50 m/sec^2. 4. $y=f(x)=\sqrt{36^2-x^2}$, $f(21.6\pm0.2)\approx 28.8$ in $\pm$ 0.15 in.

Answers to Practice First Midterm #4

1. (a) $-35<x<15$; (b) $-7\le x\le 7$. 2. (b) when $t=12$ (parachute opens); when $t=0$ (drop from plane); right after parachute opens. 3. 12.8. 4. (a) $A=\frac{\pi}{4}r^2+rs$; (b) $100\pi+600$; (c) $s=f(r)=\frac{100\pi+600}{r}-\frac{\pi}{4}r$, $f(21)\approx f(20)+1\cdot f'(20)=26.93$ cm.

Answers to Practice Second Midterm #1

1. 19.861. 2. (a) $(-1,-1.5)$ is max; (b) $x<-1$ and $x>0$; (c) y-axis; (d) none; (e) see bottom of page. 3. (b) $y=20\sin\left(\frac{2\pi}{12}(t-9.5)\right)+55$; (c) mid-March to mid-November. 4. b, $C,\cos C$ are all constants, a and c are variables, take d/dt of everything, you know $da/dt=-5.70$, you get $dc/dt=-5.55$ m/sec.

Answers to Practice Second Midterm #2

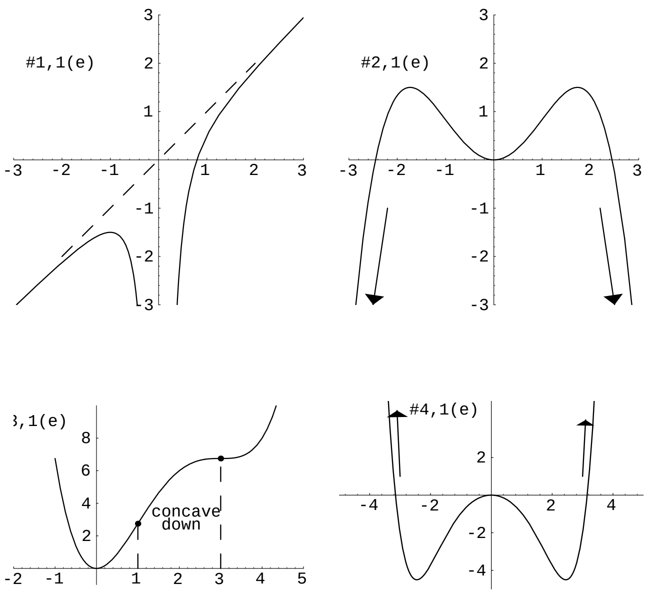
1. (a) $(0,0)$, $(\sqrt{3},1.5)$, $(-\sqrt{3},1.5)$; (b) $-\infty<x<-\sqrt{3}$, and $0<x<\sqrt{3}$; (c) $(1,5/6)$, $(-1,5/6)$; (d) $-1<x<1$; (e) see below.
2. (a) 5π rad/sec; (b) $\dot{x}=5(1-\cos\theta)$, $\dot{y}=5\sin\theta$; (c) $\dot{x}=2.50$ m/sec, $\dot{y}=-4.33$ m/sec, $dy/dx=-1.73$. 3. (a) $x=\sqrt{100-4y^2}$; (b) $A=\pi(100-4y^2)$; (c) 0.63 m^2/sec.
4. (a) $\sqrt{(x+5)^2+y^2}+\sqrt{x^2+(y+5)^2}=20$; (b) $y'=1/5$.

Answers to Practice Second Midterm #3

1. (a) $(0,0)$; (b) $(1,2.75)$ and $(3,6.75)$; (c) see bottom of page; (d) $1 < x < 3$.
2. $C = f(c)$ (a, b are const), $\pm 0.6175°$. 3. $y = 5\cot\theta$, 0.0436 ft/min.
4. (a) $y = 19.6 - 4.9t^2$; (b) $x = 2\sin(\pi(t - \frac{1}{2})) + 2$; (c) $\dot{x} = -4.443$ m/sec, $\dot{y} = -12.25$ m/sec, $dy/dx = 2.757$.

Answers to Practice Second Midterm #4

1. (a) max: $(0,0)$, min: $(\pm\sqrt{6},-4.5)$; (b) $-\sqrt{6}<x<0$ and $\sqrt{6}<x<\infty$; (c) $(\pm\sqrt{3},-2.925)$; (d) $-\sqrt{3}<x<\sqrt{3}$; (e) see bottom of page. 2. $\pi R^2T = 1$ m^3; take d/dt of both sides; 0.804 m/hr. 3. $B = f(A)$ (a, b are const), 61.732°. 4. (a) $V = \frac{1000\pi}{3}\tan^2(\frac{1}{2}\theta)$; (b) 349 ft^3 $\pm$ 14 ft^3.



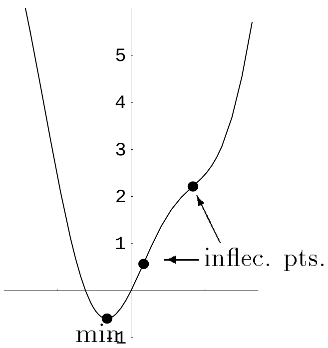
3. Answers to Practice Final Exams

Answers to Practice Final Exam #1

1: (a) -15/4; (b) $\frac{5}{\pi}\sin^2\left(\frac{\pi}{10}(t-1)\right)+C$, or else $-\frac{5}{\pi}\cos^2\left(\frac{\pi}{10}(t-1)\right)+C$. 2: -78.125 m/sec.
3: (b) $v=\frac{1}{2}t^2,\ 0\le t\le 1;\ v=t-\frac{1}{2},\ 1\le t\le 2;\ v=3/2,\ 2\le t\le 3$; (c) 8/3 m.
4: area of hexagon $=2\sqrt{3}r^2$, optimal ratio h/r is $4\sqrt{3}/\pi = 2.2053\cdots$.
5: (a) 4.60555 cm; (b) $x = 2\cos(7\pi t)+\sqrt{16-4\sin^2(7\pi t)}$;
(c) $dx/dt = -14\pi\sin(7\pi t) - 28\pi\sin(7\pi t)\cos(7\pi t)\Big(16-4\sin^2(7\pi t)\Big)^{-1/2}$;
(d) $f'(1/21) = -48.654$, $f(1/20)\approx 4.4897$ cm.
6: (a) $x = 5t-10\sin(t/4)$; (b) 8.18361 cm; (c) $t_1 = 3.57285$ sec, $t_2 = 3.55148$ sec, $x(t_2) = 10.00011$ cm.

Answers to Practice Final Exam #2

1: (a) $\frac{1}{4}(1-2t)^{-2}+C$; (b) 2/9. 2: (a) $20+\frac{10}{\pi}$ rev; (b) $40\pi+20$ rad; (c) 109.25 ft. 3. 16 ft/sec. 4: 2.033 ft. 5: $s = 1.59$ ft, $l = 4.76$ ft.
6: (b) $2x+2\cos(2x) = 0$; (c) $(\pi/12, \pi^2/144+1/2)$ and $(5\pi/12, 25\pi^2/144+1/2)$; (d) $-\pi/4$; (e) $x_1 = -0.5236$, $x_2 = -0.5150$; (f) no other max/min; (g) graph at right:



7: (a) $\pi/6$ rad/sec; (b) $x = 20\cos(\pi t/6)$, $y = 20+20\sin(\pi t/6)$;
(c) $v_{0,\text{horiz}} = -30$ ft/sec, $v_{0,\text{vert}} = 52$ ft/sec;
(d) $x = 75-30t$, $y = -16t^2+52t$; (e) 5.6737 ft;
(f) dist$=\sqrt{(75-30t-20\cos(\pi t/6))^2+(-16t^2+52t-20-20\sin(\pi t/6))^2}$
(g) let $f(t) = \frac{d}{dt}(\text{dist}^2) = 2(75-30t-20\cos(\pi t/6))(-30+(20\pi/6)\sin(\pi t/6))+$
$+2(-16t^2+52t-20-20\sin(\pi t/6))(-32t+52-(20\pi/6)\cos(\pi t/6))$.

Answers to Practice Final Exam #3

1: (a) $-\frac{2}{7}\sqrt{1-7z}+C$; (b) $(4-\sqrt{2})/3 = 0.8619$.
2: (b) $2-t$ for $0 \le t \le 1$, $0.5+0.5(2-t)^2$ for $1 \le t \le 2$, 0.5 for $2 \le t \le 3$, $2-0.5t$ for $3 \le t \le 4$; (c) 2.92 m. 3: -9.823 ft/sec. 4: differentiate $x+y+\sqrt{x^2+y^2}$ =const, get $y' = -32/49$, answer is 24.013.
5: (a) vol$=\frac{1}{3}\pi r^3 x$, area$=\pi r^2\sqrt{1+x^2}$; (b) $r = \sqrt{C/\pi}\left(1+x^2\right)^{-1/4}$. (c) $x = \sqrt{2}$.
6: Choose $f(x) = x^{1/2}+x^{1/3}-x$ in Newton's method, so that $f'(x) = \frac{1}{2}x^{-1/2}+\frac{1}{3}x^{-2/3}-1$; then $x_0 = 3$, $x_1 = 3.3163$, $x_2 = 3.3094$.
7: (a) $x = \frac{1}{2}\sin(2\pi(t-\frac{1}{4}))+\frac{1}{2}$; (b) $(4-4.9t^2)^2+(\frac{1}{2}\sin(2\pi(t-\frac{1}{4}))+\frac{1}{2})^2$; (c) $f(t) = -19.6t(4-4.9t^2)+\big(\sin(2\pi(t-\frac{1}{4}))+1\big)\pi\cos(2\pi(t-\frac{1}{4}))$.

Answers to Practice Final Exam #4

1: (a) $28/15$; (b) $3B/16\pi$. 2: (a) $3+3t-\frac{3}{2}t^2$, $0 \le t \le 2$; $9-3t$, $2 \le t \le 3$; (b) 9.5 m.
3: (a) Vol$=50\cdot 4\cos\theta(14+4\sin\theta)$; (b) $\mathrm{d(Vol)}/d\theta = -400(4X^2+7X-2)$ (where $X = \sin\theta$); (c) 2904 in^3.
4: A good initial guess is $s_0 = 10$ (since vol=1000, surface area=600). Choose $f(s) = s^3+6s^2-1500$; then $s_1 = 9.762$, $s_3 = s_2 = 9.757$.
5: (a) $x_A = \cos(2\pi t/9)$, $y_A = \sin(2\pi t/9)$, $x_B = 2-t$, $y_B = t$;
(b) $2\big(2-t-\cos(2\pi t/9)\big)\big(-1+(2\pi/9)\sin(2\pi t/9)\big)+2\big(t-\sin(2\pi t/9)\big)\big(1-(2\pi/9)\cos(2\pi t/9)\big)$.
6: (a) $x = (10\cos\alpha)t$, $y = -4.9t^2+(10\sin\alpha)t+50$; (b) $T = 1.0204\big(\sin\alpha+\sqrt{\sin^2\alpha+9.8}\big)$; (c) $X = (10\cos\alpha)T$ (write T in terms of α using (b)); (d) find $\frac{\pi}{180}\cdot\frac{dX}{d\alpha}$ and then take $\alpha = 60°$: -0.5781 (the minus sign means that a positive error in α leads to a negative error in X).

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