CHAPTER 1

Techniques of Integration I: Substitution, Powers of Trig Functions

1. Substitution

By a **technique of integration** we mean a method which can be used to find the antiderivative (indefinite integral) of certain types of functions. We first review u-substitutions. We shall use an example to illustrate the thought process that goes into deciding on a substitution of the form $u = u(x)$.

EXAMPLE 1.1. Find the x-coordinate of the center of mass of the uniform plate that is bounded by the x- and y-axes, the line $x = 10$, and the curve $y = 1/(1 + 0.01x^2)$.

We must compute $\overline{x} = \int_0^{10} x\,dx/(1 + 0.01x^2) \Big/ \int_0^{10} dx/(1 + 0.01x^2)$. At first the top and bottom integrals look similar: both have $(1 + 0.01x^2)$ in the denominator. However, they are actually quite different.

In the bottom we have something very close to $\int du/(1 + u^2)$ =Arctan $u + C$. The only problem is the constant 0.01 in front of x^2. So we choose u so that $u^2 = 0.01x^2$, i.e., $u = 0.1x$. Then $x = 10u$, $dx = 10du$. The limits of integration also change: $x = 0 \Longrightarrow u = 0$, and $x = 10 \Longrightarrow u = 0.1 \cdot 10 = 1$. Hence, the bottom integral becomes $\int_0^1 10\,du/(1 + u^2) = 10\,\mathrm{Arctan}\,u \,\big|_0^1 = 10(\pi/4) = 2.5\pi$.

The top integral, on the other hand, has nothing to do with Arctan. The x next to the dx changes everything. Namely, x is essentially the derivative of the denominator. Thus, a substitution u =denominator will lead to something like $\int du/u = \ln|u|+C$. More precisely, we set $u = 1+0.01x^2$, so that $du = 0.02x\,dx$, and hence $x\,dx = 50\,du$. This gives us $50\,du/u$ inside the integral. The limits of integration change as follows: $x = 0 \Longrightarrow u = 1+0.01\cdot 0 = 1$, and $x = 10 \Longrightarrow u = 1 + 0.01 \cdot 10^2 = 2$. Thus, the top integral becomes $50\int_1^2 du/u = 50 \ln 2$. Finally, we compute $\overline{x} = 50 \ln 2/2.5\pi = 4.4127$.

2. Odd Powers of Sine and Cosine

The technique of u-substitution can be used to evaluate any integral of the form

$$\int \sin^n(ax + b)\,dx \quad\text{or}\quad \int \cos^n(ax + b)\,dx, \text{ where } n \text{ is a positive odd number.}$$

The method works as follows. Say we want to integrate $\sin^n$. First take out one sin and put it with the dx. What remains will be sin to an *even* power. Express this in terms of cos (using $\sin^2 = 1 - \cos^2$). Now make a u-substitution for the cos. You will end up with only a polynomial to integrate. (If we had $\cos^n$ to integrate, we would do the same thing, with the roles of sin and cos reversed.) We illustrate with an example.

EXAMPLE 1.2. Find $\int_0^1 \sin^5(\pi x)dx$.

We first write

$$\int_0^1 \sin^5(\pi x)dx = \int_0^1 \sin^4(\pi x)\sin(\pi x)dx = \int_0^1 \left(1-\cos^2(\pi x)\right)^2 \sin(\pi x)dx.$$

Next, we make the substitution $u = \cos(\pi x)$, so that $du = -\pi\sin(\pi x)$. Then $\sin(\pi x)dx = -\frac{1}{\pi}du$, and so our integral becomes $-\frac{1}{\pi}\int\left(1-u^2\right)^2 du$. The limits of integration change as follows: $x = 0 \Longrightarrow u = \cos 0 = 1$, and $x = 1 \Longrightarrow u = \cos\pi = -1$. To integrate the polynomial, we expand the square: $(1-u^2)^2 = 1-2u^2+u^4$. So we obtain:

$$-\frac{1}{\pi}\int_1^{-1} 1-2u^2+u^4\,du = -\frac{1}{\pi}\left(u-\frac{2}{3}u^3+\frac{1}{5}u^5\right)\Big|_1^{-1} = \frac{16}{15\pi}.$$

It is also possible to use this method to integrate *negative* odd powers of sine and cosine. However, in that case instead of a polynomial in u you end up with a *rational function* (i.e., you have a denominator with a polynomial in it). We will treat some cases of this in part III of techniques of integration.

3. Even Powers of Sine and Cosine

The above technique does not work if you want to integrate something like $\cos^2$ or $\sin^4$. To handle even powers one needs a trig identity, namely, the $\cos(2A)$ formula: $\cos(2A) = \cos^2 A - \sin^2 A$. When combined with the trig identity $1 = \cos^2 A + \sin^2 A$, this gives formulas for either $\cos^2 A$ or $\sin^2 A$ in terms of $\cos(2A)$. If we add these two identities and divide by 2, we get $\cos^2 A$; if we subtract them and divide by 2, we get $\sin^2 A$:

$$\begin{array}{rl} & \cos^2 A+\sin^2 A = 1 \\ + & \underline{\cos^2 A-\sin^2 A} = \underline{\cos(2A)} \\ & 2\cos^2 A = 1+\cos(2A) \end{array} \qquad \begin{array}{rl} & \cos^2 A+\sin^2 A = 1 \\ - & \underline{\cos^2 A-\sin^2 A} = \underline{\cos(2A)} \\ & 2\sin^2 A = 1-\cos(2A)\,, \end{array}$$

i.e.,

$$\cos^2 A = \frac{1+\cos(2A)}{2}, \qquad \sin^2 A = \frac{1-\cos(2A)}{2}.$$

We now illustrate the use of these two identities in integration problems.

Example 1.3. Find the average value of $\sin^2 x$ between 0 and π.

The average value of a function $f(x)$ on the interval $[a,b]$ is the integral of the function divided by the length of the interval: $\frac{1}{b-a}\int_a^b f(x)dx$. So the average value of $\sin^2 x$ is

$$\begin{aligned} \frac{1}{\pi}\int_0^\pi \sin^2 x\,dx &= \frac{1}{\pi}\int_0^\pi \left(\frac{1-\cos(2x)}{2}\right)dx \\ &= \frac{1}{\pi}\left(\frac{1}{2}x-\frac{1}{2}\cdot\frac{1}{2}\sin(2x)\right)\Big|_0^\pi \\ &= \frac{1}{\pi}\left(\frac{\pi}{2}-\frac{1}{4}(0-0)\right) = \frac{1}{2}. \end{aligned}$$

Example 1.4. Find the antiderivative of $\cos^4 x$.

We have

$$\int \cos^4 x \, dx = \int \left(\cos^2 x\right)^2 \, dx = \int \left(\frac{1+\cos(2x)}{2}\right)^2 \, dx$$
$$= \int \left(\frac{1+2\cos(2x)+\cos^2(2x)}{4}\right) \, dx.$$

We now use the identity for $\cos^2(A)$ once more, this time with $A = 2x$: $\cos^2(2x) = (1 + \cos(4x))/2$. In this way we end up replacing all powers of cosine with expressions of the form $\cos(ax)$, which are easy to integrate. After writing

$$1+2\cos(2x)+\cos^2(2x) = 1+2\cos(2x)+(1+\cos(4x))/2$$
$$= \frac{3}{2}+2\cos(2x)+\frac{1}{2}\cos(4x)$$

and dividing this by 4, we obtain

$$\int \cos^4 x \, dx = \int \left(\frac{3}{8}+\frac{1}{2}\cos(2x)+\frac{1}{8}\cos(4x)\right) dx$$
$$= \frac{3}{8}x+\frac{1}{4}\sin(2x)+\frac{1}{32}\sin(4x)+C.$$

HOMEWORK 1.

1. Find the x-coordinate of the center of mass of the flat plate bounded by the x- and y-axes, the line $x = 5$, and the curve $y = 1/\sqrt{1 - 0.01x^2}$.

2. Find the anti-derivative of

(a) $\sin(2\pi t)/\sqrt{2 + \cos(2\pi t)}$;

(b) $\cos(ax + b)/\sin^8(ax + b)$.

3. Find the volume of the solid obtained by revolving $y = \cos x$ around the x-axis between $x = 0$ and $x = \pi/4$.

4. Find the average value of $\sin^n x$ between 0 and $\pi/2$ when n is

(a) 3, (b) 4, (c) 5, (d) 6, (e) 7.