The Lattice of Knaster Continua

Carl Eberhart[1], J. B. Fugate[2], and Shannon Schumann[3]

It is shown that the set of Knaster continua possesses a natural lattice ordering and a description of this lattice is included.

MOS Subject Classification
Primary:54F15
Secondary:06B23

Key Words: lattice, indecomposable, continuum, open mapping

[1]carl@ms.uky.edu, University of Kentucky, Lexington, Kentucky
[2]fugate@ms.uky.edu, University of Kentucky, Lexington, Kentucky
[3]schumann@math.uccs.edu, University of Colorado at Colorado Springs, Colorado Springs, Colorado

1. Lattice orderings

A **quasi-ordering** of a set X is a relation $\leq$ on X which is transitive and reflexive. If in addition $\leq$ is antisymmetric then $\leq$ is a **partial order** on X. A partial order $\leq$ of a set X is called a **lattice** order of X if each pair (x, y) of elements of X has a least upper bound $x \vee y$ and greatest lower bound $x \wedge y$ in X. A lattice ordering is **distributive** if the operations $\vee$ and $\wedge$ distribute over each other, and is **complete** if each nonempty bounded subset has a greatest lower bound and least upper bound. For more information and terminology on lattices, consult with any text on lattices, for example [1].

Now, take any set $\mathcal{S}$ of subcontinua of the Hibert Cube and any set $\mathcal{M}$ of mappings between members of $\mathcal{S}$ such that $\mathcal{M}$ is closed under composition and contains all homeomorphisms between member of $\mathcal{S}$. Then it is easy to see that the relation $\leq$ on $\mathcal{S}$

defined by

$$X \leq Y \iff \text{there is a map } f : X \to Y \text{ with } f \text{ in } \mathcal{M}$$

is a quasi-order of $\mathcal{S}$.

Further, let $\sim$ be the equivalence relation on $\mathcal{S}$ defined by

$$X \sim Y \iff X \leq Y \text{ and } Y \leq X.$$

Then we see that the quotient set $\mathcal{S}/\sim$ is partially ordered by the relation $[X] \leq [Y] \iff X \leq Y$, where $[X]$ denotes the $\sim$-equivalence class of X. The requirement that $\mathcal{M}$ contain all homeomorphisms between members of $\mathcal{S}$ guarantees that the $\sim$-equivalence classes are unions of homeomorphism classes, and so the relation $X \leq Y$ is topologically invariant. We will call this partial order the $\mathcal{M}$ **partial-order of** $\mathcal{S}/\sim$.

Usually, this partial order of $\mathcal{S}/\sim$ is not a lattice order, and we are delighted to find cases when it is. One such case, demonstrated here, is that if $\mathcal{S}$ is the set of Knaster subcontinua (see below) of the Hilbert cube and $\mathcal{M}$ is the set of open maps between them, then the partial order is a lattice order. We will describe that lattice order in some detail, and use it to motivate some questions about open maps on Knaster continua.

Let $\mathbb{P}$ denote the set of primes $\{2, 3, 5, \cdots\}$, and let ω denote the first infinite ordinal $[0, 1, 2, \cdots, \infty]$. A **trivial function** is a function $\phi : \mathbb{P} \to \omega$ such that $\phi(p) < \infty$ for all primes p and $\phi(p) = 0$ for all but finitely many primes p.

1.1. Theorem *The set $\omega^{\mathbb{P}}$ of functions $\eta : \mathbb{P} \to \omega$, is a complete distributive lattice when ordered by the relation*

$$\eta \leq \lambda \iff \eta(p) \leq \lambda(p) \text{ for all primes } p.$$

The relation $\sim$ on this lattice defined by

$$\eta \sim \lambda \iff \exists \text{ a trivial function } \phi \text{ such that } \eta \vee \phi = \lambda \vee \phi$$

is a lattice congruence on $\omega^{\mathbb{P}}$, and the quotient lattice $\omega^{\mathbb{P}}/\sim$ is distributive and complete.

Proof: The order on $\omega^{\mathbb{P}}$ is the product order. Since ω is well ordered, it is a complete distributive lattice, so $\omega^{\mathbb{P}}$ is a complete distributive lattice. Use the facts that the set of trivial functions in $\omega^{\mathbb{P}}$ is a sublattice and the distributivity, to see that the relation $\sim$ is a lattice congruence, and the quotient lattice $\omega^{\mathbb{P}}/\sim$ is distributive and complete. ■

2. Knaster Continua

Following the notation of Rogers in [5], for each positive integer n, let $w_n : I \to I$ be defined by

$$w_n(x) = \begin{cases} nx - i & \text{if } i \text{ is even and } 0 \leq \dfrac{i}{n} \leq x \leq \dfrac{i+1}{n} \leq 1 \\ i + 1 - nx & \text{if } i \text{ is odd and } 0 < \dfrac{i}{n} \leq x \leq \dfrac{i+1}{n} \leq 1 \end{cases}$$

The map w_n will be called **the standard map of degree n**.

If π is any sequence in $\mathbb{P} \cup \{1\}$ and K_π denotes the inverse limit $\varprojlim\{I_k, \pi_k^{k+1}\}$, where $I_k = I$ and $\pi_k^{k+1} = w_{\pi(k)}$, then K_π is an indecomposable continuum (a continuum is a compact connected metric space) except in the case that $\pi(i) = 1$ for all but finitely many i (see Nadler [4]). We refer to the continua K_π as **Knaster continua**, even in this last case where the inverse limit is homeomorphic with I. Note that K_π is a subcontinuum of Q, the Hilbert cube.

In [2], Dębski provides a classification theorem for Knaster continua.

2.1. Theorem *Dębski's Classification: Two Knaster continua K_π and K_ρ are homeomorphic if and only if for all but finitely many primes p, p occurs in the sequences π and ρ the same number of times. In the exceptional cases, the number of occurences of p in each sequence is finite.*

Every sequence π in $\mathbb{P} \cup \{1\}$ has associated with it an **occurence function** occ_π, defined on the set $\mathbb{P}$ of primes by $occ_\pi(p)$ is the number of occurences of p in the sequence.

Note that the occurence function of a sequence π with $K_\pi \sim I$ is a trivial function.

So the set $\mathbb{K}$ of homeomorphism classes of Knaster continua is in 1-1 correspondence with lattice $\omega^{\mathbb{P}}/\sim$ under the function $[K_\pi] \to [occ_\pi]$. The next theorem shows that if the Knaster continua are ordered with the open-map quasi-order, this correspondence is an order isomorphism.

A map $f : K_\pi \to K_\rho$ is said to be an **induced map** provided that there is an increasing sequence of subscripts i_k and maps $f_k : I_{i_k} \to I_k$ so that $\rho_k f = f_k \pi_{i_k}$ for each $k = 1, 2, \cdots$. The sequence is called a **defining sequence** of coordinate maps for f.

In [3], we proved (Theorem 4.7, p 143) that the open induced maps are dense in the space of open maps from K_π to K_ρ.

We can use this to prove the following theorem.

2.2. Theorem *The open-map partial order on $\mathbb{K}$ is a lattice order which is order isomorphic with the lattice $\omega^{\mathbb{P}}/\sim$ under the function $[K_\pi] \to [occ_\pi]$.*

Proof: As we have observed above, the function $[K_\pi] \to [occ_\pi]$ is a 1-1 correspondence between the sets $\mathbb{K}$ and $\omega^{\mathbb{P}}/\sim$. We will show that the function and its inverse are order-preserving. Suppose that $[occ_\rho] \leq [occ_\pi]$. Then we construct an open induced map f from K_π to K_ρ as follows: Let n be the product of the finite number of occurences of primes that occur in ρ that do not occur in π, and let $f_1 = w_n : I_1 \to I_1$. If $\rho(1)$ divides n, let $f_2 = w_{n\pi(1)/\rho(1)} : I_2 \to I_2$; otherwise choose k_2 so that $\pi(k_2 - 1)$ is the first occurence of $\rho(1)$ in the sequence π, and let $f_2 = w_n \pi_1^{k_2-1}$. From the definition of $[occ_\rho] \leq [occ_\pi]$, we are guaranteed that we can continue to define $f_3, f_4, \cdots$ so that the induced map is open. Hence $K_\rho \leq K_\pi$.

Conversely, suppose that $K_\rho \leq K_\pi$. Then there is an open map $f : K_\pi \to K_\rho$. Then by Theorem 4.7 in [3], there is an open induced map $g : K_\pi \to K_\rho$. By the structure theorem for induced open maps (Theorem 3.16, p. 136, in [3]), $g = hwu$, where h and u are homeomorphisms and $w : K_\pi \to K_\rho$ is an induced map whose coordinate maps $w_{n_i} : I_{k_i} \to I_i$ are all standard open maps.

Consider the first coordinate map w_{n_1} of w. Suppose that p is a prime that occurs at least $r+s$ times ($s > 0$) in the sequence ρ and only r times in the sequence π. Choose N so large that p occurs $r+s$ times from ρ_1 to ρ_{N-1}. Then p^s divides M where $w_M = w_{\rho_1} \cdots w_{\rho_N} w_{n_N}$. Since w is induced, $w_{\rho_1} \cdots w_{\rho_N} w_{n_N} = w_{n_1} w_{\pi_{k_1+1}} \cdots w_{\pi_{k_N - 1}}$, and so p^s divides n_1 the subscript of the first coordinate map w_{n_1} of w. From this we conclude that no prime occurs infinitely often in the sequence ρ but not in π and only finitely many primes can occur more often in ρ than they do in π. Hence $occ_\rho \leq occ_\pi$, and we have shown that the correspondence $[K_\pi] \to [occ_\pi]$ is an order isomorphism. ∎

3. The structure of the lattice

If for all but finitely many $p \in \mathbb{P}$, $occ_\pi(p)$ is either 0 or ∞, then occ_π is said to be **full**. If ∞ is not a value of occ_π, then occ_π is said to be **sparse**.

An occurrence function occ_π always decomposes into the join

$$occ_\pi = full_\pi \vee sps_\pi$$

of a full occurrence function $full_\pi$, and a sparse one sps_π, given by

$$full_\pi(p) = 0 \text{ if } occ_\pi(p) < \infty, full_\pi(p) = occ_\pi(p) \text{ otherwise},$$

$$sps_\pi(p) = 0 \text{ if } occ_\pi(p) = \infty, sps_\pi(p) = occ_\pi(p) \text{ otherwise}.$$

So, by the lattice isomorphism, $[K_\pi] \to [occ_\pi]$, A Knaster continuum K_π always decomposes into the join according to how occ_π decomposes. We will say K_π is **full** (**sparse**) if occ_π is full (sparse). So, for example if $\pi = 2, 3, 2, 5, 2, 7, \cdots$ is the sequence whose odd terms are all 2 and whose even terms are the odd primes, then $K_\pi = K_2 \vee K_\sigma$, where K_2 is the bucket-handle (a full Knaster continuum) and $\sigma = 3, 5, \cdots ,$, the sequence of odd primes (so K_σ is a sparse Knaster continuum).

Define $\mathbb{K}_F$ and $\mathbb{K}_S$ to be the set of full (resp. sparse) homeomorphism classes of Knaster continua.

Define a function $\Phi\colon \mathbb{K} \to 2^{\mathbb{P}}$, the lattice of subsets of $\mathbb{P}$ by

$$\Phi([K_\pi]) = \{p \in \mathbb{P} | occ_\pi(p) = \infty\}$$

We see that the function Φ is a lattice homomorphism.

3.1. Theorem *The set of full Knaster continua $\mathbb{K}_F$ is a sublattice of $\mathbb{K}$, isomorphic with the lattice of subsets of the prime numbers. Hence $\mathbb{K}_F$ is a complete distributive lattice.*

Proof: It is easily verified that Φ takes $\mathbb{K}_F$ isomorphically onto $2^{\mathbb{P}}$. ■

The bottom element of $\mathbb{K}$ is K_1, which is homeomorphic with the unit interval I. Let γ be the sequence of primes $2, 3, 2, 3, 5, 2, 3, 5, 7, \cdots$, where $occ_\pi(p) = \infty$ for each prime p. It is natural to call K_γ the **universal** Knaster continuum, because by Theorem 2.2 K_γ maps openly onto any Knaster continuum. It is the largest element of the lattice of Knaster continua.

The full Knaster continua K_π for which $occ_\pi^{-1}(\infty)$ is finite is a sublattice (in fact a $\wedge$-ideal) of $\mathbb{K}_F$ isomorphic with the lattice of finite subsets of $\mathbb{P}$. We call this sublattice $\mathbb{FI}$. At the other end of the lattice $\mathbb{K}_F$ are the full Knaster continua K_π for which $\mathbb{P} \setminus occ_\pi^{-1}(\infty)$ is finite, that is the Knaster continua in which all but finitely many primes occur infinitely many times. These form a sublattice (in fact, a $\vee$-ideal of $\mathbb{K}_F$ which we call $\mathbb{IF}$. The remaining continua do not form a sublattice; we call this set $\mathbb{II}$. (See the diagram below for the position of these three sets in $\mathbb{K}$.)

The set of sparse Knaster continua forms a lattice also. However, it is very different from the lattice of full Knaster continua.

3.2. Theorem *The set of sparse Knaster continua $\mathbb{K}_S$ is a $\wedge$-ideal of $\mathbb{K}$, that is, if $K_\pi \in \mathbb{K}$ and $K_\rho \in \mathbb{K}_S$, then $K_\pi \wedge K_\rho \in \mathbb{K}_S$. Also $\mathbb{K}_S$ is not a complete lattice.*

Proof: Clearly, $\mathbb{K}_S = \Phi^{-1}(\emptyset)$, and hence is a $\wedge$-ideal. The fact that it is far from complete can be seen as follows: Let K_π be any sparse Knaster continuum, other than K_1. For each positive integer n, define $K_{n\pi}$ to be the continuum where $occ_{n\pi}(p) = np$ for each prime p.

Clearly, we have

$$K_\pi < K_{2\pi} < \cdots < K_{n\pi} \cdots$$

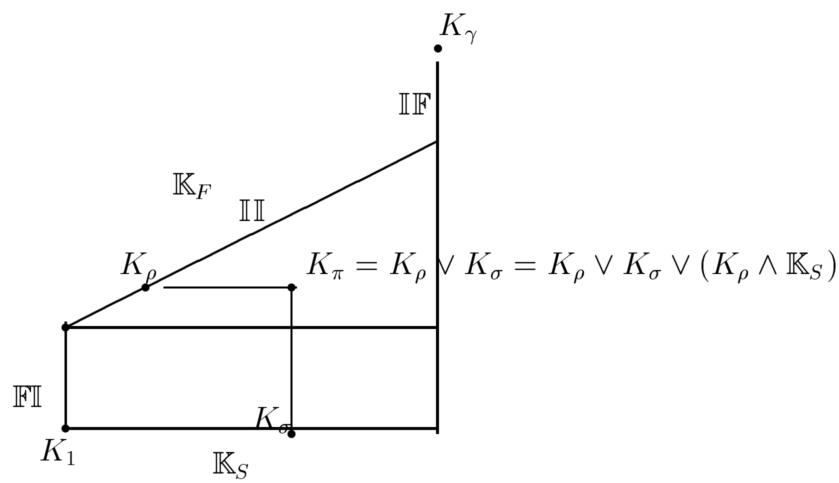
in the lattice order on $\mathbb{K}$. In $\mathbb{K}$, the least upper bound of this chain is K_γ and the, which is not sparse, so $\mathbb{K}_S$ is not complete. ■

3.3. Theorem *For each $K_\pi \in \mathbb{K}$ there are $K_\rho \in \mathbb{K}_F$ and $K_\sigma \in \mathbb{K}_S$ such that $K_\pi = K_\rho \vee K_\sigma$. This decomposition is unique in the sense that if $K_{\rho'} \in \mathbb{K}_F$ and $K_{\sigma'} \in \mathbb{K}_S$ such that $K_\pi = K_{\rho'} \vee K_{\sigma'}$, then $K_\rho = K_{\rho'}$ and $K_\sigma \leq K_{\sigma'}$. Furthermore if $K_\rho \in \mathbb{FI}$, then $K_\sigma = K_{\sigma'}$.*

Proof: Choosing K_ρ and K_σ to be any Knaster continua with $occ_\rho = full_\pi$ and $occ_\sigma = sps_\pi$ satisfies the first sentence. Clearly, there is no leeway in the choice $full_\pi$; however, if $occ_{\pi(p)} = \infty$ for infinitely many primes p then we can augment σ by throwing in an infinite subset of those primes with arbitrarily chosen finite occurence values, thus creating a sequence σ' greater than σ. If $full_\pi^{-1}(\infty)$ is finite, we cannot augment in this manner, so K_σ is unique. ■

Here is a diagram that we can use to visualize the lattice of Knaster Continua.

FIGURE 1. The lattice of Knaster continua



In the same paper where he classifies the Knaster continua, Dębski also shows that there is an uncountable set of Knaster continua, no one of which is the open image of the other. His example is in fact a collection of sparse Knaster continua. By modifying his example, it is not hard to prove that there are many such collections of incomparable Knaster continua.

3.4. Theorem *There exist an uncountable set of full Knaster continua no one of which is the open image the other. For each full Knaster continuum K_π such that $occ_\pi^{-1}(0)$ is infinite, there is an uncountable set of incomparable Knaster continua with full part K_π.*

4. Some questions motivated by the lattice structure

One of the uses of lattices in topology is to organize information about spaces. This in turn exposes gaps in the information and may prompt questions about those spaces. The following questions about open maps on Knaster continua arose in this way. (That is not to say they might not have arisen in some other way.)

In [3], p. 138, we showed that all the the <u>induced</u> open maps $f : K_\pi \to K_\pi$ are at most n to 1, for some positive integer n.

4.1. Question *Must an open map between homeomorphic Knaster continua be at most n to 1, for some positive integer n?*

By contrast, we can show the following

4.2. Theorem *If K_π and K_ρ are not homeomorphic, then each open induced map $f : K_\pi \to K_\rho$ is uncountable to 1.*

Proof: Let f be an induced open map from K_π to K_ρ. Then by the structure theorem for open induced maps ([3], page 136), $f = hwu$ where h and u are homeomorphisms and $w : K_\pi \to K_\rho$ is an induced map whose coordinate maps are standard open maps. Since K_π and K_ρ are not homeomorphic, either some prime p occurs infinitely often in the sequence π and only finitely often in the sequence ρ, or there are infinitely many primes p which occur in the sequence π but occur less often in the sequence ρ. Hence infinitely many of the standard open maps w_n in the defining sequence of w have subscripts arbitrarily large. This is enough to show that given a point x in K_ρ, $w^{-1}(x)$ contains a Cantor set. ■

4.3. Question *Must an open map between nonhomeomorphic Knaster continua be uncountable to one?*

In [3], p. 133, we showed that the induced open maps $f : K_\gamma \to K_\gamma$, the largest Knaster continuum, are all homeomorphisms. In light of the fact that the induced open maps are dense, it is natural to ask:

4.4. Question *Is there an open map $f : K_\gamma \to K_\gamma$ which is not a homeomorphism?*

We have investigated the set of solenoids (= inverse limits of the circle whose bonding maps are power maps) and found that they have the same lattice structure under the open-map quasi-ordering. It would be interesting to have some answers to the following question.

4.5. Question *For what other sets $\mathcal{S}$ of spaces and maps $\mathcal{M}$ between them is the $\mathcal{M}$ partial-order of $\mathcal{S}/\sim$ a lattice ordering?*

References

[1] P. Crawley, R.P. Dilworth, **Algebraic Theory of Lattices**, Prentice-Hall, 1973.

[2] W. Dębski, *On topological types of the simplest indecomposable continua*, Colloquium Mathematicum, Vol. XLIX, No. 2, 1985.

[3] Carl Eberhart, J.B. Fugate, and Shannon Schumann, *Open maps between Knaster continua*, Fund. Math. 162(1999), 119-148.

[4] S.B. Nadler, Jr., **Continuum Theory**, Dekker Press, 1992.

[5] J. W. Rogers, *On mapping indecomposable continua onto certain chainable indecomposable continua*, Proc. Amer. Math. Soc., **25** (1969), 449-456.